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Characterizations of timelike slant helices in Minkowski 3-space

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Abstract. In this paper, we investigate the tangent indicatrix, the principal normal indicatrix and the binormal indicatrix of a timelike curve in Minkowski 3-space $\mathbb{E}_1^3$ and we construct their Frenet equations and curvature functions. Moreover, we obtain some differential equations which characterize a timelike curve to be a slant helix by using the Frenet apparatus of a spherical indicatrix of the curve. Also, related examples and their illustrations are given.
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1. Introduction

In classical differential geometry, a general helix in Euclidean 3-space $\mathbb{E}^3$ is a curve with constant slope, that is, a curve which makes a constant angle with some fixed direction (the axis of the helix). The classical result stated by M. A. Lancret in 1802 and first proved by B. de Saint Venant in 1845 (for details see [16, 20]) is: A necessary and sufficient condition for a curve to be a general helix is that the ratio of its curvature and torsion is constant. In particular, circular helices with constant curvature and torsion as well as plane curves with vanishing torsion provide two subclasses of general helices.

The Lancret theorem was revisited and solved by M. Barros ([5]) in three-dimensional real space forms by using the notion of Killing vector fields along curves. Characterizations for helices and Cornu spirals in those backgrounds were also obtained by J. Arroyo, M. Barros and O. J. Garay in [1].

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For general helices in semi-Riemannian settings, including Lorentzian ones, we refer the reader to [6, 7, 8, 9, 10, 12] .

Recently, in [11] Izumiya and Takeuchi, have introduced the concept of slant helix in Euclidean 3-space. A slant helix in Euclidean space $\mathbb{E}^3$ was defined by the property that the principal normal makes a constant angle with a fixed direction. Moreover, Izumiya and Takeuchi showed that γ is a slant helix in E^3 if and only if the geodesic curvature of the principal normal of a space curve γ is a constant function.

In [14], L. Kula and Y. Yayli studied spherical images under both tangent and binormal indicatrices of slant helices and obtained that spherical images of a slant helix are spherical helices. In [15], the authors characterized slant helices by certain differential equations verified for each of the obtained spherical indicatrices in Euclidean 3-space. Recently, in [2] Ali and Lopez have studied a slant helix in Minkowski 3-space. They showed that the spherical indicatrix of a slant helix in $\mathbb{E}^3_1$ are helices. Also in [3], Ali and Turgut studied the position vector of a timelike slant helix in $\mathbb{E}^3_1$.

In this paper, we consider a timelike curve in Minkowski 3-space and obtain its spherical indicatrix and their Frenet apparatus. Finally, we obtain some differential equations for a timelike curve to be a slant helix by the help of a spherical indicatrix of the curve and well known results obtained by Ali and Lopez in [2] .

2. Preliminaries

Minkowski 3−space $\mathbb{E}^3_1$ is Euclidean 3-space $\mathbb{E}^3$ equipped with indefinite flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3_1$. Recall that a vector $v \in \mathbb{E}^3_1$ is called *spacelike* if $g(v, v) > 0$ or $v = 0$, *timelike* if $g(v, v) < 0$ and *null (lightlike)* if $g(v, v) = 0$ and $v \neq 0$. The norm of a vector v is given by $||v|| = \sqrt{|g(v, v)|}$ and two vectors v and w are said to be orthogonal if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in $\mathbb{E}^3_1$ can be locally *spacelike*, *timelike* or *null* (*lightlike*), if all its velocity vectors $\alpha'(s)$ are spacelike, timelike or null, respectively. A Spacelike or timelike curve α has unit speed, if $g(\alpha'(s), \alpha'(s)) = \pm 1$. A null curve α is parameterized by pseudo-arc s if $g(\alpha''(s), \alpha''(s)) = 1$ ([17]). For non-null unit speed space curve $\alpha(s)$ in the space $\mathbb{E}^3_1$ with non-null normals, the following Frenet formulae are given in [7, 10]

$$\begin{aligned} T'(s) &= \varkappa(s) N(s), \\ N'(s) &= -\varepsilon_0 \varepsilon_1 \varkappa(s) T(s) + \tau(s) B(s), \\ B'(s) &= -\varepsilon_1 \varepsilon_2 \tau(s) N(s), \end{aligned} \tag{1}$$

and the Minkowski vector products of Frenet vectors are given as

$$\begin{aligned} T(s) \times N(s) &= B(s), \\ N(s) \times B(s) &= -\varepsilon_1 T(s), \\ B(s) \times T(s) &= -\varepsilon_0 N(s), \end{aligned}$$

where $g(T(s), T(s)) = \varepsilon_0 = \pm 1$, $g(N(s), N(s)) = \varepsilon_1 = \pm 1$ and $g(B(s), B(s)) = \varepsilon_2 = \pm 1$ and two ε_i's are equal to 1, the other ε_i is -1.

For a null space curve $\alpha(s)$ with $\varkappa$ and τ curvature functions in the space $\mathbb{E}_1^3$, the following Frenet formulae are given in ([18, 19])

$$T' = \varkappa N, \quad N' = \tau T - \varkappa B, \quad B' = -\tau N$$
$$g(T,T) = g(B,B) = 0, \quad g(N,N) = 1, \quad g(T,N) = g(N,B) = 0, \quad g(T,B) = 1.$$

In this case, $\varkappa$ can take only two values: $\varkappa = 0$ when α is a straight null line or $\varkappa = 1$ in all other cases.

It is well known that the pseudo - Riemannian sphere with radius $r = 1$ and centered at the origin and defined by

$$S_1^2 = \left\{ p \in \mathbb{E}_1^3 : g(p,p) = 1 \right\}, \tag{2}$$

the pseudohyperbolic space of radius $r = 1$ and centered at the origin and defined by

$$H_0^2 = \left\{ p \in \mathbb{E}_1^3 : g(p,p) = -1 \right\} \tag{3}$$

are the hyperquadrics with dimension 2 and index 1 and with dimension 2 and index 0, respectively,([17]).

3. The spherical indicatrix of a timelike curve in Minkowski 3-space

In Euclidean geometry, a spherical indicatrix of a space curve is defined as follows: Let α be a unit speed regular curve in Euclidean 3 -space with Frenet vectors T, N and B. Unit tangent vectors along the curve α generate a curve (T) on the sphere of radius 1 about the origin. The curve (T) is called a spherical indicatrix of T or more commonly, (T) is called the tangent indicatrix of the curve α. If $\alpha = \alpha(s)$ is a natural representation of α, then $(T) = T(s)$ will be a representation of (T). Similarly, one considers the principal normal indicatrix $(N) = N(s)$ and the binormal indicatrix $(B) = B(s)$. It is clear that this definition is related with spherical curve ([20]).

In Minkowski 3-space $\mathbb{E}_1^3$, the definition of a spherical indicatrix of a space curve is similar to the Euclidean case but richer than the Euclidean case. For example, for a timelike curve, if its position vector is spacelike, then the curve lies on the pseudo-Riemannian sphere S_1^2; if its position vector is timelike, then the curve lies on pseudohyperbolic space H_0^2. In Minkowski space, for the characterizations of spherical curves, we refer to the papers of Petrović -Torgašev and Šućurović, ([18, 19]) and Inoguchi and Lee ([13]).

In this section, we investigate the Frenet apparatus of the tangent indicatrix, the principal normal indicatrix and the binormal indicatrix of a timelike curve in Minkowski 3-space. Here, by ε we consider

$$\varepsilon = \begin{cases} 1, & \tau^2 - \varkappa^2 > 0 \\ -1, & \tau^2 - \varkappa^2 < 0 \end{cases}.$$

Lemma 1. *Let α be a unit speed timelike curve in $\mathbb{E}_1^3$. Geodesic curvature of the spherical image of the spacelike principal normal indicatrix (N) of α is*

$$\sigma_1 = \frac{\varkappa^2}{\left(\tau^2 - \varkappa^2\right)^{3/2}} \left(\frac{\tau}{\varkappa}\right)'$$

and geodesic curvature of the spherical image of the timelike principal normal indicatrix (N) of α is

$$\sigma_2 = \frac{\varkappa^2}{\left(\varkappa^2 - \tau^2\right)^{3/2}} \left(\frac{\tau}{\varkappa}\right)',$$

where $\tau^2 - \varkappa^2$ does not vanish [2, 4].

