Math. Commun. **19**(2014), 159–182

Braided strict Ann-categories and commutative extensions of rings

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Received August 13, 2013; accepted March 24, 2014

Abstract. A braided E-system is a version of a braided crossed module for rings. In this paper, we present applications of the braided Ann-category theory to the problem of classifications of braided regular E-systems, of ring extensions of the type of a strong braided E-system, and of extensions of commutative rings.

AMS subject classifications: 18D10, 16E40, 16S70

Key words: braided strict Ann-category, crossed module, E-system, ring extension, ring cohomology

1. Introduction

The relationship among crossed modules, extensions of groups and strict categorical groups was presented in the works of Brown et al. Brown - Spencer [4] proved the equivalence between the category of crossed modules (the morphisms are morphisms of crossed modules) and the category of strict categorical groups (the morphisms are strict monoidal functors). The notion of group extension of the type of a crossed module is a generalization of the notion of group extension. The problem of group extensions of the type of a crossed module was dealt with by Brown and Mucuk in [5].

The notion of a braided crossed module appeared in the works of Brown - Gilbert [3], and Joyal and Street [8]. In [8], the authors showed that braided crossed modules are determined by braided strict categorical groups. The problem of classification of braided crossed modules is solved by Quang - Phung - Cuc in [17], which is a consequence of the classification of braided Γ-modules. The basic notions and results on braided Γ-graded categorical groups and fibred braided categorical groups are studied by Cegarra et al. in [7], [6].

The notion of a crossed module has many versions, such as crossed modules over **k**-algebras, crossed bimodules, E-systems. Crossed modules over **k**-algebras which is **k**-split with the same kernel M and coker B were classified by Baues - Minian in [1] thanks to Hochschild cohomology $H^3_{Hoch}(B,M)$. Baues - Pirashivili [2] replaced the field **k** with a commutative ring $\mathbb{K}$, then crossed modules over $\mathbb{K}$-algebras are termed *crossed bimodules*. In particular, when $\mathbb{K}=\mathbb{Z}$, one obtains crossed bimodules over rings. The class of E-systems, which is larger than the class of crossed bimodules

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over rings, is classified by Quang - Cuc in [14] thanks to strict Ann-categories. In this paper, we consider E-systems with a braiding.

The paper is organized as follows. In Section 3, we prove that each braided regular Ann-category is braided Ann-equivalent to a braided strict Ann-category (Theorem 2). Then, we introduce the notion of a braided E-system and show the determination of braided regular E-systems and braided strict Ann-categories. Then, we indicate the relation between these notions and the notion of internal category in the commutative rings. We also prove in Section 4 that the category of braided regular E-systems is equivalent to the category of braided strict Ann-categories and almost strict braided Ann-functors (Theorem 3). A morphism in the category of braided E-systems consists of a morphism $(f_1, f_0) : \mathcal{M} \to \mathcal{M}'$ of braided E-systems and an element of the group of symmetric 2-cocycles $Z^2_{Shab}(\pi_0\mathcal{M}, \pi_1\mathcal{M}')$ in the sense of [16]. We also give a result on E-systems (Theorem 4), which contains a corresponding theorem in [14]. (Braided) regular E-systems having the same two invariants, $\pi_0\mathcal{M}, \pi_1\mathcal{M}$, are classified by means of the group of Shukla cohomology H^3_{Sh} (H^3_{Shab}). In Section 5, we construct the theory of obstructions of ring extension of the type of a strong E-system. In the last section, we apply the results of Section 4 to solve the problem of commutative ring extensions.

2. Preliminaries

2.1. Braided Ann-categories

The notions of Ann-category and of Ann-functor can be found in [12, 13].

Definition 1 ([15]). *A braided Ann-category $\mathcal{A}$ is an Ann-category $\mathcal{A}$ together with a braiding $\mathbf{c} = \mathbf{c}_{X,Y} : X \otimes Y \to Y \otimes X$ such that $(\mathcal{A}, \otimes, \mathbf{a}, \mathbf{c}, (1, \mathbf{l}, \mathbf{r}))$ is a braided monoidal category and $\mathbf{c}$ satisfies the diagram*

$$\begin{array}{ccc} A(X\oplus Y) & \xrightarrow{\breve{L}^A_{X,Y}} & AX \oplus AY \\ \downarrow{\scriptstyle \mathbf{c}} & & \downarrow{\scriptstyle \mathbf{c}\oplus\mathbf{c}} \\ (X\oplus Y)A & \xrightarrow{\breve{R}^A_{X,Y}} & XA\oplus YA \end{array} \tag{1}$$

with $\mathbf{c}_{0,0} = id$.

A *braided* Ann-functor $(F, \breve{F}, \widetilde{F}, F_*) : \mathcal{A} \to \mathcal{A}'$ between two braided Ann-categories is an Ann-functor which is compatible with braidings. If F is an equivalence of categories, then the (braided) Ann-functor $(F, \breve{F}, \widetilde{F}, F_*)$ is said to be a *(braided) Ann-equivalence.*

Each braided Ann-category $\mathcal{A}$ determines three invariants [16]:
- the set $R = \pi_0\mathcal{A}$ of isomorphic classes of objects in $\mathcal{A}$ is a commutative ring,
- the set $M = \pi_1\mathcal{A}$ of automorphisms of 0 is a $\pi_0\mathcal{A}$-bimodule with the same two-sided actions,
- an element $h_\mathcal{A} \in H^3_{ab}(R, M)$ (this group is mentioned in Section 3).

Let R be a ring and M an R-bimodule. From the definition of Mac Lane cohomology of rings [10], we obtain the description of elements in the group $Z^3_{MacL}(R,M)$. Each normalized 3-cocycle k of R with coefficients in M consists of a quadruple $(\sigma,\alpha,\lambda,\rho)$ of the maps:

$$\sigma : R^4 \to M; \quad \alpha,\lambda,\rho : R^3 \to M$$

satisfying the conditions (for details, see Section 2.3 [13]). For each $k \in Z^3_{MacL}(R,M)$, Quang [13] constructed an Ann-category denoted by $\mathcal{S} = \int(R,M,k)$. The objects of $\mathcal{S}$ are elements of R and its morphisms are automorphisms, $(s,a) : s \to s,\ s \in R, a \in M$. The composition of morphisms is given by

$$(s,a) \circ (s,b) = (s,a+b).$$

Two operations on $\mathcal{S}$ are given by

$$\begin{gathered} s \oplus t = s+t,\ (s,a) \oplus (t,b) = (s+t, a+b), \\ s \otimes t = st,\ (s,a) \otimes (t,b) = (st, sb+at), \end{gathered}$$

where $s,t \in R,\ a,b \in M$. The unit constraints of $\mathcal{S}$ are identities. The 3-cocycle $k = (\xi,\eta,\alpha,\lambda,\rho)$ defines other constraints.

In the case when R is commutative, two-sided actions on M are equal, and $h = (k,\beta) \in Z^3_{ab}(R,M)$ (see Section 3), then $\mathcal{S} = \int(R,M,h)$ is a braided Ann-category whose braiding is induced by the function $\beta : R^2 \to M$. The braided Ann-category $\int(\pi_0\mathcal{A}, \pi_1\mathcal{A}, h_{\mathcal{A}})$, which is equivalent to $\mathcal{A}$, is called a *reduction* of $\mathcal{A}$.

We now recall some results on braided Ann-functors from [16]. Each braided Ann-functor $(F,\breve{F},\widetilde{F}) : \mathcal{A} \to \mathcal{A}'$ induces one S_F of their reductions. A functor $F : \int(R,M,h) \to \int(R',M',h')$ is of *type* (p,q) if

$$F(s) = p(s), \qquad F(s,a) = (p(s), q(a))$$

where $p : R \to R'$ is a ring homomorphism and $q : M \to M'$ is a group homomorphism such that

$$q(sa) = p(s)q(a), \quad s \in R, a \in M.$$

One can regard M' as an R-bimodule with the action $sa' = p(s)a'$, then q is a homomorphism of R-bimodules. In this case we say that (p,q) is a *pair of homomorphism*, and the function

$$\zeta = q_*h - p^*h' \tag{2}$$

is an *obstruction* of F, where p^*, q_* are canonical homomorphisms

$$Z^3_{ab}(R,M) \xrightarrow{q_*} Z^3_{ab}(R,M') \xleftarrow{p^*} Z^3_{ab}(R',M').$$

Denote by

$$\mathrm{Hom}^{BrAnn}_{(p,q)}[\mathcal{S},\mathcal{S}']$$

the set of all homotopy classes of braided Ann-functors of type (p,q) from $\mathcal{S}$ to $\mathcal{S}'$.

