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Functions of triples of positive real numbers and their use in study of bicentric polygons II

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Abstract. The article deals with some functions which play a key role in the study of bicentric polygons where conics are circles. This article can be considered as a companion article to [8]. We report on new functions and new results concerning these functions of positive real triples and their use for studying bicentric polygons. Finally, some new conjectures are posed.

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1. Introduction

A polygon which is both chordal and tangential is called a *bicentric polygon.* The relation (condition) that an n-sided polygon be a bicentric one is the *Fuss' relation for bicentric n-gons* and will be denoted by $F_n(R, r, d) = 0$ in honor of Swiss mathematician Nicolaus Fuss who first found the relation for a bicentric quadrilateral. This relation reads

$$(R^2 - r^2)^2 - 2r^2(R^2 + d^2) = 0,$$

where R and r are radii of the circumcircle and the incircle, respectively, and d is the distance between centers of circumcircle and incircle, see [2]. Fuss also found relations for bicentric n–gons for $4 \le n \le 8$, consult [3].

The keystone result in the theory of bicentric polygons is Poncelet's famous closure theorem which can be stated as follows:

> *Let C and D be two nested conics such that there is an n-sided polygon inscribed in D and circumscribed around C. Then for every point x on D there is an n-sided polygon inscribed in D and circumscribed around C such that the point x is one of its vertices. Hence, for every starting point x there is a polygon with the same n-periodicity*, [4].

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Many mathematicians have worked for centuries on a number of problems related to this inspiring result. However, we work here with some functions and their properties important in the theory of bicentric polygons where conics are circles. Some of those functions, like f_1, f_2 and g, are already considered and used in [8] and also in [5, 6]. In the present article we have established another properties of those functions and we have also found some novel functions for the same purposes, e.g. φ_1, φ_2, σ_1, σ_2, τ_1, τ_2. Using these functions, f_1, f_2 and g we have stated certain conjectures (Conjectures 2, 3, 4). Although these conjectures can be considered as a main result in the article, it can also be said that the obtained functions play on essential role in posing these conjectures.

We point out that some of the main motivations for writing this article upon the further research, bearing in mind the traces and the achievements of [5, 6, 8], are the inspiring relations (22) and (48) which in a way are the starting points of a set of other new relations and functions of numerical characteristics, like the radii r, R of the incircle and the circumcircle of bicentric polygons, and the distance d between their centers.

Now we recall the definition of f_1, f_2 according to [5, 6] and [8].

Definition 1 ([8, Definition 1])**.** *Let $\mathbb{S}$ be the set given by*

$$\mathbb{S} = \left\{(R, r, d) \in \mathbb{R}^3_+ \colon R > r + d\right\}.$$

Let f_1, $f_2 : \mathbb{S} \to \mathbb{S}$ be functions on the set $\mathbb{S}$ defined as follows. Let $(R_0, r_0, d_0) \in \mathbb{S}$. Then

$$f_1(R_0, r_0, d_0) = (R_1, r_1, d_1), \tag{1}$$

where

$$\begin{aligned} R_1^2 &= R_0\left(R_0 + r_0 + \sqrt{(R_0 + r_0)^2 - d_0^2}\right), \\ r_1^2 &= (R_0 + r_0)^2 - d_0^2, \\ d_1^2 &= R_0\left(R_0 + r_0 - \sqrt{(R_0 + r_0)^2 - d_0^2}\right), \end{aligned} \tag{2}$$

and

$$f_2(R_0, r_0, d_0) = (R_2, r_2, d_2), \tag{3}$$

where

$$\begin{aligned} R_2^2 &= R_0\left(R_0 - r_0 + \sqrt{(R_0 - r_0)^2 - d_0^2}\right), \\ r_2^2 &= (R_0 - r_0)^2 - d_0^2, \\ d_2^2 &= R_0\left(R_0 - r_0 - \sqrt{(R_0 - r_0)^2 - d_0^2}\right). \end{aligned} \tag{4}$$

The following statements hold true. Let (R_i, r_i, d_i), $i = 0, 1, 2$, and let f_1, f_2 be as in Definition 1. Then

$$R_1 > r_1 + d_1, \qquad R_2 > r_2 + d_2, \tag{5}$$

$$R_1 d_1 = R_2 d_2 = R_0 d_0, \tag{6}$$

$$R_1^2 + d_1^2 - r_1^2 = R_2^2 + d_2^2 - r_2^2 = R_0^2 + d_0^2 - r_0^2, \tag{7}$$

$$\frac{R_1^2 - d_1^2}{2r_1} = \frac{R_2^2 - d_2^2}{2r_2} = R_0 \tag{8}$$

$$\frac{2R_1 r_1 d_1}{R_1^2 - d_1^2} = \frac{2R_2 r_2 d_2}{R_2^2 - d_2^2} = d_0, \tag{9}$$

$$-\left(R_1^2 + d_1^2 - r_1^2\right) + \left(\frac{R_1^2 - d_1^2}{2r_1}\right)^2 + \left(\frac{2R_1 r_1 d_1}{R_1^2 - d_1^2}\right)^2 = -\left(R_2^2 + d_2^2 - r_2^2\right) + \left(\frac{R_2^2 - d_2^2}{2r_2}\right)^2 + \left(\frac{2R_2 r_2 d_2}{R_2^2 - d_2^2}\right)^2 = r_0^2. \tag{10}$$

The proof of these assertions is straightforward, thus it is omitted.

Let $\mathbb{K}$ denote the set given by

$$\mathbb{K} = \left\{(R, r, d) \in \mathbb{S} \colon (R^2 - d^2)^2 - 2r^2(R^2 + d^2) = 0\right\}.$$

In other words, $\mathbb{K}$ denotes the set of all (positive) solutions of Fuss' relations for bicentric quadrilaterals.

Theorem 1 ([8, Theorem 2]). *Let (R, r, d) be a triple of the set $\mathbb{S} \setminus \mathbb{K}$ and let g be a function on the set $\mathbb{S} \setminus \mathbb{K}$ given by*

$$g(R, r, d) = (\hat{R}, \hat{r}, \hat{d}),$$

where

$$\hat{R} = \frac{R^2 - d^2}{2r}, \tag{11}$$

$$\hat{r} = \sqrt{-\left(R^2 + d^2 - r^2\right) + \left(\frac{R^2 - d^2}{2r}\right)^2 + \left(\frac{2Rdr}{R^2 - d^2}\right)^2}, \tag{12}$$

$$\hat{d} = \frac{2Rdr}{R^2 - d^2}. \tag{13}$$

Then $\mathbb{S} \setminus \mathbb{K}$ is a maximal subset of $\mathbb{S}$ such that

$$(\hat{R}, \hat{r}, \hat{d}) \in \mathbb{S} \setminus \mathbb{K} \implies (R, r, d) \in \mathbb{S} \setminus \mathbb{K}.$$

We give here only the sketch of the proof. Firstly, it is clear from (11) and (13) that $\hat{R} > 0$ and $\hat{d} > 0$ since $(R, r, d) \in \mathbb{S}$. The fact that $\hat{r} > 0$ follows from the equality

$$\left[-\left(R^2+d^2-r^2\right)+\left(\frac{R^2-d^2}{2r}\right)^2+\left(\frac{2Rdr}{R^2-d^2}\right)^2\right]\cdot 4r^2(R^2-d^2)^2$$
$$=\left[\left(R^2-d^2\right)^2-2r^2\left(R^2+d^2\right)\right)\right]^2. \tag{14}$$

Thus $\hat{r}=0$ only if $(R,r,d)\in\mathbb{K}$ since by (14) exactly then we have $\hat{r}=0$. Now, using relations (11)-(13), there follows $\hat{R}>\hat{r}+\hat{d}$.