In the next three theorems, we obtain Frenet formulae of *the tangent indicatrix β, the principal normal indicatrix γ* and *the binormal indicatrix δ of the timelike curve α* in $\mathbb{E}_1^3$.

Theorem 1. *Let α be a timelike curve in Minkowski space $\mathbb{E}_1^3$ with Frenet vectors T, N, B and curvatures $\varkappa$, τ. If the Frenet frame of the tangent indicatrix β of the space curve α is $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$, then we have the Frenet-Serret formulae:*

$$\begin{aligned} D_{\mathbf{T}}\mathbf{T} &= \varkappa_\beta \mathbf{N}, \\ D_{\mathbf{T}}\mathbf{N} &= -\varepsilon \varkappa_\beta \mathbf{T} + \tau_\beta \mathbf{B}, \\ D_{\mathbf{T}}\mathbf{B} &= \varepsilon \tau_\beta \mathbf{N}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} \mathbf{T} &= N, \\ \mathbf{N} &= \frac{1}{\sqrt{\varepsilon\left(\tau^2 - \varkappa^2\right)}}\left(\varkappa T + \tau B\right), \\ \mathbf{B} &= \frac{1}{\sqrt{\varepsilon\left(\tau^2 - \varkappa^2\right)}}\left(-\tau T - \varkappa B\right), \end{aligned} \tag{5}$$

and $\varkappa_\beta = \frac{\sqrt{\varepsilon(\tau^2 - \varkappa^2)}}{\varkappa}$ is the curvature of β, $\tau_\beta = \frac{\varkappa\left(\frac{\tau}{\varkappa}\right)'}{\tau^2 - \varkappa^2}$ is the torsion of β.

Proof. Let s be an arc-parameter of α and s_β an arc-parameter of β.

$$\beta\left(s_\beta\right) = T\left(s\right) \tag{6}$$

Differentiating equation (6) with respect to s and by using Frenet formulas given in equation (1), we get

$$\begin{aligned} \frac{d\beta}{ds_\beta} \cdot \frac{ds_\beta}{ds} &= \frac{dT\left(s\right)}{ds}, \\ \mathbf{T}\left(s_\beta\right) \cdot \frac{ds_\beta}{ds} &= \varkappa\left(s\right) N\left(s\right). \end{aligned} \tag{7}$$

Since $\varepsilon_0 = -1$, we have $\varepsilon_1 = 1$. Hence we have

$$g\left(\mathbf{T}\left(s_\beta\right), \mathbf{T}\left(s_\beta\right)\right) = \left(\frac{ds}{ds_\beta}\right)^2 \varkappa^2\left(s\right). \tag{8}$$

Since $g\left(\mathbf{T}\left(s_\beta\right), \mathbf{T}\left(s_\beta\right)\right) > 0$, β is a spacelike curve, that is, $g\left(\mathbf{T}\left(s_\beta\right), \mathbf{T}\left(s_\beta\right)\right) = 1$.

If we consider this fact in equation (8), we see that

$$\frac{ds_\beta}{ds} = \varkappa\left(s\right). \tag{9}$$

So, we can rewrite equation (7)

$$\mathbf{T}\left(s_\beta\right) = N\left(s\right). \tag{10}$$

Differentiating equation (10) with respect to s

$$\frac{d\mathbf{T}\left(s_\beta\right)}{ds_\beta} = T\left(s\right) + \frac{\tau}{\varkappa} B\left(s\right). \tag{11}$$

Since $\varepsilon_2 = 1$, the norm of $\frac{d\mathbf{T}(s_\beta)}{ds_\beta}$ is

$$\left\|\frac{d\mathbf{T}\left(s_\beta\right)}{ds_\beta}\right\| = \frac{\sqrt{\varepsilon\left(\tau^2 - \varkappa^2\right)}}{\varkappa}.$$

If we consider

$$\mathbf{N}\left(s_\beta\right) = \frac{\frac{d\mathbf{T}(s_\beta)}{ds_\beta}}{\left\|\frac{d\mathbf{T}(s_\beta)}{ds_\beta}\right\|},$$

we can write

$$\mathbf{N}\left(s_\beta\right) = \frac{1}{\sqrt{\varepsilon\left(\tau^2 - \varkappa^2\right)}}\left(\varkappa\left(s\right) T\left(s\right) + \tau\left(s\right) B\left(s\right)\right). \tag{12}$$

Now we know that $\mathbf{B}\left(s_\beta\right) = \mathbf{T}\left(s_\beta\right) \times \mathbf{N}\left(s_\beta\right)$ and using equations (10) and (12) we show that

$$\mathbf{B}\left(s_\beta\right) = \frac{1}{\sqrt{\varepsilon\left(\tau^2 - \varkappa^2\right)}}\left(-\varkappa\left(s\right) B\left(s\right) - \tau\left(s\right) T\left(s\right)\right). \tag{13}$$

We can easily see that $g\left(\mathbf{N}\left(s_\beta\right),\ \mathbf{N}\left(s_\beta\right)\right) = \varepsilon$ and $g\left(\mathbf{B}\left(s_\beta\right),\ \mathbf{B}\left(s_\beta\right)\right) = -\varepsilon$. Moreover, the Frenet formulas of β is given by

$$\begin{aligned} D_{\mathbf{T}}\mathbf{T} &= \varkappa_\beta \mathbf{N}, \\ D_{\mathbf{T}}\mathbf{N} &= -\varepsilon\varkappa_\beta \mathbf{T} + \tau_\beta \mathbf{B}, \\ D_{\mathbf{T}}\mathbf{B} &= \varepsilon\tau_\beta \mathbf{N}, \end{aligned}$$

By using the equality of $D_{\mathbf{T}}\mathbf{T} = \frac{d\mathbf{T}(s_\beta)}{ds_\beta} = \varkappa_\beta \mathbf{N}$ with equations (11) and (12), we get the curvature of β

$$\varkappa_\beta = \frac{\sqrt{\varepsilon\left(\tau^2 - \varkappa^2\right)}}{\varkappa}. \tag{14}$$

Similarly, from the equation $D_{\mathbf{T}}\mathbf{N} = -\varepsilon\varkappa_\beta \mathbf{T} + \tau_\beta \mathbf{B}$ we can easily see that the torsion of β is

$$\tau_\beta = \frac{\varkappa\left(\frac{\tau}{\varkappa}\right)'}{\tau^2 - \varkappa^2}. \tag{15}$$

This completes the proof. □

Theorem 2. *Let α be a timelike curve in $\mathbb{E}_1^3$ with Frenet vectors T, N, B and curvatures $\varkappa$, τ. Then we have the Frenet-Serret formulae of the principal normal indicatrix γ of the space curve α for three cases:*

Case I: If $\tau^2 > \varkappa^2$, γ is a spacelike curve.