Theorem 1 ([16, Theorem 5.5]). *A functor $F : \int(R, M, h) \to \int(R', M', h')$ of type (p, q) realizes a braided Ann-functor if and only if the obstruction $\overline{\zeta} = 0$ in $H^3_{ab}(R, M')$. Then, there exists a bijection*

$$\mathrm{Hom}^{BrAnn}_{(p,q)}[\mathcal{S}, \mathcal{S}'] \leftrightarrow H^2_{ab}(R, M').$$

2.2. E-systems

The notion of E-system is presented in [14]. For convenience, we recall some notions and terminologies. The category of E-systems and their morphisms is equivalent to the homotopy category of strict Ann-category and single Ann-functors. According to Mac Lane [10], we call a *bimultiplication* of a ring B a pair of mappings $a \to \zeta a, a \to a\zeta$ of B into itself which satisfy the rules

$$\zeta(a + b) = \zeta a + \zeta b, (a + b)\zeta = a\zeta + b\zeta,$$
$$\zeta(ab) = (\zeta a)b,\ (ab)\zeta = a(b\zeta),\ a(\zeta b) = (a\zeta)b,$$

for all elements $a, b \in B$. The sum $\zeta + \omega$ and the product $\zeta\omega$ of two bimultiplications ζ and ω are defined by the equations for all a in B

$$(\zeta + \omega)a = \zeta a + \omega a,\ a(\zeta + \omega) = a\zeta + a\omega,$$
$$(\zeta\omega)a = \zeta(\omega a),\ a(\zeta\omega) = (a\zeta)\omega.$$

The set of all bimultiplications of B is a ring denoted by M_B. For each element c of B, an *inner bimultiplication* τ_c is defined by

$$\tau_c b = cb, b\tau_c = bc, b \in B.$$

The kernel C_B of the homomorphism $\tau : B \to M_B$ is a two-sided ideal in B consisting of elements $c \in B$ such that $cb = bc = 0$ for all $b \in B$. We call C_B a *bicenter* of B. The bimultiplications σ and σ' are called *permutable* if for every $a \in B$,

$$\sigma(a\sigma') = (\sigma a)\sigma',\quad \sigma'(a\sigma) = (\sigma' a)\sigma. \tag{3}$$

The quotient ring $P_B = M_B/\tau B$ is called the ring of *outer bimultiplication* of B, and a ring homomorphism $\psi : R \to P_B$ is called *regular* if $\psi(1) = 1$ and two any elements of $\psi(R)$ are *permutable.* Then, C_B is an R-bimodule under the operations

$$xc = (\psi x)c, cx = c(\psi x), c \in C_B, x \in R.$$

Definition 2 ([14]). *(i) An E-system is a quadruple $\mathcal{M} = (B, D, d, \theta)$, where $d : B \to D$, $\theta : D \to M_B$ are ring homomorphisms satisfying:*

E_1. $\theta \circ d = \tau$,

E_2. $d(\theta_x b) = xd(b),\ d(b\theta_x) = d(b)x,\quad x \in D, b \in B$.

(ii) An E-system (B, D, d, θ) is termed regular *if θ is an 1-homomorphism (a homomorphism carries the identity to the identity), and elements of $\theta(D)$ are permutable.*

(iii) A morphism $(f_1, f_0) : (B, D, d, \theta) \to (B', D', d', \theta')$ of E-systems consists of

ring homomorphisms $f_1 : B \to B'$*,* $f_0 : D \to D'$ *such that*

H_1. $f_0 d = d' f_1$,

H_2. $f_1(\theta_x b) = \theta'_{f_0(x)} f_1(b)$, $f_1(b\theta_x) = f_1(b)\theta'_{f_0(x)}$, *for all* $x \in D, b \in B$.

In this paper, an E-system (B, D, d, θ) is sometimes denoted by $B \stackrel{d}{\to} D$, or $B \to D$. The following result follows from the definition of an E-system.

Proposition 1. *Let* $\mathcal{M} = (B, D, d, \theta)$ *be a regular E-system. Then,* $\mathrm{Ker}d \subset C_B$ *and* $\mathrm{Ker}d$ *is a* $\mathrm{Coker}d$*-bimodule with the actions*

$$sa = \theta_x(a), \quad as = (a)\theta_x, \ a \in \mathrm{Ker}d, \ x \text{ is a representative of } r \in \mathrm{Coker}d.$$

We denote the groups $\mathrm{Coker}d and \mathrm{Ker}d$ by $\pi_0\mathcal{M} and \pi_1\mathcal{M}$, respectively.

3. Braided E-systems and braided strict Ann-categories

In this section, we study the relationship among braided strict Ann-categories, braided regular E-systems and internal categories in the commutative rings.

Definition 3. *(i) An Ann-category is strict if its constraints are identities and for each object there is a strict inverse with respect to* $\oplus$ *(*$x \oplus y = 0$*).*

(ii) A braided Ann-category $\mathcal{A}$ *is strict if it is a strict Ann-category.*

According to [13], an Ann-category $\mathcal{A}$ is termed *regular* if the commutativity constraint with respect to $\oplus$, $c^+_{X,Y} : X \oplus Y \to Y \oplus X$, satisfies the condition $c^+_{X,X} = id_X$. A braided Ann-category $\mathcal{A}$ is called a braided *regular* Ann-category if it is a regular Ann-category. The following theorem is necessary to classify braided regular E-systems (Definition 4).

Theorem 2. *Each braided regular Ann-category is braided Ann-equivalent to a braided strict Ann-category.*

Proof. Let $\mathcal{A}$ be a braided regular Ann-category with the braiding **c**. According to [21], $\mathcal{A}$ is Ann-equivalent to an Ann-category $\mathcal{A}'$ whose constraints are strict. Paper [21] is only published in the form of a short report (in Communication of the Moscow Mathematical Society). This statement can be proved analogously to [18] (Lemmas 13, 14, Proposition 15). We sketch this proof as follows.

1. Let $\mathcal{A}$ be a regular Ann-category whose reduction one is $\int_{\mathcal{A}} = \int(\Pi, C, k)$. By the theorem on realization of the obstruction ([10], Theorem 8) there exists a ring B with the bicenter $C_B = C$ and a ring homomorphism $\psi : \Pi \to P_B$, $(P_B = M_B/\tau B)$, such that the obstruction of ψ is k. Consider the category $\mathcal{M}_B$ whose objects are elements of the ring M_B, and if φ, λ are bimultiplications of B, then let us denote:

$$\mathrm{Hom}(\varphi, \lambda) = \{c \in B \mid \lambda = \tau_c + \varphi\}.$$

The composition of two arrows is the operation $+$ in B. The operations $\oplus$ and $\otimes$

are given by:

$$\begin{array}{lll} \varphi \oplus \lambda = \varphi + \lambda, & \varphi, \lambda \in \mathcal{M}_B \\ c \oplus d \ = c + d, & c, d \in B \\ \varphi \otimes \lambda = \varphi \circ \lambda, & \text{(the composition of two maps)} \\ c \otimes d \ = cd + c\lambda + \varphi d, & \text{where } c : \varphi \to \varphi', d : \lambda \to \lambda' \end{array}$$

With these two operations, $\mathcal{M}_B$ becomes an Ann-category with the constraints naturally determined to be strict. Then, the reduced Ann-category of $\mathcal{M}_B$ is $\int(P_B, C, h)$, where $[\psi^* h] = [k] \in H^3_{MacL}(\Pi, C)$.