Let $(R_0,r_0,d_0)\in\mathbb{R}_+^3$ be a solution of Fuss' relation $F_n(R,r,d)=0$. Let $\mathcal{C}_1,\mathcal{C}_2$ be such circles that $\mathcal{C}_2$ is completely inside of $\mathcal{C}_1$ and let

$$\begin{aligned} R_0 &= \text{ radius of } \mathcal{C}_1,\ r_0 = \text{ radius of } \mathcal{C}_2,\\ d_0 &= \text{distance between centers of } \mathcal{C}_1 \text{ and } \mathcal{C}_2. \end{aligned}$$

Then the set of all bicentric n-gons whose circumcircle is $\mathcal{C}_1$ and incircle $\mathcal{C}_2$ constitute a class of bicentric n-gons determined by the triple (R_0,r_0,d_0); we denote this class by $C(R_0,r_0,d_0)$.

Let $A_1\cdots A_n$ be a bicentric n-gon from this class and let $T_1,\dots,T_n$ be touching points of its sides (segments) $A_1A_2,\dots,A_nA_1$ and circle $\mathcal{C}_2$, respectively. Then $|A_iT_i|$, $i=1,\dots,n$, are the so-called *tangent lengths* of the n-gon $A_1\cdots A_n$. If

$$\sum_{i=1}^{n}\arctan\frac{|A_iT_i|}{r_0}=k\pi,$$

where $k\in\mathbb{N}$. The n-gon $A_1\cdots A_n$ is *k-circumscribed* and k is the *rotation number for n*.

The term cycle will also be used what follows. Let $(R_{k_1},r_{k_1},d_{k_1})\in\mathbb{R}_+^3$ be a solution of Fuss' relation $F_n(R,r,d)=0$, where $n\geq 3$ is an odd integer. Then there is an integer $m\geq 1$ such that

$$\begin{aligned} g\left(R_{k_1},r_{k_1},d_{k_1}\right) &= \left(R_{k_2},r_{k_2},d_{k_2}\right),\\ g\left(R_{k_2},r_{k_2},d_{k_2}\right) &= \left(R_{k_3},r_{k_3},d_{k_3}\right),\\ g\left(R_{k_m},r_{k_m},d_{k_m}\right) &= \left(R_{k_1},r_{k_1},d_{k_1}\right), \end{aligned}$$

that is,

$$g^m\left(R_{k_1},r_{k_1},d_{k_1}\right)=\left(R_{k_1},r_{k_1},d_{k_1}\right),$$

where $k_2,\dots,k_m$ are also rotation numbers for n. Then $(k_1,\dots,k_m)$ is called a *cycle for n*. For example, the cycles for $n=3,5,7,9$ are (1), $(1,2)$, $(1,2,3)$, $(1,2,4)$, respectively.

Now we formulate the conjecture [8, Conjecture 2], rewritten into a form suitable for our current purposes.

Conjecture 1. *Let (R_k,r_k,d_k) be a solution of Fuss' relation $F_n(R,r,d)=0$, where $n\geq 3$ is an odd integer. Let*

$$g(R_k,r_k,d_k)=(R_l,r_l,d_l),$$

where k and l are rotation numbers for n. Then

$$f_1(R_l, r_l, d_l) = (R_k, r_k, d_k), \text{ if } l \text{ is even},$$
$$f_2(R_l, r_l, d_l) = (R_k, r_k, d_k), \text{ if } l \text{ is odd}.$$

Moreover,

$$F_n\left(f_1(R_l, r_l, d_l)\right) = 0, \text{ if } l \text{ is even},$$
$$F_n\left(f_2(R_l, r_l, d_l)\right) = 0, \text{ if } l \text{ is odd}.$$

But

$$F_{2n}\left(f_1(R_l, r_l, d_l)\right) = 0, \text{ if } l \text{ is odd},$$
$$F_{2n}\left(f_2(R_l, r_l, d_l)\right) = 0, \text{ if } l \text{ is even}.$$

So, if (R_l, r_l, d_l) is a triple with rotation number l for odd $n \geq 3$, then

$$\text{either } F_n\left(f_i(R_l, r_l, d_l)\right) = 0 \text{ or } F_{2n}\left(f_i(R_l, r_l, d_l)\right) = 0, \quad i = 1, 2.$$

For further subsequent information about functions f_1 and f_2, consult [8] .

2. Another properties of functions f_1, f_2 and g and new functions which refer to bicentric polygons with the incircle

Firstly, from the relations (2) and (4) we conclude

$$(R, r, d) \in \mathbb{S} \implies f_i(R, r, d) \in \mathbb{S}, \quad i = 1, 2.$$

Theorem 2. *Let $(R_0, r_0, d_0) \in \mathbb{K}$. Then there is no $(R, r, d) \in \mathbb{S}$ for which*

$$f_1(R, r, d) = (R_0, r_0, d_0).$$

Proof. From the system

$$R\left(R + r + \sqrt{(R+r)^2 - d^2}\right) = R_0^2, \quad (R+r)^2 - d^2 = r_0^2$$
$$R\left(R + r - \sqrt{(R+r)^2 - d^2}\right) = d_0^2, \tag{15}$$

using the first and the third equation, we get

$$R_0^2 - d_0^2 = 2R\sqrt{(R+r)^2 - d^2} = 2Rr_0,$$

from which there follows

$$\frac{R_0^2 - d_0^2}{2r_0} = R \quad \text{i.e.} \quad \hat{R}_0 = R. \tag{16}$$

Also, using the first and the third equation and relation (16) we can write

$$R_0 d_0 = Rd, \ d = \frac{R_0 d_0}{R} = \frac{2R_0 r_0 d_0}{R_0^2 - d_0^2} = \hat{d}_0, \ \hat{d}_0 = d. \tag{17}$$

Now, by (16) and (17), the second equation of system (15), can be written as

$$r^2 + 2\hat{R}_0 r + \hat{R}_0^2 - \hat{d}_0^2 - r_0^2 = 0.$$

This equation in r has the root

$$r = -\hat{R}_0 + \sqrt{\hat{d}_0^2 + r_0^2}. \tag{18}$$

However, it is $r = 0$, that is,

$$\hat{R}_0 = \sqrt{\hat{d}_0^2 + r_0^2}. \tag{19}$$

The proof can be sketched in the following lines. Since the triple (R_0, r_0, d_0) is a solution of $F_4(R, r, d) = 0$, it is sufficient to show that (19) can be written as

$$(R_0^2 - d_0^2)^2 = 2r_0^2(R_0^2 + d_0^2). \tag{20}$$

By (16), (17), (18) and (20), it is

$$\begin{aligned}
\hat{R}_0^2 &= \hat{d}_0^2 + r_0^2,\\
\left(\frac{R_0^2 - d_0^2}{2r_0}\right)^2 &= \left(\frac{2R_0 r_0 d_0}{R_0^2 - d_0^2}\right)^2 + r_0^2,\\
\frac{2r_0^2(R_0^2 + d_0^2)}{4r_0^2} &= \frac{4R_0^2 r_0^2 d_0^2}{2r_0^2(R_0^2 + d_0^2)} + r_0^2,\\
(R_0^2 - d_0^2)^2 &= 2r_0^2(R_0^2 + d_0^2).
\end{aligned}$$