(a) If $-1 < \sigma_1 < 1$, the Frenet frame of the curve γ is $\{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1\}$, then its Frenet-Serret formulas satisfy

$$\begin{aligned} D_{\boldsymbol{T}_1}\boldsymbol{T}_1 &= \varkappa_\gamma \boldsymbol{N}_1, \\ D_{\boldsymbol{T}_1}\boldsymbol{N}_1 &= -\varkappa_\gamma \boldsymbol{T}_1 + \tau_\gamma \boldsymbol{B}_1, \\ D_{\boldsymbol{T}_1}\boldsymbol{B}_1 &= \tau_\gamma \boldsymbol{N}_1, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \boldsymbol{T}_1 &= \frac{1}{\sqrt{\tau^2 - \varkappa^2}}\left(\varkappa T + \tau B\right), \\ \boldsymbol{N}_1 &= \frac{1}{\sqrt{1 - (\sigma_1)^2}}\left(-\frac{\sigma_1 \tau}{\sqrt{\tau^2 - \varkappa^2}} T - N - \frac{\sigma_1 \varkappa}{\sqrt{\tau^2 - \varkappa^2}} B\right), \\ \boldsymbol{B}_1 &= \frac{1}{\sqrt{(1-(\sigma_1)^2)(\tau^2 - \varkappa^2)}}\left(-\tau T - \sigma_1 \sqrt{\tau^2 - \varkappa^2} N - \varkappa B\right). \end{aligned} \tag{17}$$

Moreover, the curvature of γ is

$$\varkappa_\gamma = \sqrt{1 - (\sigma_1)^2} \tag{18}$$

and the torsion of γ is

$$\tau_\gamma = \frac{1}{(1 - (\sigma_1)^2)\sqrt{\tau^2 - \varkappa^2}} \sigma_1'. \tag{19}$$

(b) If $\sigma_1 < -1$ or $\sigma_1 > 1$, the Frenet frame of the curve γ is $\{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1\}$, then its Frenet-Serret formulas satisfy

$$\begin{aligned} D_{\boldsymbol{T}_1}\boldsymbol{T}_1 &= \varkappa_\gamma \boldsymbol{N}_1, \\ D_{\boldsymbol{T}_1}\boldsymbol{N}_1 &= \varkappa_\gamma \boldsymbol{T}_1 + \tau_\gamma \boldsymbol{B}_1, \\ D_{\boldsymbol{T}_1}\boldsymbol{B}_1 &= \tau_\gamma \boldsymbol{N}_1, \end{aligned} \tag{20}$$

where

$$\begin{aligned} \boldsymbol{T}_1 &= \frac{1}{\sqrt{\tau^2 - \varkappa^2}}\left(\varkappa T + \tau B\right), \\ \boldsymbol{N}_1 &= \frac{1}{\sqrt{(\sigma_1)^2 - 1}}\left(-\frac{\sigma_1 \tau}{\sqrt{\tau^2 - \varkappa^2}} T - N - \frac{\sigma_1 \varkappa}{\sqrt{\tau^2 - \varkappa^2}} B\right), \\ \boldsymbol{B}_1 &= \frac{1}{\sqrt{((\sigma_1)^2 - 1)(\tau^2 - \varkappa^2)}}\left(-\tau T - \sigma_1 \sqrt{\tau^2 - \varkappa^2} N - \varkappa B\right). \end{aligned} \tag{21}$$

Moreover, the curvature of γ is

$$\varkappa_{\gamma} = \sqrt{(\sigma_1)^2 - 1} \tag{22}$$

and the torsion of γ is

$$\tau_{\gamma} = \frac{1}{((\sigma_1)^2 - 1)\sqrt{\tau^2 - \varkappa^2}}\sigma_1'. \tag{23}$$

Case II: If $\tau^2 < \varkappa^2$, γ is a timelike curve and the Frenet frame of the curve γ is $\{\boldsymbol{T}_2, \boldsymbol{N}_2, \boldsymbol{B}_2\}$, then we have the Frenet-Serret formulae:

$$\begin{aligned} D_{\boldsymbol{T}_2}\boldsymbol{T}_2 &= \varkappa_{\gamma}\boldsymbol{N}_2, \\ D_{\boldsymbol{T}_2}\boldsymbol{N}_2 &= \varkappa_{\gamma}\boldsymbol{T}_2 + \tau_{\gamma}\boldsymbol{B}_2, \\ D_{\boldsymbol{T}_2}\boldsymbol{B}_2 &= -\tau_{\gamma}\boldsymbol{N}_2, \end{aligned} \tag{24}$$

where

$$\begin{aligned} \boldsymbol{T}_2 &= \frac{1}{\sqrt{\varkappa^2 - \tau^2}}\left(\varkappa T + \tau B\right), \\ \boldsymbol{N}_2 &= \frac{1}{\sqrt{1 + (\sigma_2)^2}}\left(\frac{\sigma_2\tau}{\sqrt{\varkappa^2 - \tau^2}}T + N + \frac{\sigma_2\varkappa}{\sqrt{\varkappa^2 - \tau^2}}B\right), \\ \boldsymbol{B}_2 &= \frac{1}{\sqrt{(1+(\sigma_2)^2)(\varkappa^2-\tau^2)}}\left(\tau T + \sigma_2\sqrt{\varkappa^2 - \tau^2}N + \varkappa B\right). \end{aligned} \tag{25}$$

Moreover, the curvature of γ is

$$\varkappa_{\gamma} = \sqrt{1 + (\sigma_2)^2} \tag{26}$$

and the torsion of γ is

$$\tau_{\gamma} = \frac{1}{(1 + (\sigma_2)^2)\sqrt{\varkappa^2 - \tau^2}}\sigma_2'. \tag{27}$$

Case III: If $\tau^2 = \varkappa^2$, γ is a null curve and the Frenet frame of the curve γ is $\{\boldsymbol{T}_3, \boldsymbol{N}_3, \boldsymbol{B}_3\}$, then

$$\varkappa_{\gamma} = 0.$$

So γ is a null straight line.

Proof. Case I: Let s be an arc-parameter of α and s_{γ} an arc-parameter of γ.

$$\gamma\left(s_{\gamma}\right) = N\left(s\right). \tag{28}$$

Differentiating equation (28) with respect to s and by using Frenet formulas given in equation (1), we get

$$\begin{aligned} \frac{d\gamma}{ds_{\gamma}} \cdot \frac{ds_{\gamma}}{ds} &= \frac{dN\left(s\right)}{ds}, \\ \boldsymbol{T}_1\left(s_{\gamma}\right)\frac{ds_{\gamma}}{ds} &= \varkappa\left(s\right)T\left(s\right) + \tau\left(s\right)B\left(s\right), \end{aligned} \tag{29}$$

because $\varepsilon_0 = -1$, $\varepsilon_1 = 1$. Thus we have

$$g\left(\boldsymbol{T}_1\left(s_\gamma\right),\boldsymbol{T}_1\left(s_\gamma\right)\right) = \left(\frac{ds}{ds_\gamma}\right)^2\left(\tau^2\left(s\right) - \varkappa^2\left(s\right)\right). \tag{30}$$

Since $g\left(\boldsymbol{T}_1\left(s_\gamma\right),\boldsymbol{T}_1\left(s_\gamma\right)\right) > 0$, γ is a spacelike curve, that is, $g\left(\boldsymbol{T}_1\left(s_\gamma\right),\boldsymbol{T}_1\left(s_\gamma\right)\right) = 1$.

If we consider this fact in equation (30), we see that

$$\frac{ds_\gamma}{ds} = \sqrt{\tau^2\left(s\right) - \varkappa^2\left(s\right)}.$$

So, we can rewrite equation (29)

$$\boldsymbol{T}_1\left(s_\gamma\right) = \frac{1}{\sqrt{\tau^2\left(s\right) - \varkappa^2\left(s\right)}}\left(\varkappa\left(s\right)T\left(s\right) + \tau(s)B\left(s\right)\right). \tag{31}$$

Differentiating equation (31) with respect to s

$$\frac{d\boldsymbol{T}_1\left(s_\gamma\right)}{ds_\gamma} = -\frac{\tau\varkappa^2\left(\frac{\tau}{\varkappa}\right)'}{\left(\tau^2\left(s\right) - \varkappa^2\left(s\right)\right)^2}T\left(s\right) - N\left(s\right) - \frac{\varkappa^3\left(\frac{\tau}{\varkappa}\right)'}{\left(\tau^2\left(s\right) - \varkappa^2\left(s\right)\right)^2}B\left(s\right). \tag{32}$$

(a) If $-1 < \sigma_1 < 1$, the norm of $\frac{d\boldsymbol{T}_1(s_\gamma)}{ds_\gamma}$ is

$$\left\|\frac{d\boldsymbol{T}_1\left(s_\gamma\right)}{ds_\gamma}\right\| = \sqrt{1-(\sigma_1)^2}.$$