2. For the Ann-category $\mathcal{H} = M_B$ and the homomorphism ψ, we construct a braided strict Ann-category $\mathcal{H}_\psi$ as follows:

$$\mathrm{Ob}(\mathcal{H}_\psi) = \{(x, X)|\ x \in \Pi, X \in \psi(x)\},$$
$$\mathrm{Hom}((x, X), (x, Y)) = \{x\} \times \mathrm{Hom}_{\mathcal{H}}(X, Y).$$

Two operations $\oplus$ and $\otimes$ on objects and on morphism in $\mathbb{G}$ are given by

$$(x, X) \oplus (y, Y) = (x + y, X \oplus Y), \quad (x, u) \oplus (y, v) = (x + y, u + v),$$
$$(x, X) \otimes (y, Y) = (xy, X \otimes Y), \qquad (x, u) \otimes (y, v) = (xy, u \otimes v).$$

The zero object and the unit object of $\mathcal{H}_\psi$ are $(0, O)$ and $(1, I)$, respectively, where O and I are the zero object and the unit object of $\mathcal{H}$, respectively. Since the constraints in $\mathcal{H}$ are strict, so are the reducing constraints in $\mathcal{H}_\psi$. Further, the inverse of the object (x, X) is $(-x, -X)$, so $\mathcal{H}_\psi$ is a strict Ann-category. This Ann-category is Ann-equivalent to $\mathcal{A}$.

3. Finally, let $\mathbf{c}$ be the braiding in $\mathcal{A}$ and $(F, \breve{F}, \widetilde{F}, F_*) : \mathcal{H}_\psi \to \mathcal{A}$ the above Ann-equivalence. According to Proposition 3.9 [16], F induces a braiding $\mathbf{c}'$ in $\mathcal{H}_\psi$ by

$$F(\mathbf{c}'_{X,Y}) = (\widetilde{F}_{Y,X})^{-1} \circ \mathbf{c}_{X,Y} \circ \widetilde{F}_{X,Y}$$

thus F is a braided Ann-equivalence. □

The construction of the notion of braided E-systems is similar to that of braided crossed modules of Brown and Gilbert [3] (Example on p. 55), of Joyal and Street [8] (p. 47).

Definition 4. *A braided E-system consists of an E-system $\mathcal{M} = (B, D, d, \theta)$ and a function $\eta : D \times D \to B$ called a braiding satisfying the following conditions:*

B_1. $d(\eta(x, y)) = yx - xy$,

B_2. $\eta(db, y) = \theta_y(b) - b\theta_y, \quad \eta(x, db) = -\eta(db, x)$,

B_3. *η is a biadditive function,*

B_4. $\eta(x, yz) = \eta(x, y)\theta_z + \theta_y \eta(x, z)$,

B_5. $\eta(xy, z) = \eta(x, z)\theta_y + \theta_x \eta(y, z)$,

for all $x, y, z \in D,\ b \in B$.

A braided E-system is termed regular if it is a regular E-system.

Proposition 2. *Let (B, D, d, θ, η) be a braided regular E-system.*

(i) The function η *is normalized in the sense that* $\eta(x, y) = 0$ *if either* x *or* y *is equal to* 0 *or* 1.

(ii) Coker d *is a commutative ring and two-sided actions of* Coker d *on* Ker d *as in Proposition 1 are coincident.*

Proof. (i) It follows from conditions B_3 - B_5 in Definition 4.

(ii) It follows from condition B_1 that Coker d is commutative. For $a \in \operatorname{Ker} d$ and $s \in \operatorname{Coker} d$, then Coker d acts on Ker d by

$$sa = \theta_x(a),\ as = (a)\theta_x, x \in s.$$

By condition B_2 and the normality of η, one has $sa = as$. □

Example 1. *For any ring extension*

$$0 \to B \xrightarrow{j} E \xrightarrow{p} R \to 0,$$

the E-system (B, E, j, τ)*, where* τ *is given by multiplications, is a regular E-system if* p *is a 1-homomorphism. It is a braided E-system with a braiding* η *if and only if* R *is a commutative ring and* η *is given by*

$$\eta(x, y) = \{x, y\} = yx - xy,\ x, y \in E. \tag{4}$$

According to [14], each regular E-system $\mathcal{M} = (B, D, d, \theta)$ defines a strict Ann-category $\mathcal{A}_{\mathcal{M}}$ *associated* to the E-system $\mathcal{M}$. $\mathrm{Ob}(\mathcal{A}_{\mathcal{M}}) = D$ and for two objects x, y of $\mathcal{A}_{\mathcal{M}}$, then

$$\mathrm{Hom}(x, y) = \{b \in B \mid y = d(b) + x\}.$$

The composition of morphisms is given by the addition in B. Two operations $\oplus, \otimes$ on objects are given by the operations $+, \times$ on the ring D. For the morphisms, we set

$$(x \xrightarrow{b} y) \oplus (x' \xrightarrow{b'} y') = (x + x' \xrightarrow{b+b'} y + y'). \tag{5}$$

$$(x \xrightarrow{b} y) \otimes (x' \xrightarrow{b'} y') = (xx' \xrightarrow{bb' + b\theta_{x'} + \theta_x b'} yy'). \tag{6}$$

The fact that elements of $\theta(D)$ are permutable is equivalent to the associativity of the operation $\otimes$ on morphisms, so one can choose the associativity constraint $\mathbf{a}$ to be strict. The remaining constraints of $\mathcal{A}_{\mathcal{M}}$ are strict. Now, if $\mathcal{M}$ has a braiding η, then $\mathcal{A}_{\mathcal{M}}$ is a braided strict Ann-category whose braiding $\mathbf{c}$ is given by

$$\mathbf{c}_{x,y} = \eta(x, y) : xy \to yx.$$

Indeed, condition B_1 in Definition 4 shows that $\mathbf{c}_{x,y}$ is a morphism in $\mathcal{A}_{\mathcal{M}}$. Condition B_3 shows that $\mathbf{c}$ satisfies diagram (1) in the definition of braided Ann-category. By conditions B_4, B_5, the braiding $\mathbf{c}$ is compatible with the associativity constraint $\mathbf{a}$. The naturality of $\mathbf{c}$ follows from the conditions B_2, B_3.

Conversely, for any braided strict Ann-category $(\mathcal{A}, \oplus, \otimes)$ one can define a braided regular E-system $\mathcal{M}_\mathcal{A} = (B, D, d, \theta)$. Indeed, let

$$D = \mathrm{Ob}(\mathcal{A}),\ \ B = \{0 \xrightarrow{b} x |\ x \in D\}.$$

Then, B and D are rings with the units, and their corresponding operations are given by

$$b + c = b \oplus c,\ \ bc = b \otimes c,$$
$$x + y = x \oplus y,\ xy = x \otimes y.$$

The homomorphisms $d : B \to D$ and $\theta : D \to M_B$ are defined by

$$d(0 \xrightarrow{b} x) = x,$$

$$\theta_y(0 \xrightarrow{b} x) = (0 \xrightarrow{id_y \otimes b} yx),\quad (0 \xrightarrow{b} x)\theta_y = (0 \xrightarrow{b \otimes id_y} xy).$$

The braiding of E-system $\mathcal{M}_\mathcal{A}$ is given by

$$\eta(x, y) = \mathbf{c}_{x,y} \oplus id_{-xy},\ \ x, y \in D.$$

Definition 5. *An E-system (B, D, d, θ, η) is strong if θ is a 1-homomorphism and*

$$\theta_x b = b\theta_x,\ \ x \in D, b \in B. \tag{7}$$

Example 2. *In Example 1, the braided E-system $(B, E, j, \tau, \{,\})$ is strong if and only if R is commutative and $B \subset Z(E)$, where the center $Z(E)$ consists of elements $z \in E$ such that $zx = xz$ for all $x \in E$.*

Proposition 3. *Let $\mathcal{M} = (B, D, d, \theta, \eta)$ be a strong braided E-system.*
(i) $\mathcal{M}$ is a regular E-system.
(ii) $d(B) \subset Z(D)$.
(iii) B is a commutative ring.
(iv) In the Ann-category $\mathcal{A}_\mathcal{M}$ associated to the E-system $\mathcal{M}$, the braiding constraint induces a function $\beta : (\mathrm{Coker}\, d)^2 \to \mathrm{Ker}\, d$, and $b \otimes b' = b' \otimes b$ for any two morphisms $x \xrightarrow{b} y$, $x' \xrightarrow{b'} y'$ in $\mathrm{Mor}(\mathcal{A}_\mathcal{M})$.