Thus, the triple (R, r, d) is not in $\mathbb{S}$ since $r = 0$. □

Corollary 1. *The solution of the system given by* (15) *can be written as* $(R, r, d) = (\hat{R}_0, \hat{r}_0, \hat{d}_0)$, *where*

$$\hat{R}_0 > 0,\ \hat{d}_0 > 0,\ \hat{r}_0 = \sqrt{-(R_0^2 + d_0^2 - r_0^2) + \hat{R}_0^2 + \hat{d}_0^2} = 0.$$

Proof. If $(R_0, r_0, d_0) \in \mathbb{K}$, then

$$-\hat{R}_0 + \sqrt{\hat{d}_0^2 + r_0^2} = -\left(R_0^2 + d_0^2 - r_0^2\right) + \hat{R}_0^2 + \hat{d}_0^2.$$

The rest is clear, following the lines of the proof for (19). □

Fromrelation (14) we have

Corollary 2. *Let* $(R_0, r_0, d_0) \in \mathbb{S} \setminus \mathbb{K}$. *Then the solution of the system given by* $(R, r, d) = (R_0, r_0, d_0)$ *is* $(\hat{R}_0, \hat{r}_0, \hat{d}_0) \in \mathbb{S} \setminus \mathbb{K}$.

In turn, we point out that g is a left inverse of f_1, that is $gf_1(R_0, r_0, d_0) = (R_0, r_0, d_0)$.

Theorem 3. *Let* $(R_0, r_0, d_0) \in \mathbb{S}$. *Then there are two triples in* $\mathbb{S}$ *which maps* g *into* (R_0, r_0, d_0); *these are* $f_1(R_0, r_0, d_0)$ *and* $f_2(R_0, r_0, d_0)$.

Proof. Let $f_i(R_0, r_0, d_0) = (R_i, r_i, d_i),\ i = 1, 2$. It is easy to show that from

$$R_1\left(R_1 + r_1 + \sqrt{(R_1 + r_1)^2 - d_1^2}\right) = R_0^2, \quad (R_1 + r_1)^2 - d_1^2 = r_0^2,$$
$$R_1\left(R_1 + r_1 - \sqrt{(R_1 + r_1)^2 - d_1^2}\right) = d_0^2,$$

there follows $R_1 = \hat{R}_0$, $r_1 = \hat{r}_0$, $d_1 = \hat{d}_0$. The same holds for (R_2, r_2, d_2). □

Let $\mathbb{L}$ denotes a subset of $\mathbb{S}$ defined as

$$\mathbb{L} = \{(R, r, d) \in \mathbb{S} :\ \text{there is odd } n \geq 3 \text{ such that } F_n(R, r, d) = 0\}.$$

In other words, let $\mathbb{L}$ denotes the set of all (positive) solutions of every Fuss' relation $F_n(R, r, d) = 0$ where $n \geq 3$ is an odd integer.

Conjecture 2. *The function g is one-to-one function on the set $\mathbb{L}$ and if $(R_0, r_0, d_0) \in \mathbb{L}$, then only one of the triples $f_1(R_0, r_0, d_0)$ and $f_2(R_0, r_0, d_0)$ belongs to $\mathbb{L}$.*

Definition 2. *Let $(R_0, r_0, d_0) \in \mathbb{S}$. Then $(\tilde{R}_0, \tilde{r}_0, \tilde{d}_0)$ is a triple obtained from (R_0, r_0, d_0) such that R_0 is replaced by d_0 and* vice versa. *Thus*

$$(\tilde{R}_0, \tilde{r}_0, \tilde{d}_0) = (d_0, r_0, R_0).$$

This kind two triples will be called conjugate.

Our next goal is the composition of the functions f_1 and f_2.
Let $(R_0, r_0, d_0) \in \mathbb{S}$ and $i_1, \ldots, i_n \in \{1, 2\}$. Then the triple

$$(R_{i_1 \ldots i_n}, r_{i_1 \ldots i_n}, d_{i_1 \ldots i_n}) = f_{i_n} \ldots f_{i_1}(R_0, r_0, d_0),$$

compare Figure 1.

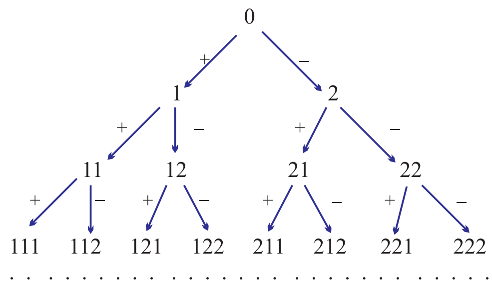

Figure 1: *The arrow + refers to $f_1(R_i, r_i, d_i)$, the arrow − refers to $f_2(R_i, r_i, d_i)$*

It can be shown that

$$\frac{R_{i_1 \ldots i_n}^2 + d_{i_1 \ldots i_n}^2 - r_{i_1 \ldots i_n}^2}{2R_{i_1 \ldots i_n} d_{i_1 \ldots i_n}} = \frac{R_0^2 + d_0^2 - r_0^2}{2R_0 d_0} = I,$$

where I is the invariant of the corresponding pencil.

It is sufficient to show that

$$\frac{R_1^2+d_1^2-r_1^2}{2R_1d_1}=\frac{R_2^2+d_2^2-r_2^2}{2R_2d_2}=\frac{R_0^2+d_0^2-r_0^2}{2R_0d_0},$$

since the analogy is complete.

It is often more convenient to use the triple $(1,\rho,\delta)$, normalized with respect to R, instead of (R,r,d), writing $\rho=\frac{r}{R}$, $\delta=\frac{d}{R}$, see e.g. [1].

Let (R_0,r_0,d_0) be a solution of Fuss' relation $F_n(R,r,d)=0$, where $n\geq 3$ is an odd integer. Then from

$$\frac{R_0^2+d_0^2-r_0^2}{2R_0d_0}=I \quad \text{or} \quad \frac{1+\delta_0^2-\rho_0^2}{2\delta_0}=I$$

it follows

$$r_0^2=R_0^2-2R_0d_0I+d_0^2 \quad \text{or} \quad \rho_0^2=1-2I\delta_0+\delta_0^2.$$

The triple $g(1,\rho_0,\delta_0)$ can be obtained by using the relation

$$\hat{\delta}_0=\frac{4\delta_0(1-2I\delta_0+\delta_0^2)}{(1-\delta_0^2)^2}. \tag{21}$$

Indeed, the above display follows from

$$\hat{d}_0=\frac{2R_0r_0d_0}{R_0^2-d_0^2},$$

when both sides are divided by $\hat{R}_0$, that is, by $\frac{R_0^2-d_0^2}{2r_0}$. Thus

$$\hat{\delta}_0=\frac{4\delta_0\rho_0^2}{(1-\delta_0^2)^2}, \tag{22}$$

since $\frac{\hat{d}_0}{\hat{R}_0}=\hat{\delta}_0$. Accordingly, $\hat{\rho}_0^2=1-2\hat{\delta}_0I+\hat{\delta}_0^2$, see [8, Eq. (17)].