If we consider

$$\boldsymbol{N}_1\left(s_\gamma\right) = \frac{\frac{d\boldsymbol{T}_1(s_\gamma)}{ds_\gamma}}{\left\|\frac{d\boldsymbol{T}_1(s_\gamma)}{ds_\gamma}\right\|},$$

we can write

$$\boldsymbol{N}_1\left(s_\gamma\right) = \frac{1}{\sqrt{1-(\sigma_1)^2}}\left(-\frac{(\sigma_1)\tau}{\sqrt{\tau^2-\varkappa^2}}T\left(s\right) - N\left(s\right) + -\frac{(\sigma_1)\varkappa}{\sqrt{\tau^2-\varkappa^2}}B\left(s\right)\right). \tag{33}$$

Now we know that $\boldsymbol{B}_1\left(s_\gamma\right) = \boldsymbol{T}_1\left(s_\gamma\right) \times \boldsymbol{N}_1\left(s_\gamma\right)$ and using equations (31) and (33) we obtain

$$\begin{aligned}\boldsymbol{B}_1\left(s_\gamma\right) = {}& \frac{1}{\sqrt{(1-(\sigma_1)^2)(\tau^2\left(s\right)-\varkappa^2\left(s\right))}}\\ &\times\left(-\tau T\left(s\right) + \sigma_1\sqrt{\tau^2\left(s\right)-\varkappa^2\left(s\right)}N\left(s\right) - \varkappa B\left(s\right)\right).\end{aligned}$$

We can easily see that $g\left(\boldsymbol{N}_1\left(s_\gamma\right),\ \boldsymbol{N}_1\left(s_\gamma\right)\right) = 1$ and $g\left(\boldsymbol{B}_1\left(s_\gamma\right),\ \boldsymbol{B}_1\left(s_\gamma\right)\right) = -1$, that is, $\boldsymbol{N}_1$ is a spacelike vector and $\boldsymbol{B}_1$ is a timelike vector. So, γ is a timelike curve with a spacelike principal normal and a timelike binormal. Moreover, the Frenet formulas of γ are given by

$$\begin{aligned}D_{\boldsymbol{T}_1}\boldsymbol{T}_1 &= \varkappa_\gamma\boldsymbol{N}_1,\\ D_{\boldsymbol{T}_1}\boldsymbol{N}_1 &= -\varkappa_\gamma\boldsymbol{T}_1 + \tau_\gamma\boldsymbol{B}_1,\\ D_{\boldsymbol{T}_1}\boldsymbol{B}_1 &= \tau_\gamma\boldsymbol{N}_1.\end{aligned}$$

By using the equality of $D_{\boldsymbol{T}_1}\boldsymbol{T}_1 = \frac{d\boldsymbol{T}_1(s_\gamma)}{ds_\gamma} = \varkappa_\gamma \boldsymbol{N}_1$ with equations (32) and (33) we get

$$\varkappa_\gamma = \sqrt{1-(\sigma_1)^2}.$$

Similarly, from the equation $D_{\boldsymbol{T}_1}\boldsymbol{N}_1 = -\varkappa_\gamma \boldsymbol{T}_1 + \tau_\gamma \boldsymbol{B}_1$ we can easily see that

$$\tau_\gamma = \frac{1}{(1-(\sigma_1)^2)\sqrt{\tau^2-\varkappa^2}}\sigma_1'.$$

(b) By using the method in a) the proof of b) is obvious.

Case II: If $\tau^2 < \varkappa^2$, γ is a timelike curve. By using the method in Case (1) the proof of Case (2) is obvious.

Case III: Let s be an arc-parameter of α and s_γ an parameter of the curve γ.

$$\gamma(s_\gamma) = N(s). \tag{34}$$

Differentiating equation (34) with respect to s and by using Frenet formulas given in equation (1), we have

$$\begin{aligned} \frac{d\gamma}{ds_\gamma}\cdot\frac{ds_\gamma}{ds} &= \frac{dN(s)}{ds}, \\ \mathcal{V}_\gamma \boldsymbol{T}_3(s_\gamma)\frac{ds_\gamma}{ds} &= \varkappa(s)T(s) + \tau(s)B(s), \end{aligned} \tag{35}$$

where $\mathcal{V}_\gamma = \left\| \frac{d\gamma}{ds_\gamma} \right\|$. Using equation (35) we get

$$\mathcal{V}_\gamma^2 g(\boldsymbol{T}_3(s_\gamma), \boldsymbol{T}_3(s_\gamma)) = \left(\frac{ds}{ds_\gamma}\right)^2 \left(\tau^2(s) - \varkappa^2(s)\right).$$

Since $\tau^2 = \varkappa^2$, we have $\mathcal{V}_\gamma^2 g(\boldsymbol{T}_3(s_\gamma), \boldsymbol{T}_3(s_\gamma)) = 0$. So, $\mathcal{V}_\gamma = 0$ or $g(\boldsymbol{T}_3(s_\gamma), \boldsymbol{T}_3(s_\gamma)) = 0$. In both cases we can easily see that γ is a null curve.

We assume that s_γ is an arc-parameter of the curve γ, i.e. $g(\gamma''(s_\gamma), \gamma''(s_\gamma)) = 1$. Differentiating equation (35) with respect to s and using Frenet formulas of the null curve γ we get

$$\begin{aligned} &\varkappa_\gamma \boldsymbol{N}_3(s_\gamma)\frac{ds_\gamma}{ds}.\frac{ds_\gamma}{ds} + \boldsymbol{T}_3(s_\gamma)\frac{d^2 s_\gamma}{ds^2} \\ &\quad = \varkappa'(s)T(s) + \varkappa^2(s)N(s) + \tau'(s)B(s) - \tau^2(s)N(s) \end{aligned}$$

or

$$\varkappa_\gamma \boldsymbol{N}_3(s_\gamma)\left(\frac{ds_\gamma}{ds}\right)^2 + \boldsymbol{T}_3(s_\gamma)\frac{d^2 s_\gamma}{ds^2} = \varkappa'(s)T(s) + \tau'(s)B(s)$$

From the last equation we have

$$\begin{aligned} &g\left(\varkappa_\gamma \boldsymbol{N}_3(s_\gamma)\left(\frac{ds_\gamma}{ds}\right)^2 + \boldsymbol{T}_3(s_\gamma)\frac{d^2 s_\gamma}{ds^2},\ \varkappa_\gamma \boldsymbol{N}_3(s_\gamma)\left(\frac{ds_\gamma}{ds}\right)^2 + \boldsymbol{T}_3(s_\gamma)\frac{d^2 s_\gamma}{ds^2}\right) \\ &\quad = (\tau')^2 - (\varkappa')^2, \end{aligned}$$

or using $g\left(\boldsymbol{N}_3\left(s_\gamma\right), \boldsymbol{N}_3\left(s_\gamma\right)\right)=1$ from the Frenet formulas of γ null curve

$$\varkappa_\gamma^2\left(\frac{ds_\gamma}{ds}\right)^4=0,$$

where $\dfrac{ds_\gamma}{ds}\neq 0$. So, we get $\varkappa_\gamma=0$, that is, γ is a null straight line. This completes the proof. □

As a consequence of Theorem 2, we give the following corollary.