Proof. (i) For all $x, y \in D$,

$$\theta_x(b\theta_y) = \theta_x(\theta_y b) = \theta_{xy} b = b\theta_{xy} = (b\theta_x)\theta_y = (\theta_x b)\theta_y.$$

Thus, θ_x, θ_y are permutable.
(ii) It follows from (7) and E_2 that

$$(db)x = x(db), x \in D,$$

that means $\mathrm{Im} d \subset Z(D)$.
(iii) and (iv) According to condition B_2 in Definition 4,

$$\eta(db, y) = 0 = \eta(x, db). \tag{8}$$

For $(x \xrightarrow{b} x'), (y \xrightarrow{c} y')$ two morphisms in the associated Ann-category $\mathcal{A}_{\mathcal{M}}$, the following diagram commutes by the naturality of braiding constraint $\mathbf{c}$,

$$\begin{array}{ccc} xy & \xrightarrow{\mathbf{c}_{x,y}} & yx \\ {\scriptstyle b\otimes c}\downarrow & & \downarrow{\scriptstyle c\otimes b} \\ x'y' & \xrightarrow[\mathbf{c}_{x',y'}]{} & y'x'. \end{array}$$

It follows from (6) that

$$b \otimes c = bc + b\theta_y + \theta_x c,$$
$$c \otimes b = cb + c\theta_x + \theta_y b.$$

By (7), two above equalities imply

$$b \otimes c - c \otimes b = bc - cb.$$

Note that $x' = d(b) + x, y' = d(c) + y$. Since η is biadditive (condition B_3) and by (8), then $\eta(x', y') = \eta(x, y)$. It follows that $\mathbf{c}_{x',y'} = \mathbf{c}_{x,y}$. This defines a function $\beta : \operatorname{Coker} d \times \operatorname{Coker} d \to \operatorname{Ker} d$ by

$$\beta(r, s) = \mathbf{c}_{x,y}, \ x \in r, y \in s.$$

The above commutative diagram leads to $b \otimes c - c \otimes b = bc - cb = 0$, that is, the ring B is commutative and the tensor product of morphisms in $\mathcal{A}$ is commutative. □

Obviously, one has the following result.

Proposition 4. *If $\mathcal{A}$ a braided strict Ann-category whose operation $\otimes$ on morphisms is aben, then the associated braided E-system $\mathcal{M}_{\mathcal{A}}$ is strong.*

The above construction leads to the relation to the notion of internal category in the commutative rings in the sense of [11].

Proposition 5. *Each braided strict Ann-category associated with a strong braided E-system (B, D, d, θ), where D is commutative, is an internal category in the commutative rings.*

Proof. Let us prove that *each strict Ann-category is an internal category in the rings.*

Let $\mathcal{M} = (B, D, d, \theta)$ be a strong braided E-system and $\mathcal{A}_{\mathcal{M}}$ its associated braided strict Ann-category. Then, by Proposition 3, B is a commutative ring and is a D-module with the same two-sided actions

$$xb = \theta_x(b) = (b)\theta_x = bx, \ b \in B, x \in D.$$

Since elements of $\theta(D)$ are permutable, then there exists a semidirect product $B \rtimes D$ with two operations:

$$(b, x) + (b', x') = (b + b', r + r'),$$

$$(b,x).(b',x') = (b.b' + bx' + xb', rr').$$

The exact sequence of rings

$$0 \to B \xrightarrow{i} B \rtimes D \underset{e}{\overset{p}{\rightleftarrows}} D \to 0, \tag{9}$$

for $e(x) = (0,x)$ is split. The ring homomorphisms $s,t : B \rtimes D \to D$ are given by $s(b,x) = x,\ t(b,x) = d(b) + x$. Clearly, $se = te = id_D$. So $(B \rtimes D, s, t, e)$ is an internal category in the commutative rings. □

Remark 1. *It follows from the above proof that each strong braided E-system (B, D, d, θ), where D is commutative, is a $\mathcal{C}$-crossed module, where $\mathcal{C}$ is the category of the commutative rings. In general, a $\mathcal{C}$-crossed module is not a strong braided E-system since D-modules are not unita.*

4. Classification thoerems

In order to classify braided regular E-systems, we first indicate the determination of braided Ann-functors corresponding to morphisms of braided E-systems. This relates to Mac Lane cohomology groups ([9]) and the group $Z^2_{ab}(R, M)$ of a commutative ring R. Quang and Hanh [16] define groups $H^n_{ab}(R, M)$ $(n = 1, 2, 3)$ of a commutative ring R and an R-module M in terms of the complex $C_{ab}(R, M)$ as follows

$$0 \longrightarrow C^1_{ab}(R,M) \xrightarrow{\partial} C^2_{ab}(R,M) \xrightarrow{\partial} Z^3_{ab}(R,M) \longrightarrow 0,$$

where $C^1_{ab}(R, M)$ consists of all normalized maps $t : R \to M$, $C^2_{ab}(R, M)$ consists of all normalized maps $\mu, \nu : R^2 \to M$, and $Z^3_{ab}(R, M)$ consists of pairs (h, β) in which $h \in Z^3_{MacL}(R, M)$ and $\beta : R \to M$ satisfies conditions of a 3-cocycle:

$$\alpha(x,y,z) - \alpha(x,z,y) + \alpha(z,x,y) + x\beta(y,z) - \beta(xy,z) + y\beta(x,z) = 0,$$
$$\alpha(x,y,z) - \alpha(y,x,z) + \alpha(y,z,x) - y\beta(x,z) + \beta(x,yz) - z\beta(x,y) = 0,$$
$$\beta(x,y) - \beta(x,y+z) + \beta(x,z) = \rho(y,z,x) - \lambda(x,y,z),$$

for all $x, y, z \in R$.

For each $t \in C^1_{ab}(R, M)$, the coboundary $\partial_{ab} t$ is given by

$$\partial_{ab} t = \partial_{MacL} t,$$

and for each $g = (\mu, \nu) \in C^2_{ab}(R, M)$, the coboundary $\partial_{ab} g$ is given by

$$\partial_{ab} g = (\partial_{MacL} g, \beta),$$

where $\beta(x, y) = \nu(x, y) - \nu(y, x)$.

Remark 2. *(i) If the function η of a Mac Lane 3-cocycle $k = (\xi, \eta, \alpha, \lambda, \rho)$ satisfies the regular condition $\eta(x, x) = 0$, for all $x \in R$, then k is a Shukla 3-cocycle* [19]. *The group of such (k, β) is denoted by $Z^3_{Shab}(R, M)$.*

(ii) When $n = 2$ then $Z^2_{Shu}(R, M) = Z^2_{MacL}(R, M)$.

Definition 6. *A morphism $(f_1, f_0) : (B, D, d, \theta, \eta) \to (B', D', d', \theta', \eta')$ of braided E-systems is a morphism of E-systems and it satisfies*
H_3. $f_1(\eta(x, y)) = \eta'(f_0(x), f_0(y)),\ x, y \in D$.

Lemma 1. *Let $(f_1, f_0) : (B, D, d, \theta, \eta) \to (B', D', d', \theta', \eta')$ be a morphism of a braided regular E-system.*
(i) There is a functor $F : \mathcal{A}_{B\to D} \to \mathcal{A}_{B'\to D'}$ defined by

$$F(x) = f_0(x),\ F(b) = f_1(b),\ x \in \mathrm{Ob}\mathcal{A}, b \in \mathrm{Mor}\mathcal{A}.$$

(ii) Natural isomorphisms $\breve{F}_{x,y} : F(x + y) \to F(x) + F(y)$ and $\widetilde{F}_{x,y} : F(xy) \to F(x)F(y)$ together with F is a braided Ann-functor if and only if $\breve{F}_{x,y} = \mu(\overline{x}, \overline{y})$, $\widetilde{F}_{x,y} = \nu(\overline{x}, \overline{y})$, for $(\mu, \nu) \in Z^2_{ab}(\mathrm{Coker}\, d, \mathrm{Ker}\, d')$.