We omit the proof of the next result due to its simplicity.

Theorem 4. *Equation* (21) *has four solutions in* δ_0*:*

$$(\delta_0)_1=\frac{1+\hat{\rho}_0-\sqrt{2(1-I\hat{\delta}_0+\hat{\rho}_0)}}{\hat{\delta}_0},\ (\delta_0)_2=\frac{1-\hat{\rho}_0-\sqrt{2(1-I\hat{\delta}_0-\hat{\rho}_0)}}{\hat{\delta}_0},$$
$$(\delta_0)_3=\frac{1+\hat{\rho}_0+\sqrt{2(1-I\hat{\delta}_0+\hat{\rho}_0)}}{\hat{\delta}_0},\ (\delta_0)_4=\frac{1-\hat{\rho}_0+\sqrt{2(1-I\hat{\delta}_0-\hat{\rho}_0)}}{\hat{\delta}_0},$$

where

$$\hat{\rho}_0=\sqrt{1-2I\hat{\delta}_0+\hat{\delta}_0^2},$$

and I *stands for the invariant of the corresponding pencil.*

These solutions define the following functions

$$f_1(\hat{\delta}_0) = \frac{1+\hat{\rho}_0 - \sqrt{2(1-I\hat{\delta}_0+\hat{\rho}_0)}}{\hat{\delta}_0}, \tag{23}$$

$$f_2(\hat{\delta}_0) = \frac{1-\hat{\rho}_0 - \sqrt{2(1-I\hat{\delta}_0-\hat{\rho}_0)}}{\hat{\delta}_0}, \tag{24}$$

$$\varphi_1(\hat{\delta}_0) = \frac{1+\hat{\rho}_0 + \sqrt{2(1-I\hat{\delta}_0+\hat{\rho}_0)}}{\hat{\delta}_0}, \tag{25}$$

$$\varphi_2(\hat{\delta}_0) = \frac{1-\hat{\rho}_0 + \sqrt{2(1-I\hat{\delta}_0-\hat{\rho}_0)}}{\hat{\delta}_0}, \tag{26}$$

where $\hat{\rho}_0$ is described above.

Corollary 3. *Functions f_1, f_2 given by* (23) *and* (24) *are only rewritten functions f_1 and f_2 given by* (1) *and* (3).

The next result is the consequence of (23)-(26) .

Corollary 4. *It holds*

$$\begin{aligned}
f_1(\delta)+\varphi_1(\delta)+f_2(\delta)+\varphi_2(\delta) &= \frac{4}{\delta},\\
f_1(\delta_1)\varphi_1(\delta_1) = f_1(\delta_2)\varphi_1(\delta_2) &= f_1(\delta_3)\varphi_1(\delta_3) = f_2(\delta_1)\varphi_2(\delta_1) = f_2(\delta_2)\varphi_2(\delta_2)\\
&= f_2(\delta_3)\varphi_2(\delta_3) = 1,\\
\delta_1 + 1/\delta_1 + \delta_3 + 1/\delta_3 &= 4/\delta_2,
\end{aligned}$$

where $\delta = \delta_1$.

Conjecture 3. *Let (R_0, r_0, d_0) be a positive triple for which $R_0 > r_0 + d_0$ and $F_n(R_0, r_0, d_0) = 0$. Then there is Fuss' relation $\tilde{F}_n(R, r, d) = 0$ so that $\tilde{F}_n(\tilde{R}_0, \tilde{r}_0, \tilde{d}_0) = 0$. This relation is obtained such that R and d in the relation $F_n(R, r, d) = 0$ are mutually interchanged. Also, there hold:*

(*i*) *If $A_1 \cdots A_n$ is a bicentric n-gon from the class $C(R_0, r_0, d_0)$ and $t_1, \dots, t_n$ are its tangent lengths, then there is a bicentric n-gon $A_1 \cdots A_n$ from the class $C(\tilde{R}_0, \tilde{r}_0, \tilde{d}_0)$ such that its tangent lengths are $\tilde{t}_1, \dots, \tilde{t}_n$, where*

$$\tilde{t}_i = \begin{cases} t_i, & \text{if } i \text{ is odd},\\ -t_i, & \text{if } i \text{ is even}.\end{cases}$$

(*ii*) *Let $n \ge 3$ be an odd integer. Then both relations $F_n(R, r, d) = 0$ and $\tilde{F}_n(R, r, d) = 0$ have the same rotation numbers for n and the same cycle. (Of course, t_i needs to be taken instead of $-t_i$.)*

(iii) *Let* $(k_1, \ldots, k_m)$ *be a cycle for an odd* $n \geq 3$ *and let*

$$(1, \rho_{k_i}, \delta_{k_i}), \quad i = 1, \ldots, m,$$

be the corresponding solutions of Fuss' relation $F_n(R, r, d) = 0$. *Let* k_i *be even and let* $f_1(\delta_{k_i}) = \delta_{k_j}$. *Then*

$$\varphi_1(\delta_{k_i}) = 1/\delta_{k_j}. \tag{27}$$

But, if k_i *is odd and* $f_2(\delta_{k_i}) = \delta_{k_j}$, *then*

$$\varphi_2(\delta_{k_i}) = 1/\delta_{k_j}. \tag{28}$$

Thus

$$\varphi_1(\delta_{k_i}) = \frac{1}{f_1(\delta_{k_i})}, \text{ if } k_i \text{ is even }, \tag{29}$$

$$\varphi_2(\delta_{k_i}) = \frac{1}{f_2(\delta_{k_i})}, \text{ if } k_i \text{ is odd }. \tag{30}$$

In both cases, when k_i *is odd and when* k_i *is even, it holds*

$$\tilde{F}_n(1, \tilde{\rho}, \tilde{\delta}) = 0, \text{ where } \tilde{\delta} = 1/\delta_{k_j}, \ \tilde{\rho} = \sqrt{1 - 2I\tilde{\delta} + \tilde{\delta}^2}.$$

Let us remark here that $F_n(1, \rho_{k_j}, \delta_{k_j}) = 0$ *implies* $\tilde{F}_n(\delta_{k_j}, \rho_{k_j}, 1) = 0$ *and also* $\tilde{F}_n(1, \rho_{k_j}/\delta_{k_j}, 1/\delta_{k_j}) = 0$.