Corollary 1. *If α is a timelike general helix with non-zero curvatures $\varkappa$, τ in $\mathbb{E}_1^3$, then the principal normal indicatrix of α is null-geodesic lying in pseudo sphere $\mathbb{S}_1^2$.*

Remark 1. *From [12] we know that at every point $p\in S_1^2$ there exist two null lines contained in S_1^2.*

Theorem 3. *Let α be a timelike curve in $\mathbb{E}_1^3$ with Frenet vectors T, N, B and curvatures $\varkappa$, τ. If the Frenet frame of the binormal indicatrix δ of the space curve α is $\{\mathbb{T},\mathbb{N},\mathbb{B}\}$, then we have the Frenet-Serret formulae:*

$$\begin{aligned} D_{\mathbb{T}}\mathbb{T} &= \varkappa_\delta\mathbb{N}, \\ D_{\mathbb{T}}\mathbb{N} &= -\varepsilon\varkappa_\delta\mathbb{T}+\tau_\delta\mathbb{B}, \\ D_{\mathbb{T}}\mathbb{B} &= \varepsilon\tau_\delta\mathbb{N}, \end{aligned} \tag{36}$$

where

$$\begin{aligned} \mathbb{T} &= -N, \\ \mathbb{N} &= -\frac{1}{\sqrt{\varepsilon\left(\tau^2-\varkappa^2\right)}}\left(\varkappa T+\tau B\right), \\ \mathbb{B} &= -\frac{1}{\sqrt{\varepsilon\left(\tau^2-\varkappa^2\right)}}\left(\tau T+\varkappa B\right), \end{aligned} \tag{37}$$

$\varkappa_\delta=\frac{\sqrt{\varepsilon(\tau^2-\varkappa^2)}}{\tau}$ *is the curvature of δ and* $\tau_\delta=\frac{\varkappa^2\left(\frac{\tau}{\varkappa}\right)'}{\tau(\tau^2-\varkappa^2)}$ *is the torsion of δ.*

Proof. Let s be an arc-parameter of α and s_δ an arc-parameter of δ.

$$\delta\left(s_\delta\right)=B\left(s\right) \tag{38}$$

Differentiating equation (6) with respect to s and by using Frenet formulas given in equation (1), we get

$$\begin{aligned} \frac{d\delta}{ds_\delta}.\frac{ds_\delta}{ds} &= \frac{dB\left(s\right)}{ds}, \\ \mathbb{T}\left(s_\delta\right).\frac{ds_\delta}{ds} &= -\tau\left(s\right)N\left(s\right) \end{aligned} \tag{39}$$

and we have

$$g\left(\mathbb{T}\left(s_\delta\right),\mathbb{T}\left(s_\delta\right)\right)=\left(\frac{ds}{ds_\delta}\right)^2\tau^2\left(s\right). \tag{40}$$

Since $g(\mathbb{T}(s_\delta), \mathbb{T}(s_\delta)) > 0$, δ is a spacelike curve, that is, $g(\mathbb{T}(s_\delta), \mathbb{T}(s_\delta)) = 1$.

If we consider this fact in equation (40), we see that

$$\frac{ds_\delta}{ds} = \tau(s). \tag{41}$$

So, we can rewrite equation (39)

$$\mathbb{T}(s_\delta) = -N(s). \tag{42}$$

Differentiating equation (42) with respect to s

$$\frac{d\mathbb{T}(s_\delta)}{ds_\delta} = -\frac{\varkappa}{\tau}T(s) - B(s). \tag{43}$$

The norm of $\frac{d\mathbb{T}(s_\delta)}{ds_\delta}$ is

$$\left\| \frac{d\mathbb{T}(s_\delta)}{ds_\delta} \right\| = \frac{\sqrt{\varepsilon(\tau^2 - \varkappa^2)}}{\tau(s)}.$$

If we consider

$$\mathbb{N}(s_\delta) = \frac{\frac{d\mathbb{T}(s_\delta)}{ds_\delta}}{\left\| \frac{d\mathbb{T}(s_\delta)}{ds_\delta} \right\|},$$

we can write

$$\mathbb{N}(s_\delta) = -\frac{1}{\sqrt{\varepsilon(\tau^2 - \varkappa^2)}}\left(\varkappa(s)T(s) + \tau(s)B(s)\right) \tag{44}$$

Now we know that $\mathbb{B}(s_\delta) = \mathbb{T}(s_\delta) \times \mathbb{N}(s_\delta)$, and using equations (42) and (44) we get

$$\mathbb{B}(s_\delta) = -\frac{1}{\sqrt{\varepsilon(\tau^2 - \varkappa^2)}}\left(\varkappa(s)B(s) + \tau(s)T(s)\right). \tag{45}$$

We can easily see that $g(\mathbb{N}(s_\delta), \mathbb{N}(s_\delta)) = \varepsilon$ and $g(\mathbb{B}(s_\delta), \mathbb{B}(s_\delta)) = -\varepsilon$. Moreover, the Frenet formulas of δ are given by

$$\begin{aligned} D_{\mathbb{T}}\mathbb{T} &= \varkappa_\delta \mathbb{N}, \\ D_{\mathbb{T}}\mathbb{N} &= -\varepsilon\varkappa_\delta \mathbb{T} + \tau_\delta \mathbb{B}, \\ D_{\mathbb{T}}\mathbb{B} &= \varepsilon\tau_\delta \mathbb{N}, \end{aligned}$$

By using the equality of $D_{\mathbb{T}}\mathbb{T} = \frac{d\mathbb{T}(s_\delta)}{ds_\delta} = \varkappa_\delta \mathbb{N}$ with equations (43) and (44), we get the curvature of δ

$$\varkappa_\delta = \frac{\sqrt{\varepsilon(\tau^2 - \varkappa^2)}}{\tau}. \tag{46}$$

Similarly, from the equation $D_{\mathbb{T}}\mathbb{N} = -\varepsilon\varkappa_\delta \mathbb{T} + \tau_\delta \mathbb{B}$ we can easily see that the torsion of δ is

$$\tau_\delta = \frac{\varkappa^2 \left(\frac{\tau}{\varkappa}\right)'}{\tau(\tau^2 - \varkappa^2)}. \tag{47}$$

This completes the proof. $\square$

4. Characterizations of timelike slant helices in Minkowski 3-space

In Minkowski 3-space, a slant helix and its properties were studied by Ali and Lopez in [2]. They proved the following theorems:

Theorem 4. *Let α be a unit speed timelike curve in $\mathbb{E}_1^3$. Then α is a slant helix if and only if either of the next two functions of geodesic curvatures for the spherical image on pseudosphere S_1^2 of the principal normal indicatrix (N) of α*

$$\sigma = \frac{\varkappa^2}{(\varkappa^2 - \tau^2)^{3/2}}\left(\frac{\tau}{\varkappa}\right)' \quad or \quad \sigma = \frac{\varkappa^2}{(\tau^2 - \varkappa^2)^{3/2}}\left(\frac{\tau}{\varkappa}\right)'$$

is constant in the domain where $\tau^2 - \varkappa^2$ does not vanish [2].

Theorem 5. *Let α be a unit speed timelike curve in $\mathbb{E}_1^3$. α is a timelike slant helix if and only if the tangent indicatrix β of the curve α is a spherical helix.*

Proof. ($\Rightarrow$): In [2], this has been proven by Ali and Lopez.
($\Leftarrow$): Let the tangent indicatrix β of the space curve α be a helix in $\mathbb{E}_1^3$. Then, $\frac{\tau_\beta}{\varkappa_\beta}$ is constant, where $\varkappa_\beta$, τ_β are curvatures of the tangent indicatrix β of the curve α. By using equations (14) and (15), we get

$$\frac{\tau_\beta}{\varkappa_\beta} = \frac{\frac{\varkappa\left(\frac{\tau}{\varkappa}\right)'}{\tau^2-\varkappa^2}}{\frac{\sqrt{\varepsilon(\tau^2-\varkappa^2)}}{\varkappa}} = \frac{\varkappa^2}{[\varepsilon\left(\varkappa^2-\tau^2\right)]^{3/2}}\left(\frac{\tau}{\varkappa}\right)' = \sigma = \text{constant},$$

which means that α is a slant helix. This completes the proof. □

Theorem 6. *Let α be a unit speed timelike curve in $\mathbb{E}_1^3$. α is a timelike slant helix if and only if the binormal indicatrix δ of the curve α is a spherical helix.*

Proof. ($\Rightarrow$): In [2], this has been proven by Ali and Lopez.
($\Leftarrow$): Let the binormal indicatrix δ of the space curve α be a helix in $\mathbb{E}_1^3$. Then, $\frac{\tau_\delta}{\varkappa_\delta}$ is constant, where $\varkappa_\delta$, τ_δ are curvatures of the binormal indicatrix δ of the curve α. By using equations (46) and (47) we get

$$\frac{\tau_\delta}{\varkappa_\delta} = \frac{\frac{\varkappa^2\left(\frac{\tau}{\varkappa}\right)'}{\tau(\tau^2-\varkappa^2)}}{\frac{\sqrt{\varepsilon(\tau^2-\varkappa^2)}}{\tau}} = \frac{\varkappa^2}{[\varepsilon\left(\varkappa^2-\tau^2\right)]^{3/2}}\left(\frac{\tau}{\varkappa}\right)' = \sigma = \text{constant},$$

which means that α is a slant helix. This completes the proof. □

In this section, by using the above results, we obtain certain differential equations for a *timelike slant helix* in $\mathbb{E}_1^3$ according to the *tangent vector field* **T**, *principal normal vector field* **N** and *binormal vector field* **B** *of the curve* β which is *the tangent indicatrix* of the curve α.