Proof. (i) Every element $b \in B$ can be considered as a morphism $(0 \stackrel{b}{\to} db)$ in $\mathcal{A}$. Then, $(F(0) \stackrel{F(b)}{\to} F(db))$ is a morphism in $\mathcal{A}'$. By the construction of the Ann-category associated to an E-system, F is a functor.
(ii) We define natural isomorphisms

$$\breve{F}_{x,y} : F(x + y) \to F(x) + F(y),\ \widetilde{F}_{x,y} : F(xy) \to F(x)F(y)$$

such that $F = (F, \breve{F}, \widetilde{F})$ becomes an Ann-functor. Since f_0 is a ring homomorphism,

$$F(x) + F(y) = F(x + y),$$

hence $d'(\breve{F}_{x,y}) = 0$. Analogously, $d'(\widetilde{F}_{x,y}) = 0$. Thus,

$$\breve{F}_{x,y}, \widetilde{F}_{x,y} \in \mathrm{Ker} d' \subset C_{B'}. \tag{10}$$

Besides, since f_1 is a ring homomorphism and by (5),

$$F(b \oplus b') = F(b + b') = F(b) + F(b') = F(b) \oplus F(b').$$

Then, the commutativity of the diagram

$$\begin{array}{ccc} F(x+y) & \xrightarrow{\breve{F}_{x,y}} & F(x)+F(y) \\ {\scriptstyle F(b\oplus b')}\downarrow & & \downarrow{\scriptstyle F(b)\oplus F(b')} \\ F(x'+y') & \xrightarrow[\breve{F}_{x',y'}]{} & F(x')+F(y') \end{array}$$

is equivalent to $\breve{F}_{x,y} = \breve{F}_{x',y'}$, where $x' = d(b) + x, y' = d(b') + y$. This defines a function $\mu : (\mathrm{Coker}\, d)^2 \to \mathrm{Ker}\, d'$ by

$$\mu(\overline{x}, \overline{y}) = \breve{F}_{x,y}.$$

The fact that homomorphisms f_0, f_1 satisfy H_2 is equivalent to

$$F(b \otimes b') = F(b) \otimes F(b').$$

The naturality of $\widetilde{F}$ is equivalent to the relation $\widetilde{F}_{x,y} = \widetilde{F}_{x',y'}$, where $x' = d(b) + x, y' = d(b') + y$. This defines a function $\nu : (\operatorname{Coker} d)^2 \to \operatorname{Ker} d'$ by

$$\nu(\overline{x}, \overline{y}) = \widetilde{F}_{x,y}.$$

Since $F(0) = 0', F(1) = 1'$, then the compatibility of $(F, \breve{F}), (F, \widetilde{F})$ with unit constraints is equivalent to the normality of μ, ν. The compatibility of $(F, \widetilde{F})$ with associativity constraints is equivalent to the relation

$$\theta'_{F(x)} \widetilde{F}_{y,z} + \widetilde{F}_{x,yz} = \widetilde{F}_{xy,z} + \widetilde{F}_{x,y} \theta_{F(z)}.$$

In terms of the homomorphism $F : D \to D'$, the abelian group $\operatorname{Ker} d'$ is a $\operatorname{Coker} d$-bimodule under the actions

$$rb' = \theta'_{F(x)} b', \; b'r = (b')\theta'_{F(x)}, \tag{11}$$

where x is a representative of $r \in \operatorname{Coker} d, b' \in \operatorname{Ker} d'$. Together with (10) and (11), one has

$$r\nu(s,t) - \nu(rs,t) + \nu(r,st) - \nu(r,s)t = 0. \tag{12}$$

Similarly, the compatibility of F with other constraints leads to

$$\mu(s,t) - \mu(r+s,t) + \mu(r,s+t) - \mu(r,s) = 0, \tag{13}$$

$$\mu(s,r) - \mu(r,s) = 0, \tag{14}$$

$$\nu(r,s+t) - \nu(r,s) - \nu(r,t) + r\mu(s,t) - \mu(rs,rt) = 0, \tag{15}$$

$$\nu(r+s,t) - \nu(r,t) - \nu(s,t) + \mu(r,s)t - \mu(rt,st) = 0. \tag{16}$$

Relations (12)–(16) show that $(\mu,\nu) \in Z^2_{MacL}(\operatorname{Coker} d, \operatorname{Ker} d')$. The compatibility of $\widetilde{F}$ with the braiding constraint $\mathbf{c}$ is given by

$$\begin{array}{ccc} F(xy) & \xrightarrow{\widetilde{F}_{x,y}} & F(x)F(y) \\ {\scriptstyle F(\mathbf{c}_{x,y})}\downarrow & & \downarrow{\scriptstyle \mathbf{c}'_{Fx,Fy}} \\ F(yx) & \xrightarrow[\widetilde{F}_{y,x}]{} & F(y)F(x). \end{array} \tag{17}$$

This diagram implies

$$\nu(r,s) + \eta'(f_0(x), f_0(y)) = f_1\eta(x,y) + \nu(s,r),$$

where $r = \overline{x}, s = \overline{y}$. It follows from the condition H_3 that $\nu(r,s) = \nu(s,r)$. This means that $(\mu,\nu) \in Z^2_{ab}(\operatorname{Coker} d, \operatorname{Ker} d')$. □

Definition 7. *(i) An Ann-functor* $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}'$ *is almost strict if the functor* F *satisfies the following conditions for* $x, y \in \mathrm{Ob}\,\mathcal{A},\ b, c \in \mathrm{Mor}\,\mathcal{A}$:

S_1. $F(x) \oplus F(y) = F(x \oplus y)$,

S_2. $F(x) \otimes F(y) = F(x \otimes y)$,

S_3. $F(b) \oplus F(c) = F(b \oplus c)$,

S_4. $F(b) \otimes F(c) = F(b \otimes c)$.

(ii) A braided Ann-functor $(F, \breve{F}, \widetilde{F})$ *is almost strict if it is a regular Ann-functor and:*

S_5. $\widetilde{F}_{x,y} = \widetilde{F}_{y,x}$.

The braided Ann-functor mentioned in Lemma 1 is almost strict. Note that if $\mathcal{A}, \mathcal{A}'$ are Ann-categories associated to regular E-systems and $F : \mathcal{A} \to \mathcal{A}'$ is a functor, then the condition S_3 always holds, which follows from the preserving of composition of morphisms.

Denote by **BrEsyst** the category of braided regular E-systems whose morphisms are quadruple (f_1, f_0, μ, ν), where (f_1, f_0) is a morphism of braided E-systems and $(\mu, \nu) \in Z^2_{ab}(\mathrm{Coker}\, d, \mathrm{Ker}\, d')$. Composition of morphisms is given by

$$(f'_1, f'_0, \mu', \nu') \circ (f_1, f_0, \mu, \nu) = (f'_1 f_1, f'_0 f_0, (f'_1)_* \mu + f_0^* \mu', (f'_1)_* \nu + f_0^* \nu').$$

This definition is compatible with the composition of braided Ann-functors.

Lemma 2. *Let* $\mathcal{A}$ *and* $\mathcal{A}'$ *be two braided strict Ann-categories associated to braided regular E-systems* (B, D, d, θ, η) *and* $(B', D', d', \theta', \eta')$*, respectively. Let* $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}'$ *be a regular braided Ann-functor. The quadruple* (f_1, f_0, μ, ν)*, where*

$$f_0(x) = F(x),\ f_1(b) = F(b),\ \mu(\overline{x}, \overline{y}) = \breve{F}_{x,y},\ \nu(\overline{x}, \overline{y}) = \widetilde{F}_{x,y},$$

for $b \in B, x, y \in D, \overline{x}, \overline{y} \in \mathrm{Coker}\, d$*, is a morphism in the category* **BrEsyst**.

Proof. We prove that (f_1, f_0) is a morphism of a braided regular E-system and $(\mu, \nu) \in Z^2_{ab}(\mathrm{Coker}\, d, \mathrm{Ker}\, d')$. It follows from S_1 and S_2 in Definition 7 that f_0 is a ring homomorphism. The condition S_3 shows that f_1 is a homomorphism of additive groups.