The following is also valid. If k_i *is odd then instead of the relation* (27) *we have relation*

$$\varphi_1(\delta_{k_i}) = \delta_1^+,$$

where δ_1^+ *is obtained in the following way. Let* (R_1^+, r_1^+, d_1^+) *be a triple given by*

$$(R_1^+, r_1^+, d_1^+) = f_1(R_{k_i}, r_{k_i}, d_{k_i}),$$

and let $(\tilde{R}_1, \tilde{r}_1, \tilde{d}_1)$ *be a triple given by*

$$(\tilde{R}_1, \tilde{r}_1, \tilde{d}_1) = (d_1^+, r_1^+, R_1^+).$$

Then

$$\delta_1^+ = \tilde{d}_1/\tilde{R}_1.$$

But if k_i *is even, then instead of relation* (28) *we have the relation*

$$\varphi_2(\delta_{k_i}) = \delta_2^+,$$

where δ_2^+ *is obtained in the following way. Let*

$$(R_2^+, r_2^+, d_2^+) = f_2(R_{k_i}, r_{k_i}, d_{k_i}),$$

and let $(\tilde{R}_2, \tilde{r}_2, \tilde{d}_2)$ *be a triple given by*

$$(\tilde{R}_2, \tilde{r}_2, \tilde{d}_2) = (d_2^+, r_2^+, R_2^+).$$

Then

$$\delta_2^+ = \tilde{d}_2/\tilde{R}_2.$$

Let us remark here that both triples (R_i^+, r_i^+, d_i^+), $i = 1, 2$, *are solutions of Fuss' relation* $F_{2n}(R, r, d) = 0$.

Example 1. *Part of this example, where* f_1, f_2 *and* g *are involved, is already known up to* (35). *In turn, there* f_1 *and* f_2 *are written in an abbreviated form with respect to the notation used in Definition 1. The remaining part of the example, concerning* φ_1 *and* φ_2 *are used, is novel.*

Let $n = 7$. *Then* $(1, 2, 3)$ *is a cycle for* $n = 7$ *and the triple*

$$(7, 4.979113505, 2)$$

is a solution of Fuss' relation $F_7(R, r, d) = 0$. *This triple has rotation number 1 for* $n = 7$ *and we write it as* (R_1, r_1, d_1). *Using* g *we get*

$$(R_2, r_2, d_2) = g(R_1, r_1, d_1) = (4.518876699, 1.345412541, 3.098115069), \quad (31)$$
$$(R_3, r_3, d_3) = g(R_2, r_2, d_2) = (4.0217886, 0.289796869, 3.481038261). \quad (32)$$

where (R_2, r_2, d_2) *has rotation number 2 for* $n = 7$, *and* (R_3, r_3, d_3) *has rotation number 3 for* $n = 7$. *It can be found that for each* $(R_i, r_i, d_i), i = 1, 2, 3$, *we have* $I = 1.007443882$. *It can also be found that*

$$\delta_1 = \frac{d_1}{R_1} = 0.285714285, \ \delta_2 = \frac{d_2}{R_2} = 0.68559467, \ \delta_3 = \frac{d_3}{R_3} = 0.865544812, \quad (33)$$
$$\rho_1 = \frac{r_1}{R_1} = 0.711301929, \ \rho_2 = \frac{r_2}{R_2} = 0.297731633, \ \rho_3 = \frac{r_3}{R_3} = 0.072056713, \quad (34)$$

where $\hat{\delta}_1 = \delta_2$, $\hat{\delta}_2 = \delta_3$, $\hat{\delta}_3 = \delta_1$, $\hat{\rho}_1 = \rho_2$, $\hat{\rho}_2 = \rho_3$, $\hat{\rho}_3 = \rho_1$.

By means of (23)-(26), *taking* $\delta_0 = \delta_1, \delta_2, \delta_3$ *and by virtue of* $\hat{\delta}_1 = \delta_2$, $\hat{\delta}_2 = \delta_3$ *we get*

$$f_1(\delta_2) = \frac{1 + \rho_2 - \sqrt{2(1 - I\delta_2 + \rho_2)}}{\delta_2} = \delta_1$$
$$f_2(\delta_3) = \frac{1 - \rho_3 - \sqrt{2(1 - I\delta_3 - \rho_3)}}{\delta_3} = \delta_2$$
$$f_2(\delta_1) = \frac{1 - \rho_1 - \sqrt{2(1 - I\delta_1 - \rho_1)}}{\delta_1} = \delta_3.$$

Thus

$$f_1(\delta_2) = \delta_1, \ f_2(\delta_3) = \delta_2, \ f_2(\delta_1) = \delta_3.$$

We also have

$$f_1(\delta_1) = 0.084686170,$$
$$f_2(\delta_2) = 0.802440024,$$
$$f_1(\delta_3) = 0.507772253. \quad (35)$$

The above three relations refer to bicentric 14-gons with an incircle. See relation given by (43), (44) *and* (45), *where* $1/11.89507519 = f_1(\delta_1)$ *etc.*

Concerning functions φ_1 *and* φ_2 *we have*

$$\begin{aligned}\varphi_1(\delta_1) &= 11.89504492,\\ \varphi_1(\delta_2) &= 3.5 = \frac{1}{\delta_1},\\ \varphi_1(\delta_3) &= 1.969386854,\end{aligned}$$

and

$$\begin{aligned}\varphi_2(\delta_1) &= 1.155341689 = \frac{1}{\delta_3},\\ \varphi_2(\delta_2) &= 1.246206948,\\ \varphi_2(\delta_3) &= 1.458601968 = \frac{1}{\delta_2}.\end{aligned}$$

The following assertion can also be verified:

$$\begin{aligned}\varphi_1(\delta_2) &= \frac{1}{\delta_1} = 3.5 \quad \textit{implies triple} \quad (1, 2.489556753, 3.5),\\ \varphi_2(\delta_1) &= \frac{1}{\delta_3} = 1.155341683 \quad \textit{implies triple} \quad (1, 0.083250127, 1.155341683), \quad (36)\\ \varphi_2(\delta_3) &= \frac{1}{\delta_2} = 1.458587769 \quad \textit{implies triple} \quad (1, 0.434281525, 1.458587769),\end{aligned}$$

The triples

$$(1, 2.489556753, 3.5), \qquad (1, 0.083250127, 1.155341683)$$

are solutions of $\tilde{F}_7^{(1,3)}(R,r,d) = 0$*, where* $F_7^{(1,3)}(R,r,d) = 0$ *is Fuss' relation for bicentric heptagons where rotation numbers for 7 are 1 and 3. This Fuss' reads*

$$\begin{aligned}F_7^{(1,3)}(R,r,d) = &-d^{12} - 4d^{10}rR + 6d^{10}R^2 + 24d^8r^3R + 4d^8r^2R^2 + 20d^8rR^3\\ &-15d^8R^4 - 32d^6r^5R + 16d^6r^4R^2 - 64d^6r^3R^3 - 16d^6r^2R^4\\ &-40d^6rR^5 + 20d^6R^6 - 32d^4r^4R^4 + 48d^4r^3R^5 + 24d^4r^2R^6\\ &+40d^4rR^7 - 15d^4R^8 - 64d^2r^6R^4 + 32d^2r^5R^5 + 16d^2r^4R^6\\ &-16d^2r^2R^8 - 20d^2rR^9 + 6d^2R^{10} - 8r^3R^9 + 4r^2R^{10} + 4rR^{11} - R^{12}.\end{aligned}$$