Theorem 7. *Let α be a unit speed timelike curve with Frenet vectors T, N, B and non-zero curvatures $\varkappa$ and τ in $\mathbb{E}_1^3$. The curve α is a timelike slant helix if and only if the tangent vector field $\mathbf{T}$ of the tangent indicatrix β of the curve α satisfies one of the following equations:*

$$D_{\mathbf{T}}^3\mathbf{T} - 3\frac{\varkappa_\beta^{'}}{\varkappa_\beta}D_{\mathbf{T}}^2\mathbf{T} - \left\{\frac{\varkappa_\beta^{''}}{\varkappa_\beta} - 3\left(\frac{\varkappa_\beta^{'}}{\varkappa_\beta}\right)^2 - \lambda_2\varepsilon\varkappa_\beta^2\right\}D_{\mathbf{T}}\mathbf{T} = 0, \tag{48}$$

or

$$D_{\mathbf{T}}^3\mathbf{T} - 3\frac{\tau_\beta^{'}}{\tau_\beta}D_{\mathbf{T}}^2\mathbf{T} - \left\{\frac{\tau_\beta^{''}}{\tau_\beta} - 3\left(\frac{\tau_\beta^{'}}{\tau_\beta}\right)^2 + \lambda_1\varepsilon\tau_\beta^2\right\}D_{\mathbf{T}}\mathbf{T} = 0, \tag{49}$$

where $\lambda_1 \in \mathbb{R}$ $(\lambda_1 = 1 - \mu^2)$, $\lambda_2 \in \mathbb{R}$ $(\lambda_2 = 1 - \frac{1}{\mu^2})$ and $\mu \in \mathbb{R}_0$. $\varkappa_\beta$ and τ_β are curvature and torsion of the curve β, respectively.

Proof. Suppose that α is a *timelike slant helix in* $\mathbb{E}_1^3$. From Theorem 5, the tangent indicatrix β of α is a spherical helix. Hence we have $\frac{\varkappa_\beta}{\tau_\beta} = \mu$, $\mu \in \mathbb{R}_0$, where $\varkappa_\beta$ and τ_β are curvature functions of β. From equation (4) we have $D_{\mathbf{T}}\mathbf{T} = \varkappa_\beta\mathbf{N}$. By differentiating twice both sides of $D_{\mathbf{T}}\mathbf{T} = \varkappa_\beta\mathbf{N}$ with respect to s_β, we get

$$D_{\mathbf{T}}^2\mathbf{T} = -\varkappa_\beta^2\mathbf{T} + \varkappa_\beta^{'}\mathbf{N} + \varkappa_\beta\tau_\beta\mathbf{B} \tag{50}$$

and again differentiating of last equality we get

$$D_{\mathbf{T}}^3\mathbf{T} = -2\varkappa_\beta\varkappa_\beta^{'}\mathbf{T} - \varkappa_\beta^2 D_{\mathbf{T}}\mathbf{T} + \varkappa_\beta^{''}\mathbf{N} + \varkappa_\beta^{'}D_{\mathbf{T}}\mathbf{N} + \varkappa_\beta^{'}\tau_\beta\mathbf{B} + \varkappa_\beta\tau_\beta^{'}\mathbf{B} + \varkappa_\beta\tau_\beta D_{\mathbf{T}}\mathbf{B}.$$

By using the Frenet equations in equation (4), we can easily get equation (48) or if we consider $\varkappa_\beta = \mu\tau_\beta$ we obtain equation (49).

Conversely let us assume that equation (48) holds. From equation (4), we have

$$\mathbf{B} = \frac{1}{\tau_\beta}D_{\mathbf{T}}\mathbf{N} + \frac{\varkappa_\beta}{\tau_\beta}\mathbf{T}. \tag{51}$$

Differentiating the last equality with respect to s_β , we have

$$D_{\mathbf{T}}\mathbf{B} = \frac{-\tau_\beta^{'}}{\tau_\beta^2}D_{\mathbf{T}}\mathbf{N} + \frac{1}{\tau_\beta}D_{\mathbf{T}}^2\mathbf{N} + \left(\frac{\varkappa_\beta}{\tau_\beta}\right)^{'}\mathbf{T} + \frac{\varkappa_\beta}{\tau_\beta}D_{\mathbf{T}}\mathbf{T}$$

or

$$\begin{aligned} D_{\mathbf{T}}\mathbf{B} = \tfrac{1}{\varkappa_\beta\tau_\beta}&\left\{D_{\mathbf{T}}^3\mathbf{T} - 3\tfrac{\varkappa_\beta^{'}}{\varkappa_\beta}D_{\mathbf{T}}^2\mathbf{T} - \left[\tfrac{\varkappa_\beta^{''}}{\varkappa_\beta} - 3\left(\tfrac{\varkappa_\beta^{'}}{\varkappa_\beta}\right)^2 - \varkappa_\beta^2 + \tau_\beta^2\right]D_{\mathbf{T}}\mathbf{T}\right\} \\ &+ \tfrac{1}{\varkappa_\beta^2}\left(\tfrac{\varkappa_\beta}{\tau_\beta}\right)^{'}D_{\mathbf{T}}^2\mathbf{T} - \left(-\tfrac{\tau_\beta}{\varkappa_\beta} + \tfrac{\varkappa_\beta^{'}}{\varkappa_\beta^3}\left(\tfrac{\varkappa_\beta}{\tau_\beta}\right)^{'}\right)D_{\mathbf{T}}\mathbf{T} - \left(\tfrac{\varkappa_\beta}{\tau_\beta}\right)^{'}\mathbf{T}. \end{aligned} \tag{52}$$

Using equation (48) in equation (52) we get

$$\begin{aligned} D_{\mathbf{T}}\mathbf{B} &= \frac{1}{\varkappa_\beta^2}\left(\frac{\varkappa_\beta}{\tau_\beta}\right)^{'}\left(-\varkappa_\beta^2\mathbf{T}+\varkappa_\beta^{'}\mathbf{N}+\varkappa_\beta\tau_\beta\mathbf{B}\right) \\ &\quad -\left(-\frac{\tau_\beta}{\varkappa_\beta}+\frac{\varkappa_\beta^{'}}{\varkappa_\beta^3}\left(\frac{\varkappa_\beta}{\tau_\beta}\right)^{'}\right)\varkappa_\beta\mathbf{N}-\left(\frac{\varkappa_\beta}{\tau_\beta}\right)^{'}\mathbf{T} \\ &= -2\left(\frac{\varkappa_\beta}{\tau_\beta}\right)^{'}\mathbf{T}+\tau_\beta\mathbf{N}+\frac{\tau_\beta}{\varkappa_\beta}\left(\frac{\varkappa_\beta}{\tau_\beta}\right)^{'}\mathbf{B} \end{aligned}$$

and then with the help of equation (4) we know that $D_{\mathbf{T}}\mathbf{B}\in\mathbf{Sp}\{\mathbf{N}\}$. So,

$$\left(\frac{\varkappa_\beta}{\tau_\beta}\right)^{'}=0 \text{ and } \frac{\varkappa_\beta}{\tau_\beta}=\sqrt{\frac{1}{1-\lambda_2}}(\text{non-zero constant}).$$

Thus, from equation (4) and (5), we obtain $\sigma=\frac{\tau_\beta}{\varkappa_\beta}$ =constant. According to Theorem 5., α is a *timelike slant helix in* $\mathbb{E}_1^3$. It is easily shown that, by using equation (49) we get the same result. □

For the next two theorems we omit their proofs since they can be done easily similarly to above proof.