For $b \in B$, then $(0 \xrightarrow{b} db)$ is a morphism in $\mathcal{A}$, and hence $(F(0) \xrightarrow{F(b)} F(db))$ is a morphism in $\mathcal{A}'$. The condition $F(0) = 0$ is just H_1: $d' f_1(b) = f_0(db)$. By the definition of the operation $\otimes$, the condition S_4 is equivalent to

$$f_1(bb') + f_1(b\theta_{x'}) + f_1(\theta_x b') = f_1(b) f_1(b') + f_1(b)\theta'_{f_0(x')} + \theta'_{f_0(x)} f_1(b'). \tag{18}$$

By taking $b = 0$ and then $b' = 0$, one obtains H_2:

$$f_1(\theta_x b') = \theta'_{f_0(x)} f_1(b'),\ f_1(b\theta_{x'}) = f_1(b)\theta'_{f_0(x)}.$$

Equality (18) becomes $f_1(bb') = f_1(b) f_1(b')$, that means f_1 is a ring homomorphism. By S_5, commutative diagram (17) implies $F(\mathbf{c}_{x,y}) = \mathbf{c'}_{Fx,Fy}$, thus the pair (f_1, f_0) satisfies H_3, and hence it is a morphism of braided E-systems.

According to the proof of Lemma 1, the natural isomorphisms $\breve{F}, \widetilde{F}$ determine an element $(\mu, \nu) \in Z^2_{ab}(\mathrm{Coker}\, d, \mathrm{Ker}\, d')$ by $\mu(\overline{x}, \overline{y}) = \breve{F}_{x,y},\ \nu(\overline{x}, \overline{y}) = \widetilde{F}_{x,y}$. □

Denote by **BrAnnstr** the category of braided strict Ann-categories and almost strict braided Ann-functors we state the following result.

Theorem 3 (Classification theorem for braided regular E-systems). *There exists an equivalence of the categories*

$$\begin{aligned} \Phi: \quad \mathbf{BrEsyst} &\to \mathbf{BrAnnstr}, \\ (B \to D) &\mapsto \quad \mathcal{A}_{(B\to D)} \\ (f_1, f_0, \mu, \nu) &\mapsto \quad (F, \breve{F}, \widetilde{F}), \end{aligned}$$

where $F(b) = f_1(b), F(x) = f_0(x), \breve{F}_{x,y} = \mu(\overline{x}, \overline{y}), \widetilde{F}_{x,y} = \nu(\overline{x}, \overline{y})$*, for* $x, y \in D, b \in B$.

Proof. By Lemmas 1, 2, the correspondence $(f_1, f_0, \mu, \nu) \mapsto (F, \breve{F}, \widetilde{F})$ defines a bijection on homsets

$$\Phi : \mathrm{Hom}_{\mathbf{BrEsyst}}(\mathcal{M}, \mathcal{M}') \to \mathrm{Hom}_{\mathbf{BrAnnstr}}(\Phi(\mathcal{M}), \Phi(\mathcal{M}')).$$

If $\mathcal{A}$ is a braided strict Ann-category and $\mathcal{M}_{\mathcal{A}}$ is the associated braided regular E-system, then $\Phi(\mathcal{M}_{\mathcal{A}}) = \mathcal{A}$ (not only isomorphic). Thus, Φ is an equivalence. □

In Lemma 1, if $B \to D$ is an E-system without a braiding, then the pair (μ, ν) is an element in $Z^2_{MacL}(\mathrm{Coker}\, d, \mathrm{Ker}\, d')(= Z^2_{Sh}(\mathrm{Coker}\, d, \mathrm{Ker}\, d')$. We obtain a similar result to Theorem 3 for the category **Esyst** of regular E-systems and the category **Annstr** of strict Ann-categories and almost strict Ann-functors.

Theorem 4 (Classification theorem for regular E-systems). *There exists an equivalence of the categories*

$$\begin{aligned} \Phi: \quad \mathbf{Esyst} &\to \mathbf{Annstr}, \\ (B \to D) &\mapsto \mathcal{A}_{(B\to D)} \\ (f_1, f_0, \mu, \nu) &\mapsto (F, \breve{F}, \widetilde{F}), \end{aligned}$$

where $F(b) = f_1(b), F(x) = f_0(x), \breve{F}_{x,y} = \mu(\overline{x}, \overline{y}), \widetilde{F}_{x,y} = \nu(\overline{x}, \overline{y})$*, for* $x, y \in D, b \in B$.

In Theorem 4, if $\mu = id, \nu = id$, then the corresponding Ann-functor F has $\breve{F} = id, \widetilde{F} = id$. Thus, this result contains Theorem 3 [14].

(Braided) regular E-systems having the same two invariants can be classified by the groups $H^3_{Sh}(R, M)$, $(H^3_{Sh}(R, M))$ as below.

Let R be a commutative ring with a unit and M an R-bimodule (regarded as a ring with the null multiplication). We say that a braided Ann-category $\mathcal{A}$ has a *pre-stick of type* (R, M) if there is a pair of ring isomorphisms, $\epsilon = (p, q)$,

$$p : R \to \pi_0\mathcal{A}, \qquad q : M \to \pi_1\mathcal{A},$$

which is compatible with the bimodule actions

$$q(su) = p(s)q(u),$$

where $s \in R, u \in M$. The pair (p, q) is called a *pre-stick of type* (R, M) of the braided Ann-category $\mathcal{A}$.

A *morphism* $(F, \breve{F}, \widetilde{F}) : \mathcal{A} \to \mathcal{A}'$ in the category $\mathbf{BrAnnstr}(R, M)$ of braided strict Ann-categories whose pre-sticks (respectively, $\epsilon = (p, q), \epsilon' = (p', q')$) are of type (R, M) is a braided Ann-functor such that the following diagrams commute

where $\pi_0(F), \pi_1(F)$ are two homomorphisms induced from $(F, \breve{F}, \widetilde{F})$. Clearly, it follows immediately from the definition that $\pi_0(F), \pi_1(F)$ are isomorphisms, and hence F is an equivalence. We write

$$\mathbf{BrAnnstr}[R, M]$$

for the set of connected components of braided strict Ann-categories whose pre-sticks are of type (R, M).

We say that a braided regular E-system $\mathcal{M} = (B, D, d, \theta, \eta)$ has a *pre-stick* of type (R, M) if there exist isomorphisms $M \to \operatorname{Coker} d$, $R \to \operatorname{Ker} d$ which are compatible with the structures of R-bimodule on M and of $\operatorname{Coker} d$-bimodule on $\operatorname{Ker} d$. Equivalently, a braided regular E-system $\mathcal{M}$ has a *pre-stick* of type (R, M) if and only if its associated braided Ann-category $\mathcal{A}_{\mathcal{M}}$ has a pre-stick of type (R, M).

A *morphism* $(f_1, f_0) : \mathcal{M} \to \mathcal{M}'$ in the category $\mathbf{BrEsyst}(R, M)$ of braided E-systems whose pre-sticks are of type (R, M) induces isomorphisms $\overline{f_1} : \operatorname{Ker} d \to \operatorname{Ker} d'$, $\overline{f_0} : \operatorname{Coker} d \to \operatorname{Coker} d'$. Then, the braided Ann-functor $(F, id, id) : \mathcal{A}_{\mathcal{M}} \to \mathcal{A}_{\mathcal{M}'}$, where $F = (f_1, f_0)$, is a morphism in the category $\mathbf{BrAnnstr}(R, M)$.

Denote by $\mathbf{BrEsyst}[R, M]$ the connected components of the category $\mathbf{BrEsyst}(R, M)$. Without braiding structures, the corresponding notations are $\mathbf{Esyst}(R, M)$ and $\mathbf{Esyst}[R, M]$.

Lemma 3. *There exist bijections*

$$\Pi : \mathbf{Esyst}[R, M] \to \mathbf{Annstr}[R, M],$$
$$\Pi' : \mathbf{BrEsyst}[R, M] \to \mathbf{BrAnnstr}[R, M].$$

Theorem 5 (Classification Theorem)**.** *There are bijections*

$$\Omega : \mathbf{Esyst}[R, M] \to H^3_{Sh}(R, M),$$
$$\Omega' : \mathbf{BrEsyst}[R, M] \to H^3_{Shab}(R, M).$$

Proof. According to Theorem 4.3 [12], one has a bijection

$$\Gamma : \mathbf{Ann}^{reg}[R, M] \to H^3_{Sh}(R, M),$$

where $\mathbf{Ann}^{reg}[R, M]$ is the set of equivalence classes of regular Ann-categories whose pre-sticks are of type (R, M). Since each regular Ann-category is Ann-equivalent to a strict Ann-category (see [21]), then there is a bijection

$$\mathbf{Ann}^{reg}[R, M] \to \mathbf{Annstr}[R, M].$$

Together with the bijection Π in Lemma 3, one obtains the bijection Ω.