It can also be verified that the triple $(1, 0.434281525, 1.458587769)$ *is a solution of* $\tilde{F}_7^{(2)}(R,r,d) = 0$*, where* $F_7^{(2)}(R,r,d) = 0$ *is Fuss' relation for bicentric heptagons where rotation number for 7 is 2. This Fuss' relation is given by*

$$\begin{aligned}F_7^{(2)}(R,r,d) = &\,d^{12} - 4d^{10}rR - 6d^{10}R^2 + 24d^8r^3R - 4d^8r^2R^2 + 20d^8rR^3\\ &+15d^8R^4 - 32d^6r^5R - 16d^6r^4R^2 - 64d^6r^3R^3 + 16d^6r^2R^4\\ &-40d^6rR^5 - 20d^6R^6 + 32d^4r^4R^4 + 48d^4r^3R^5 - 24d^4r^2R^6\\ &+40d^4rR^7 + 15d^4R^8 + 64d^2r^6R^4 + 32d^2r^5R^5 - 16d^2r^4R^6\\ &+16d^2r^2R^8 - 20d^2rR^9 - 6d^2R^{10} - 8r^3R^9 - 4r^2R^{10} + 4rR^{11} + R^{12}.\end{aligned}$$

Relations $\tilde{F}_n^{(1,3)}(R,r,d) = 0$ *and* $\tilde{F}_n^{(2)}(R,r,d) = 0$ *can be obtained such that* R *and* d *in* $F_n^{(1,3)}(R,r,d) = 0$ *and* $F_n^{(2)}(R,r,d) = 0$ *mutually interchange.*

The relations

$$\varphi_1(\delta_1) = 11.89504492, \tag{37}$$

$$\varphi_1(\delta_3) = 1.969386854, \tag{38}$$

$$\varphi_2(\delta_2) = 1.246206948, \tag{39}$$

refer to bicentric 14-gons with the excircle. Indeed, it is clear that from the above relations we get the corresponding triples

$$(1, 10.88691502, 11.89504519), \tag{40}$$

$$(1, 0.954144174, 1.969386854), \tag{41}$$

$$(1, 0.205096632, 1.246206948). \tag{42}$$

Since it can be verified that the triples

$$(11.89504519, 10.88691502, 1), \tag{43}$$

$$(1.969386854, 0.954144174, 1), \tag{44}$$

$$(1.246206948, 0.205096632, 1) \tag{45}$$

are solutions of Fuss' relation $F_{14}(R,r,d) = 0$*, we can conclude (on the condition that Conjecture 3 is true) that the triples given by* (40)-(42) *are solutions of Fuss' relation* $\tilde{F}_{14}(R,r,d) = 0$.

This can also be shown in the following way where we use the triples

$$(R_1, r_1, d_1) = (7, 4.979113505, 2),$$
$$(R_2, r_2, d_2) = (4.51887699, 1.345412541, 3.09811507),$$
$$(R_3, r_3, d_3) = (4.021789575, 0.28970865, 3.48103764).$$

(See (31) *and* (32)*).*

Also, functions f_1 *and* f_2 *will be used. So we have*

$$f_1(R_1, r_1, d_1) = (12.90467467, 11.81097627, 1.084878154),$$
$$f_1(R_3, r_3, d_3) = (5.250893088, 2.544040465, 2.66621311),$$
$$f_2(R_2, r_2, d_2) = (4.176948328, 0.687428381, 3.351729277).$$

It can be verified that each of the above three triples are solutions of Fuss' relation $F_{14}(R,r,d) = 0$*. Of course, these triples refer to bicentric 14-gons with the incircle, and the triples*

$$(1.084878154, 11.81097627, 12.90467467),$$
$$(2.66621311, 2.544040465, 5.250893088),$$
$$(3.351729277, 0.687428381, 4.176948328),$$

refer to bicentric 14-gons with the excircle. To show this, let the following be made: each member of the first triple above be divided by member 1.084878154, each member *of the second triple above be divided by member 2.66621311, each member of the third triple above be divided by member 3.351729277. Then we get triples given by* (40)-(42). *From this it follows that, for example, the polygons from the class*

$$C(1, 10.88691502, 11.89504492)$$

and

$$C(11.89504492, 10.88691502, 1)$$

are all n-sided polygons.

As will be shown now there is an interesting connection between tangent lengths which refer to a bicentric n-gon with the incircle and the corresponding tangent lengths which refer to bicentric n-gons with the excircle.

For calculation of tangent lengths we shall use the following formula

$$t_{i+1} = \frac{(R^2-d^2)t_i \pm r\sqrt{(t_M^2-t_i^2)(t_i^2-t_m^2)}}{t_i^2+r_i^2}, \tag{46}$$

where

$$t_M^2 = (R+d)^2 - r^2,\ t_m^2 = (R-d)^2 - r^2.$$

If polygons are with the incircle, then $R > d$ (in fact, $R > d + r$), but if polygons are with the excircle, then $R < d$ (in fact, $d > R + r$). In the first case, the above formula holds for calculation of tangent lengths for bicentric polygons with incircle, and in the second case, the formula holds for calculation of tangent lengths for bicentric polygons with the excircle.

More about using this formula in the case when $R > d + r$ can be seen in [7, rel. (1.5)]. It is not difficult to see that analogously holds in the case when $R + r < d$. If t_i is given then in both cases t_{i+1} can be obtained using this formula.

In this connection let us remark that this formula, using computer algebra, can be algorithmized and be very practical.

So, starting from the triple $(1, 0.74370748590576, 0.2)$*, which is a solution of Fuss' relation $F_5^{(1)}(R, r, d) = 0$, where $t_M = 0.9417532455\ldots$, $t_m = 0.2947866608\ldots$ we can take, say, $t_1 = 0.62$. By using formula* (46) *we get*

$$t_2 = 0.9415995565,\ t_3 = 0.6357184840,\ t_4 = 0.3335049102\ldots,$$
$$t_5 = 0.3281593267\ldots,\ t_6 = -0.62.$$

It can be found that the triple $(1, 0.74370748590576\ldots, 0.2)$ *has rotation number 1 for $n = 5$, that is,*

$$\sum_{i=1}^{5} \arctan \frac{t_i}{r} = \pi.$$

Now, starting from the triple $(0.2, 0.74370748590576\ldots, 1)$ *and using formula* (46)*, taking also $t_1 = 0.62$, we get*

$$t_2 = -0.9415995565,\ t_3 = 0.6357184840,\ t_4 = -0.3335049102,\ t_5 = 0.3281593267,$$

compare Figure 2. The reason why in this case we get tangent lengths which signes alternate lies in the reason that only then we have

$$|t_i + t_{i+1}| = |A_i A_{i+1}|, \quad i = 1, \ldots, 5.$$

Here is one more example where $n = 7$*. Using triple* $(1, 0.083250127, 1.155341683)$ *given by* (36)*, taking* $\tilde{t}_1 = 1$*, we get*

$$\begin{aligned} \tilde{t}_1 &= 1, \tilde{t}_2 = -0.4888545484, \\ \tilde{t}_3 &= 0.3311860163, \tilde{t}_4 = -1.412891853, \\ \tilde{t}_5 &= 0.1411162685, \tilde{t}_6 = -2.107230795, \\ \tilde{t}_7 &= 0.1761652021. \end{aligned}$$

Now, using triple $(1.155341683, 0.083250127, 1)$*, taking* $t_1 = 1$*, we get*

$$\begin{aligned} t_1 &= 1,\ t_2 = 0.4888545484,\ t_3 = 0.3311860163,\ t_4 = 1.412891853, \\ t_5 &= 0.1411162685,\ t_6 = 2.107230795,\ t_7 = 0.1761652021. \end{aligned}$$

Moreover, we have

$$\{\tilde{t}_1, |\tilde{t}_2|, \tilde{t}_3, |\tilde{t}_4|, \tilde{t}_5, |\tilde{t}_6|, \tilde{t}_7\} = \{t_1, t_2, t_3, t_4, t_5, t_6, t_7\}.$$

Analogous holds in all similar cases.