Theorem 8. *Let α be a unit speed timelike curve with Frenet vectors T, N, B and non-zero curvatures $\varkappa$ and τ in $\mathbb{E}_1^3$. The curve α is a timelike slant helix if and only if the principal normal vector field $\mathbf{N}$ of the tangent indicatrix β of the curve α satisfies one of the following equations:*

$$D_{\mathbf{T}}^2\mathbf{N}-\frac{\varkappa_\beta^{'}}{\varkappa_\beta}D_{\mathbf{T}}\mathbf{N}+\lambda_2\varepsilon\varkappa_\beta^2\mathbf{N}=0, \tag{53}$$

or

$$D_{\mathbf{T}}^2\mathbf{N}-\frac{\tau_\beta^{'}}{\tau_\beta}D_{\mathbf{T}}\mathbf{N}-\lambda_1\varepsilon\tau_\beta^2\mathbf{N}=0, \tag{54}$$

where $\lambda_1\in\mathbb{R}$ $(\lambda_1=1-\mu^2)$, $\lambda_2\in\mathbb{R}$ $(\lambda_2=1-\frac{1}{\mu^2})$ and $\mu\in\mathbb{R}_0$. $\varkappa_\beta$ and τ_β are curvature and torsion of the curve β, respectively.

Theorem 9. *Let α be a unit speed timelike curve with Frenet vectors T, N, B and non-zero curvatures $\varkappa$ and τ in $\mathbb{E}_1^3$. The curve α is a timelike slant helix if and only if the binormal vector field $\mathbf{B}$ of the tangent indicatrix β of the curve α satisfies one of the following equations:*

$$D_{\mathbf{T}}^3\mathbf{B}-3\frac{\varkappa_\beta^{'}}{\varkappa_\beta}D_{\mathbf{T}}^2\mathbf{B}-\left\{\frac{\varkappa_\beta^{''}}{\varkappa_\beta}-3\left(\frac{\varkappa_\beta^{'}}{\varkappa_\beta}\right)^2-\lambda_1\varepsilon\varkappa_\beta^2\right\}D_{\mathbf{T}}\mathbf{B}=0 \tag{55}$$

or

$$D_{\mathbf{T}}^3\mathbf{B}-3\frac{\tau_\beta^{'}}{\tau_\beta}D_{\mathbf{T}}^2\mathbf{B}-\left\{\frac{\tau_\beta^{''}}{\tau_\beta}-3\left(\frac{\tau_\beta^{'}}{\tau_\beta}\right)^2+\lambda_2\varepsilon\tau_\beta^2\right\}D_{\mathbf{T}}\mathbf{B}=0, \tag{56}$$

where $\lambda_1 \in \mathbb{R}$ $(\lambda_1 = 1 - \mu^2)$, $\lambda_2 \in \mathbb{R}$ $(\lambda_2 = 1 - \frac{1}{\mu^2})$ *and* $\mu \in \mathbb{R}_0$. $\varkappa_\beta$ *and* τ_β *are curvature and torsion of the curve* β.

In the following theorems we obtain differential equations of a *timelike slant helix* according to the *principal normal vector field of the curve* γ which is the *principal normal indicatrix* of the curve α.

Theorem 10. *Let* α *be a unit speed timelike curve with Frenet vectors* T, N, B *and non-zero curvatures* $\varkappa$ *and* τ *in* $\mathbb{E}_1^3$. *For* $\varkappa^2 < \tau^2$, *the curve* α *is a timelike slant helix if and only if the principal normal vector field* $\boldsymbol{N}_1$ *of the principal normal indicatrix* γ *of the curve* α *satisfies one of the following equations:*

(a) If $-1 < \sigma_1 < 1$,

$$D^2_{\boldsymbol{T}_1}\boldsymbol{N}_1 + \varkappa_\gamma^2\boldsymbol{N}_1 = 0. \tag{57}$$

(b) If $\sigma_1 < -1$ *or* $1 < \sigma_1$,

$$D^2_{\boldsymbol{T}_1}\boldsymbol{N}_1 - \varkappa_\gamma^2\boldsymbol{N}_1 = 0, \tag{58}$$

where $\varkappa_\gamma$ *and* τ_γ *are curvature and torsion of the curve* γ, *respectively.*

Proof. (a): Suppose that α is a timelike slant helix. The curvature of γ is

$$\varkappa_\gamma = \sqrt{1 - (\sigma_1)^2} \tag{18}$$

and the torsion of γ is

$$\tau_\gamma = \frac{1}{(1 - (\sigma_1)^2)\sqrt{\tau^2 - \varkappa^2}}\sigma_1'. \tag{19}$$

Since σ_1 is a constant function, we get

$$\varkappa_\gamma = \text{non-zero constant and } \tau_\gamma = 0.$$

From frame equation (16) we obtain that

$$D^2_{\boldsymbol{T}_1}\boldsymbol{N}_1 + \varkappa_\gamma^2\boldsymbol{N}_1 = 0.$$

Conversely, let us assume that (57) holds. We show that the curve α is a slant helix. From frame equation (16)

$$D^2_{\boldsymbol{T}_1}\boldsymbol{N}_1 + \varkappa_\gamma^2\boldsymbol{N}_1 = -\varkappa_\gamma'\boldsymbol{T}_1 + \tau_\gamma^2\boldsymbol{N}_1 + \tau_\gamma'\boldsymbol{B}_1 = 0. \tag{59}$$

Then we see that

$$\varkappa_\gamma \text{ is a constant and } \tau_\gamma = 0,$$

which means that α is a slant helix.

(b): By using the method in a) the proof of b) is obvious. □

Theorem 11. *Let* α *be a unit speed timelike curve with Frenet vectors* T, N, B *and non-zero curvatures* $\varkappa$ *and* τ *in* $\mathbb{E}_1^3$. *For* $\varkappa^2 > \tau^2$, *the curve* α *is a timelike slant helix if and only if the principal normal vector field* $\boldsymbol{N}_2$ *of the principal normal indicatrix* γ *of the curve* α *satisfies one of the following equations:*

$$D^2_{\boldsymbol{T}_2}\boldsymbol{N}_2 - \varkappa_\gamma^2\boldsymbol{N}_2 = 0, \tag{60}$$

where $\varkappa_\gamma$ *and* τ_γ *are curvature and torsion of the curve* γ, *respectively.*

Proof. If $\tau^2 < \varkappa^2$, γ is a timelike curve. By using the method in Theorem 10 the proof of Theorem 11 is obvious. □

In the following three theorems, we obtain the differential equations of a *timelike slant helix* according to the *tangent vector field* $\mathbb{T}$, *principal normal vector field* $\mathbb{N}$ and *binormal vector field* $\mathbb{B}$ *of the curve* δ which is the *binormal indicatrix* of the curve α.

The following three theorems will be given without their proofs, because they are similar to the above proofs.