According to Theorem 6.2 [16], there is a bijection

$$\Gamma : \mathbf{BrAnn}^{reg}[R, M] \to H^3_{Shab}(R, M).$$

By Theorem 2, there is a bijection

$$\mathbf{BrAnn}^{reg}[R, M] \leftrightarrow \mathbf{BrAnnstr}[R, M].$$

Together with the bijection Π' in Lemma 3, one obtains the bijection Ω'. □

Remark 3. *According to [2], each crossed bimodule over an algebra $\mathbb{A}$ is embedded into an exact sequence called a crossed extension of R by M,*

$$0 \to M \to B \xrightarrow{d} D \to R \to 0.$$

Denote by $\mathbf{Cros}(R, M)$ *connected components of the category of crossed extensions of M by R. One has a bijection ([2, Theorem 4.4.1]):*

$$\mathbf{Cros}(R, M) \leftrightarrow H^3_{Sh}(R, M).$$

When $\mathbb{A}$ is a ring (regarded as an $\mathbb{Z}$-algebra), Quang and Cuc [14] proved that the category crossed bimodules over rings is isomorphic to that of regular E-systems. Since $\mathbf{Cros}(R, M) = \mathbf{Esyst}^{reg}[R, M]$*, then the above bijection together with the bijection* Π *gives a bijection*

$$\mathbf{Esyst}[R, M] \leftrightarrow \mathbf{Esyst}^{reg}[R, M].$$

An analogous result was stated by Baues and Minian for $\mathbb{K}$-split crossed extensions and Hochschild cohomology ([1, Theorem 3.2]): there is a bijection

$$\mathbf{Cros}_{\mathbb{K}}(R, M) \leftrightarrow H^3_{Hochs}(R, M).$$

5. Strong braided E-systems and ring extensions of the type of a strong braided E-system

In this section, we apply the above results to extensions of commutative rings.

Analogously to group extensions of the type of a crossed module [5] and ring extensions of the type of an E-system [14], we consider center extensions of a commutative ring B by a commutative ring R of the type of a strong braided E-system as follows.

Definition 8. *Let $\mathcal{M} = (B, D, d, \theta, \eta)$ be a strong braided E-system and R a com-*

mutative ring. A center extension of B by R of type $\mathcal{M}$ is a diagram of ring homomorphisms

$$\begin{array}{ccccccccc} \mathcal{E} & 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & R & \longrightarrow 0, \\ & & & \| & & \downarrow{\scriptstyle\varepsilon} & & & \\ & & & B & \xrightarrow{d} & D & & & \end{array} \tag{19}$$

where the top row is exact, $B \subset Z(E)$, $(B, E, j, \tau, \{,\})$ is a strong braided E-system in which τ is given by inner bimultiplications, the braiding $\{,\}$ is given by (4)

$$\{e, e'\} = e'e - ee',$$

and the pair $(id, \varepsilon) : (B \to E) \to (B \to D)$ is a morphism of braided E-systems.

Let $q : D \to \operatorname{Coker} d$ be a canonical homomorphism. Since the top row of diagram (19) is exact and since $q \circ \varepsilon \circ j = q \circ d = 0$, there is a ring homomorphism $\psi : R \to \operatorname{Coker} d$ with $\psi \circ p = q \circ \varepsilon$, and we say that $\mathcal{E}$ *induces* ψ.

Two extensions $\mathcal{E}, \mathcal{E}'$ of type $\mathcal{M}$ are *equivalent* if the following diagram commutes

$$\begin{array}{ccccccccccc} \mathcal{E} : 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & R & \longrightarrow 0, & & E \xrightarrow{\varepsilon} D \\ & & \| & & \downarrow{\scriptstyle\alpha} & & \| & & & \\ \mathcal{E}' : 0 & \longrightarrow & B & \xrightarrow{j'} & E' & \xrightarrow{p'} & R & \longrightarrow 0, & & E' \xrightarrow{\varepsilon'} D \end{array}$$

and $\varepsilon'\alpha = \varepsilon$. Obviously, α is an isomorphism, and $\mathcal{E}, \mathcal{E}'$ induce the same ψ.

Our purpose is to describe the set of equivalence classes of ring extensions of B by R of type $\mathcal{M}$ inducing ψ,

$$\operatorname{Ext}_{\mathcal{M}}(R, B, \psi).$$

We deal with this problem by the method done for ring extensions of the type of an E-system in [14], Section 5. Let $\mathcal{A}$ be the braided Ann-category associated to the strong braided E-system $\mathcal{M}$. Since $\pi_0\mathcal{A} = \operatorname{Coker} d$ and $\pi_1\mathcal{A} = \operatorname{Ker} d$, then its third invariant is $\overline{k} \in H^3_{ab}(\operatorname{Coker} d, \operatorname{Ker} d)$. The homomorphism $\psi : R \to \operatorname{Coker} d$ induces an element

$$\overline{\psi^* k} \in H^3_{ab}(R, \operatorname{Ker} d), \tag{20}$$

called the *obstruction* of the pair $(\mathcal{M}, \psi)$.

Lemma 4. *Let $\mathcal{M} = (B, D, d, \theta, \eta)$ be a strong braided E-system, R a commutative ring and $\psi : R \to \operatorname{Coker} d$ a ring homomorphism. Then, each braided Ann-functor $(F, \breve{F}, \widetilde{F}) : \operatorname{Dis} R \to \mathcal{A}_{\mathcal{M}}$ with $Fr \in \psi(r)$ determines a ring extension $\mathcal{E}_F$ of B by R of type $\mathcal{M}$ inducing $\psi : R \to \operatorname{Coker} d$.*

Proof. We write $\breve{F} = f, \widetilde{F} = g$. The constraints of $\operatorname{Dis} R$ and $\mathcal{A}_{\mathcal{M}}$, except for the braiding, are all strict. Thus, the compatibility of F with these constraints implies that $f, g : R^2 \to B$ are normalized functions satisfing the rules

$$f(r, s+t) + f(s,t) - f(r,s) - f(r+s,t) = 0, \tag{21}$$
$$f(r,s) - f(s,r) = 0, \tag{22}$$
$$\varphi(r)g(s,t) - g(rs,t) + g(r,st) - g(r,s)\varphi(t) = 0, \tag{23}$$
$$g(r,s+t) - g(r,s) - g(r,t) + \varphi(r)f(s,t) - f(rs,rt) = 0, \tag{24}$$
$$g(r+s,t) - g(r,t) - g(s,t) + f(r,s)\varphi(t) - f(rt,st) = 0, \tag{25}$$

where $\varphi(r) = \theta_{Fr}$. Act θ on the equalities

$$Fr + Fs = d(f(r,s)) + F(r+s),$$
$$FrFs = d(g(r,s)) + F(rs),$$

one obtains

$$\varphi(r) + \varphi(s) = \tau_{f(r,s)} + \varphi(r+s), \tag{26}$$
$$\varphi(r)\varphi(s) = \tau_{g(r,s)} + \varphi(rs). \tag{27}$$

Since functions (φ, f, g) satisfy (21) - (27), then $E = B \times R$ is a ring with the operations

$$(b,r) + (b',r') = (b + b' + f(r,r'), r + r'),$$
$$(b,r).(b',r') = (b.b' + b\varphi(r') + \varphi(r)b' + g(r,r'), rr'),$$

denoted by $E_{(f,g)}$. Note that the associativity of the multiplication in $E_{(f,g)}$ holds if and only if the E-system $(B \to D)$ is regular. It is obvious by Proposition 3. The sequence of ring homomorphisms

$$\mathcal{E}_F : 0 \to B \overset{j_0}{\to} E_{(f,g)} \overset{p_0}{\to} R \to 0$$

is exact, where j_0, p_0 are canonical homomorphisms. By the multiplication in the ring $E_{(f,g)}$,

$$(b,r)(c,0) = (bc + \varphi(r)c, 0), \tag{28}$$
$$(c,0)(b,r) = (cb + c\varphi(r), 0). \tag{29}$$

Since $\mathcal{M}$ is a strong braided E-system, then $\varphi(r)c = c\varphi(r)$. By Proposition 3, the ring B is commutative. Therefore, $(b,r)(c,0) = (c,0)(b,r)$, that means $j_0 B \subset Z(E_{(f,g)})$, and it follows from Example 2 that the braided E-system $(B, E_{(f,g)}, j_0, \tau, \{,\})$ is strong.