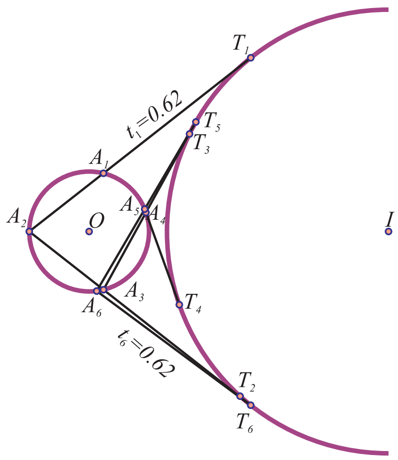

Figure 2: $t_i = |A_i T_i|, i = 1, \ldots, 5,$ *are tangent lengths.*

3. Some functions which refer to bicentric polygons with the excircle

Definition 3. *Let* $\tilde{\mathbb{S}}$ *denote the set obtained from the set* $\mathbb{S}$ *such that every triple* (R, r, d) *of* $\mathbb{S}$ *is replaced by triple* $(\tilde{R}, \tilde{r}, \tilde{d})$*. Also, let* $\tilde{\mathbb{K}}$ *denote the set obtained from* *the set* $\mathbb{K}$ *such that every triple* (R,r,d) *of* $\mathbb{K}$ *is replaced by triple* $(\tilde{R},\tilde{r},\tilde{d})$. *Let* $\tilde{g}:\tilde{\mathbb{S}}\setminus\tilde{\mathbb{K}}\to\tilde{\mathbb{S}}\setminus\tilde{\mathbb{K}}$ *be function defined on the set* $\tilde{\mathbb{S}}\setminus\tilde{\mathbb{K}}$ *as follows. For a triple* $(R_0,r_0,d_0)\in\tilde{\mathbb{S}}\setminus\tilde{\mathbb{K}}$,

$$\tilde{g}(R_0,r_0,d_0)=(\hat{R}_0,\hat{r}_0,\hat{d}_0),$$

where

$$\hat{R}_0=\frac{2R_0r_0d_0}{d_0^2-R_0^2},$$

$$\hat{r}_0=\sqrt{-\left(R_0^2+d_0^2-r_0^2\right)+\left(\frac{R_0^2-d_0^2}{2r_0}\right)^2+\left(\frac{2R_0r_0d_0}{d_0^2-R_0^2}\right)^2},$$

$$\hat{d}_0=\frac{d_0^2-R_0^2}{2r_0}.$$

Of course, here $d_0>R_0$, Cf. with relations given by (11)-(13).

Now, let (R_0,r_0,d_0) be a solution of Fuss' relation $\tilde{F}_n(R,r,d)=0$, where $n\geq 3$ is an odd integer. Then, as it can be easily seen, it holds

$$\frac{R_0^2+d_0^2-r_0^2}{2R_0d_0}=\frac{\tilde{R}_0^2+\tilde{d}_0^2-\tilde{r}_0^2}{2\tilde{R}_0\tilde{d}_0}=I,$$

where I is invariant of the corresponding cycles.

In this case instead of the relation given by (21) we have the following relation

$$\hat{\delta}_0=\frac{(1-\delta_0^2)^2}{4\delta_0\left(1-2\delta_0I+\delta_0^2\right)} \tag{47}$$

or

$$\hat{\delta}_0=\frac{\left(1-\delta_0^2\right)^2}{4\delta_0\rho_0^2}. \tag{48}$$

This relation is obtained from the relation

$$\hat{d}_0=\frac{d_0^2-R_0^2}{2r_0}$$

such that both of its sides are divided by $\hat{R}_0$, that is, by $(2R_0d_0)/(d_0^2-R_0^2)$, since $\hat{d}_0/\hat{R}_0=\hat{\delta}_0$.

We give the following theorem without proof.

Theorem 5. *The equation in* δ_0 *given by* (47) *and* (48) *has the following four solutions*

$$(\delta_0)_1=\hat{\delta}_0-\hat{\rho}_0-\sqrt{2\hat{\delta}_0\left(\hat{\delta}_0-\hat{\rho}_0-I\right)},$$

$$(\delta_0)_2=\hat{\delta}_0+\hat{\rho}_0-\sqrt{2\hat{\delta}_0\left(\hat{\delta}_0+\hat{\rho}_0-I\right)},$$

$$(\delta_0)_3=\hat{\delta}_0-\hat{\rho}_0+\sqrt{2\hat{\delta}_0\left(\hat{\delta}_0-\hat{\rho}_0-I\right)},$$

$$(\delta_0)_4=\hat{\delta}_0+\hat{\rho}_0+\sqrt{2\hat{\delta}_0\left(\hat{\delta}_0+\hat{\rho}_0-I\right)},$$

where I is the same as in relation (21).

These solutions determine four functions σ_1, σ_2, τ_1, τ_2 given by

$$\sigma_1(\hat{\delta}) = \hat{\delta} - \hat{\rho} - \sqrt{2\hat{\delta}\left(\hat{\delta} - \hat{\rho} - I\right)},$$

$$\sigma_2(\hat{\delta}) = \hat{\delta} + \hat{\rho} - \sqrt{2\hat{\delta}\left(\hat{\delta} + \hat{\rho} - I\right)},$$

$$\tau_1(\hat{\delta}) = \hat{\delta} - \hat{\rho} + \sqrt{2\hat{\delta}\left(\hat{\delta} - \hat{\rho} - I\right)},$$

$$\tau_2(\hat{\delta}) = \hat{\delta} + \hat{\rho} + \sqrt{2\hat{\delta}\left(\hat{\delta} + \hat{\rho} - I\right)}.$$

Example 2. *Let (R_1, r_1, d_1) be as in Example 1. (See* (31) *and* (32)*.) Then we have the following triples*

$$(\tilde{R}_1, \tilde{r}_1, \tilde{d}_1) = (2, 4.979113505, 7),$$
$$(\tilde{R}_2, \tilde{r}_2, \tilde{d}_2) = (3.09811507, 1.345412541, 4.51887699),$$
$$(\tilde{R}_3, \tilde{r}_3, \tilde{d}_3) = (3.48103764, 0.28970865, 4.021789575),$$

which are conjugate to the triples given in Example 1. In this case we use the notation $\tilde{\delta}_i$ and $\tilde{\rho}_i$, $i = 1, 2$, so that (cf. (33) *and* (34)*)*

$$\tilde{\delta}_1 = \frac{\tilde{d}_1}{\tilde{R}_1} = \frac{7}{2} = 3.5, \ \tilde{\rho}_1 = \frac{r_1}{\tilde{R}_1} = \frac{4.979113505}{2} = 2.489556753.$$

$$\tilde{\delta}_2 = 1.458589138, \ \tilde{\rho}_2 = 0.434268098,$$
$$\tilde{\delta}_3 = 1.55342167, \ \tilde{\rho}_3 = 0.083250139.$$

Of course, $I = 1.007443882$ is the same as in Example 1.