Theorem 12. *Let α be a unit speed timelike curve with Frenet vectors T, N, B and non-zero curvatures $\varkappa$ and τ in $\mathbb{E}_1^3$. The curve α is a timelike slant helix if and only if the tangent vector field $\mathbb{T}$ of the binormal indicatrix δ of the curve α satisfies one of the following equations:*

$$D_{\mathbb{T}}^3\mathbb{T} - 3\frac{\varkappa_\delta^{'}}{\varkappa_\delta}D_{\mathbb{T}}^2\mathbb{T} - \left\{\frac{\varkappa_\delta^{''}}{\varkappa_\delta} - 3\left(\frac{\varkappa_\delta^{'}}{\varkappa_\delta}\right)^2 - \lambda_1\varepsilon\varkappa_\delta^2\right\}D_{\mathbb{T}}\mathbb{T} = 0, \tag{61}$$

or

$$D_{\mathbb{T}}^3\mathbb{T} - 3\frac{\tau_\delta^{'}}{\tau_\delta}D_{\mathbb{T}}^2\mathbb{T} - \left\{\frac{\tau_\delta^{''}}{\tau_\delta} - 3\left(\frac{\tau_\delta^{'}}{\tau_\delta}\right)^2 + \lambda_2\varepsilon\tau_\delta^2\right\}D_{\mathbb{T}}\mathbb{T} = 0, \tag{62}$$

where $\lambda_1 \in \mathbb{R}$ $(\lambda_1 = 1 - \mu^2)$, $\lambda_2 \in \mathbb{R}$ $(\lambda_2 = 1 - \frac{1}{\mu^2})$ and $\mu \in \mathbb{R}_0$. $\varkappa_\delta$ and τ_δ are curvature and torsion of the curve δ, respectively.

Theorem 13. *Let α be a unit speed timelike curve with Frenet vectors T, N, B and non-zero curvatures $\varkappa$ and τ in $\mathbb{E}_1^3$. The curve α is a timelike slant helix if and only if the principal normal vector field $\mathbb{N}$ of the binormal indicatrix δ of the curve α satisfies one of the following equations:*

$$D_{\mathbb{T}}^2\mathbb{N} - \frac{\varkappa_\delta^{'}}{\varkappa_\delta}D_{\mathbb{T}}\mathbb{N} + \lambda_2\varepsilon\varkappa_\delta^2\mathbb{N} = 0, \tag{63}$$

or

$$D_{\mathbb{T}}^2\mathbb{N} - \frac{\tau_\delta^{'}}{\tau_\delta}D_{\mathbb{T}}\mathbb{N} - \lambda_1\varepsilon\tau_\delta^2\mathbb{N} = 0, \tag{64}$$

where $\lambda_1 \in \mathbb{R}$ $(\lambda_1 = 1 - \mu^2)$, $\lambda_2 \in \mathbb{R}$ $(\lambda_2 = 1 - \frac{1}{\mu^2})$ and $\mu \in \mathbb{R}_0$. $\varkappa_\delta$ and τ_δ are curvature and torsion of the curve δ, respectively.

Theorem 14. *Let α be a unit speed timelike curve with Frenet vectors T, N, B and non-zero curvatures $\varkappa$ and τ in $\mathbb{E}_1^3$. The curve α is a timelike slant helix if and only if the binormal vector field $\mathbb{B}$ of the binormal indicatrix δ of the curve α satisfies one of the following equations:*

$$D_{\mathbb{T}}^3\mathbb{B} - 3\frac{\varkappa_\delta^{'}}{\varkappa_\delta}D_{\mathbb{T}}^2\mathbb{B} - \left\{\frac{\varkappa_\delta^{''}}{\varkappa_\delta} - 3\left(\frac{\varkappa_\delta^{'}}{\varkappa_\delta}\right)^2 - \lambda_1\varepsilon\varkappa_\delta^2\right\}D_{\mathbb{T}}\mathbb{B} = 0, \tag{65}$$

or

$$D_{\mathbb{T}}^{3}\mathbb{B}-3\frac{\tau_{\delta}^{'}}{\tau_{\delta}}D_{\mathbb{T}}^{2}\mathbb{B}-\left\{\frac{\tau_{\delta}^{''}}{\tau_{\delta}}-3\left(\frac{\tau_{\delta}^{'}}{\tau_{\delta}}\right)^{2}+\lambda_{2}\varepsilon\tau_{\delta}^{2}\right\}D_{\mathbb{T}}\mathbb{B}=0, \tag{66}$$

where $\lambda_1 \in \mathbb{R}$ $(\lambda_1 = 1-\mu^2)$, $\lambda_2 \in \mathbb{R}$ $(\lambda_2 = 1-\frac{1}{\mu^2})$ *and* $\mu \in \mathbb{R}_0$. $\varkappa_\delta$ *and* τ_δ *are curvature and torsion of the curve* δ*, respectively.*

5. Example

In this section we give an example of a timelike slant helix in Minkowski 3 -space and draw its pictures and its the tangent indicatrix and the binormal indicatrix by using *Mathematica*.

We consider a timelike slant helix α is defined by

$$\begin{aligned}
\alpha_1(s) &= \frac{9}{400}\sinh(25s)+\frac{25}{144}\sinh(9s)\\
\alpha_2(s) &= \frac{9}{400}\cosh(25s)-\frac{25}{144}\cosh(9s)\\
\alpha_3(s) &= \frac{15}{136}\sinh(17s).
\end{aligned}$$

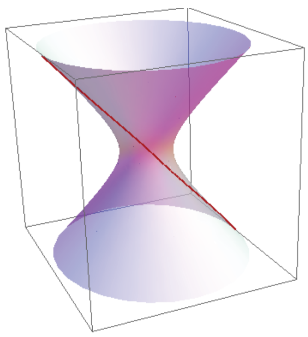

Figure 1: *A timelike slant helix* α

The curve lies on the hyperboloid of one sheet

$$-\frac{x^2}{\left(\frac{34}{225}\right)^2}+\frac{y^2}{\left(\frac{34}{225}\right)^2}+\frac{z^2}{\left(\frac{2}{15}\right)^2}=1.$$

The picture of the curve α is rendered in Figure 1.

The parametrization of the tangent indicatrix $\beta = (\beta_1, \beta_2, \beta_3)$ of the timelike slant helix α is

$$\beta_1(s) = \frac{9}{16}\cosh(25s) + \frac{25}{16}\cosh(9s)$$
$$\beta_2(s) = \frac{9}{16}\sinh(25s) - \frac{25}{16}\sinh(9s)$$
$$\beta_3(s) = \frac{30}{16}\cosh(17s)$$

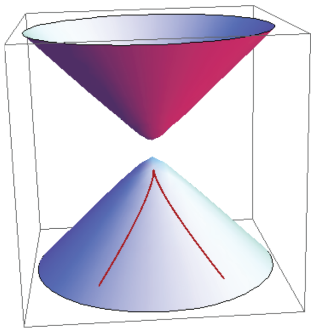

Figure 2: *The tangent indicatrix* $\beta = (\beta_1, \beta_2, \beta_3)$ *of the timelike slant helix* α

We can easily show that the curve β satisfies Theorems 7, 8 and 9.
The tangent indicatrix β lies on the pseudohyperbolic space H_0^2. The picture of the helix β is rendered in Figure 2.

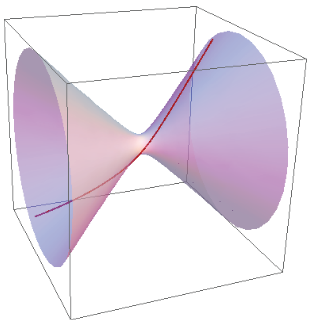

Figure 3: *The binormal indicatrix* $\delta = (\delta_1, \delta_2, \delta_3)$ *of the timelike slant helix* α

The parametrization of the binormal indicatrix $\delta = (\delta_1, \delta_2, \delta_3)$ of the timelike slant helix α is

$$\delta_1(s) = \frac{9}{16}\sinh(25s) + \frac{25}{16}\sinh(9s)$$
$$\delta_2(s) = \frac{9}{16}\cos(25s) - \frac{25}{16}\cosh(9s)$$
$$\delta_3(s) = \frac{30}{16}\sinh(17s)$$

The binormal indicatrix δ lies on the pseudo-Riemannian sphere S_1^2. The picture of the helix δ is rendered in Figure 3.

We can easily show that the curve δ satisfies Theorems 12, 13 and 14.

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