It can be found that

$$\sigma_1(\tilde{\delta}_1) = \delta_3, \ \sigma_1(\tilde{\delta}_3) = \delta_2, \ \sigma_2(\tilde{\delta}_2) = \delta_1,$$
$$\tau_1(\tilde{\delta}_1) = \tilde{\delta}_3, \ \tau_1(\tilde{\delta}_3) = \tilde{\delta}_2, \ \tau_2(\tilde{\delta}_2) = \tilde{\delta}_1,$$

$$\sigma_1(\tilde{\delta}_2) = \frac{1}{\varphi_2(\delta_2)}, \ \sigma_2(\tilde{\delta}_1) = \frac{1}{\varphi_1(\delta_1)}, \ \sigma_2(\tilde{\delta}_3) = \frac{1}{\varphi_1(\delta_3)},$$
$$\tau_1(\tilde{\delta}_2) = \varphi_2(\delta_2), \ \tau_2(\tilde{\delta}_1) = \varphi_1(\delta_1), \ \tau_2(\tilde{\delta}_3) = \varphi_1(\delta_3),$$

where δ_1, δ_2, δ_3 are given by (33) *and* (34).

Conjecture 4. *Let $(k_1, \ldots, k_m)$ be a cycle for an odd $n \geq 3$. Then*

$$\sigma_1(\tilde{\delta}_{k_i}) = \frac{1}{\tau_1(\tilde{\delta}_{k_i})}, \ \text{if } k_i \text{ is even},$$
$$\sigma_2(\tilde{\delta}_{k_i}) = \frac{1}{\tau_2(\tilde{\delta}_{k_i})}, \ \text{if } k_i \text{ is odd}.$$

Note 1. *Functions $\tilde{f}_1$, $\tilde{f}_2$ defined below are only in a new fashion rewritten functions τ_1 and τ_2.*

Definition 4. *Let $\tilde{f}_1 : \tilde{\mathbb{S}} \to \tilde{\mathbb{S}}$ and $\tilde{f}_2 : \tilde{\mathbb{S}} \to \tilde{\mathbb{S}}$ be functions defined on the set $\tilde{\mathbb{S}}$ as follows. For a triple $(R_0, r_0, d_0) \in \tilde{\mathbb{S}}$ we have*

$$\tilde{f}_1(R_0, r_0, d_0) = (\tilde{R}_1, \tilde{r}_1, \tilde{d}_1),$$

where

$$\begin{aligned} \tilde{R}_1^2 &= d_0 \left(d_0 + r_0 - \sqrt{(d_0 + r_0)^2 - R_0^2} \right), \\ \tilde{r}_1^2 &= (d_0 + r_0)^2 - R_0^2, \\ \tilde{d}_1^2 &= d_0 \left(d_0 + r_0 + \sqrt{(d_0 + r_0)^2 - R_0^2} \right). \end{aligned}$$

The function $\tilde{f}_2$ is defined as follows

$$\tilde{f}_2(R_0, r_0, d_0) = (\tilde{R}_2, \tilde{r}_2, \tilde{d}_2),$$

where

$$\begin{aligned} \tilde{R}_2^2 &= d_0 \left(d_0 - r_0 - \sqrt{(d_0 - r_0)^2 - R_0^2} \right), \\ \tilde{r}_2^2 &= (d_0 - r_0)^2 - R_0^2, \\ \tilde{d}_2^2 &= d_0 \left(d_0 - r_0 + \sqrt{(d_0 - r_0)^2 - R_0^2} \right). \end{aligned}$$

Functions $\tilde{f}_1$ and $\tilde{f}_2$ have properties analogous to functions f_1 and f_2 given by Definition 1. So, for example, Cf. (5). It holds

$$\tilde{d}_1 > \tilde{r}_1 + \tilde{R}_1, \qquad \tilde{d}_2 > \tilde{r}_2 + \tilde{R}_2,$$

$$\begin{aligned} \tilde{R}_1 \tilde{d}_1 &= \tilde{R}_2 \tilde{d}_2 = R_0 d_0, \\ \tilde{R}_1^2 + \tilde{d}_1^2 - \tilde{r}_1^2 &= \tilde{R}_2^2 + \tilde{d}_2^2 - \tilde{r}_2^2 = R_0^2 + d_0^2 - r_0^2, \\ \frac{2\tilde{R}_1 \tilde{r}_1 \tilde{d}_1}{\tilde{d}_1^2 - \tilde{R}_1^2} &= \frac{2\tilde{R}_2 \tilde{r}_2 \tilde{d}_2}{\tilde{d}_2^2 - \tilde{R}_2^2} = R_0, \\ \frac{\tilde{d}_1^2 - \tilde{R}_1^2}{2\tilde{r}_1} &= \frac{\tilde{d}_2^2 - \tilde{R}_2^2}{2\tilde{r}_2} = d_0. \end{aligned}$$

Using these relations, the conjecture analogous to Conjecture A can be stated.

Let (R_k, r_k, d_k) be a solution of Fuss' relation $\tilde{F}_n(R, r, d) = 0$, where $n \geq 3$ is an odd integer. Let

$$\tilde{g}(R_k, r_k, d_k) = (R_l, r_l, d_l),$$

where k and l are rotation numbers for n. Then

$$\begin{aligned} \tilde{f}_1(R_l, r_l, d_l) &= (R_k, r_k, d_k), \text{ if } l \text{ is even}, \\ \tilde{f}_2(R_l, r_l, d_l) &= (R_k, r_k, d_k), \text{ if } l \text{ is odd}. \end{aligned}$$

It also holds

$$\tilde{F}_n\left(\tilde{f}_1(R_l, r_l, d_l)\right) = 0, \text{ if } l \text{ is even},$$
$$\tilde{F}_n\left(\tilde{f}_2(R_l, r_l, d_l)\right) = 0, \text{ if } l \text{ is odd},$$
$$\tilde{F}_{2n}\left(\tilde{f}_1(R_l, r_l, d_l)\right) = 0, \text{ if } l \text{ is odd},$$
$$\tilde{F}_{2n}\left(\tilde{f}_2(R_l, r_l, d_l)\right) = 0, \text{ if } l \text{ is even}.$$

Finally, we can conclude the following. One of the main results in the article refers to functions f_1, f_2 and g. These functions are rather investigated now and we point out some of their roles in research of bicentric polygons. Here introduced functions φ_1, φ_2, σ_1, σ_2, τ_1, τ_2 also have important roles in the undertaken study. Many essential facts are now mutually connected and new conjectures are formulated and posed.

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