According to the definition of operations in $E_{(f,g)}$, the map $\varepsilon_0 : E_{(f,g)} \to D$ given by

$$\varepsilon(b,r) = db + Fr \tag{30}$$

is a ring homomorphism. We now prove that (id_B, ε) is a morphism of braided E-systems. The conditions H_1 - H_3 of Definition 2 turn into:

H_1. $\varepsilon j = d$,
H_2. $\tau_e b = \theta_{\varepsilon e} b,\ b\tau_e = b\theta_{\varepsilon e}$,
H_3. $\{e, e'\} = \eta(\varepsilon e, \varepsilon e'),\ e, e' \in E$.

The verification of these conditions consists of pure calculations, so the readers can skip it. Clearly, H_1 holds.

- Verify H_2: For $c \in B$ and $e = (b, r) \in E_{(f,g)}$, we have

$$\theta_{\varepsilon e}(c) \overset{(30)}{=} \theta_{db} c + \theta_{Fr} c \overset{E_1}{=} bc + \theta_{Fr} c,$$
$$\tau_e c = j_0^{-1}[(b, r)(c, 0)] \overset{(28)}{=} bc + \theta_{Fr} c.$$

Similarly, it follows from (29) that $b\tau_e = b\theta_{\varepsilon e}$.

- Verify H_3: The compatibility of F with the braiding constraints implies

$$g(r, s) - g(s, r) = \mathbf{c}_{Fs,Fs} = \eta(Fr, Fs). \tag{31}$$

For $e = (b, r), e' = (b', r')$, then

$$\{e, e'\} \overset{(4)}{=} (b', r')(b, r) - (b, r)(b', r') \equiv g(r', r) - g(r, r'),$$
$$\eta(\varepsilon e, \varepsilon e') = \eta(db + Fr, db' + Fr') = \eta(Fr, Fr') \overset{(31)}{=} g(r', r) - g(r, r').$$

Thus, $E_{(f,g)}$ is a center extension of the ring B by the ring R of type $\mathcal{M}$. Since $q\varepsilon(0, r) = q(F(r)) = \psi(r)$ for all $r \in R$, then the extension $\mathcal{E}_{(f,g)}$ induces $\psi : R \to$ Coker d. □

Lemma 5. *Under the hypothesis of Lemma 4, each center extension $\mathcal{E}$ of B by R of type $\mathcal{M}$ inducing ψ defines a braided Ann-functor $(F, f, g) : \operatorname{Dis} R \to \mathcal{A}_{\mathcal{M}}$ with $Fr \in \psi(r)$. Further, $\mathcal{E}$ is equivalent to $\mathcal{E}_{(f,g)}$.*

Proof. The extension $\mathcal{E}$ of type $\mathcal{M}$ gives a strong braided E-system $\mathcal{M}' = (B, E, j, \tau, \{,\})$ as in Example 2. Let $\mathcal{A}'$ be the braided strict Ann-category associated to the strong braided E-system $\mathcal{M}'$. By Lemma 1, the morphism $(id_B, \varepsilon') : (B, E) \to (B, D)$ defines a braided Ann-functor $(K, \breve{K}, \widetilde{K}) : \mathcal{A}' \to \mathcal{A}$, where $K = (id_B, \varepsilon)$. The reduced Ann-category of $\mathcal{A}'$ is just the discrete Ann-category $\operatorname{Dis} R$. For the canonical braided Ann-functor $H : \operatorname{Dis} R \to \mathcal{A}'$, the composition

$$(F, \breve{F}, \widetilde{F}) : \operatorname{Dis} R \overset{(H,\breve{H},\widetilde{H})}{\to} \mathcal{A}' \overset{(K,\breve{K},\widetilde{K})}{\to} \mathcal{A}$$

is a braided Ann-functor with

$$Fr = \varepsilon' e_r, \tag{32}$$

where $\{e_r, r \in R\}$ is a set of representatives of R in E. Set $\breve{F} = f, \widetilde{F} = g$, we construct a ring extension $\mathcal{E}_{(f,g)}$ of type $\mathcal{M}$ as in Lemma 4. We now show that $\mathcal{E}$ and $\mathcal{E}_{(f,g)}$ are equivalent. In diagram (19), since the top row is exact, then each element of E is written uniquely in the form $b + e_r, b \in B$. The map

$$\alpha : E_{(f,g)} \to E,\ (b, r) \mapsto b + e_r,$$

is a ring isomorphism. Indeed, observe that the representatives e_r have the following properties:

$$e_r.c = \theta_{F(r)}(c),\ c.e_r = c\theta_{F(r)},\ c \in B, \tag{33}$$

$$e_r + e_s = f(r,s) + e_{r+s}, \tag{34}$$

$$e_r.e_s = g(r,s) + e_{rs}. \tag{35}$$

Relation (33) is just the condition H_2 of the morphism $(id_B, \varepsilon') : (B \to E) \to (B \to D)$. It follows from $e_r + e_s - e_{r+s} \in B$ that

$$\begin{aligned} e_r + e_s - e_{r+s} &= \varepsilon(e_r + e_s - e_{r+s}) = \varepsilon e_r + \varepsilon e_s - \varepsilon e_{r+s} \\ &= Fr + Fs - F(r+s) = f(r,s). \end{aligned}$$

Analogously, we obtain (35). By relations (33)-(35) one can verify that α is a ring homomorphism. Moreover,

$$\varepsilon'\alpha(b,r) = \varepsilon'(b + e_r) = db + \varepsilon'(e_r) \overset{(32)}{=} d(b) + F(r) \overset{(30)}{=} \varepsilon(b,r),$$

that means $\mathcal{E}$ and $\mathcal{E}_F$ are two equivalent ring extensions of type $\mathcal{M}$. □

Theorem 6 (Schreier theory for center extensions of the type of a strong braided E-system)**.** *Under the hypothesis of Lemma 4, there exists a bijection*

$$\Omega : \mathrm{Hom}^{BrAnn}_{(\psi,0)}[\mathrm{Dis}\, R, \mathcal{A}] \to \mathrm{Ext}_{\mathcal{M}}(R, B, \psi).$$

Proof. It is easy to prove that two braided Ann-functors $(F, f, g), (F', f', g') : \mathrm{Dis}\, R \to \mathcal{A}_{\mathcal{M}}$ are homotopic via $\alpha : F \to F'$ if and only if the corresponding extensions $\mathcal{E}_{(f,g)}, \mathcal{E}_{(f',g')}$ are equivalent via the isomorphism $\alpha^* : (b, r) \mapsto (b - \alpha_r, r)$. This together with Lemmas 4, 5 completes the proof. □

Theorem 7. *Under the hypothesis of Lemma 4, the vanishing of $\overline{\psi^* k}$ in $H^3_{ab}(R, \mathrm{Ker}\, d)$ is necessary and sufficient for there to exist a center extension of a ring B by a ring R of type $\mathcal{M}$ inducing ψ. Further, if $\overline{\psi^* k}$ vanishes, then there is a bijection*

$$\mathrm{Ext}_{\mathcal{M}}(R, B, \psi) \leftrightarrow H^2_{ab}(R, \mathrm{Ker}\, d).$$

Proof. Recall that $\mathcal{A}_{\mathcal{M}}$ is the braided Ann-category associated to the strong braided E-system $\mathcal{M} = (B \xrightarrow{d} D)$. Then, its reduced braided Ann-category is $S_{\mathcal{A}} = \int(\mathrm{Coker}\, d, \mathrm{Ker}\, d, k)$, where $k \in Z^3_{Shab}(\mathrm{Coker}\, d, \mathrm{Ker}\, d)$. According to (2), an obstruction of the pair

$$(\psi, 0) : \mathrm{Dis}\, R \to \int (\mathrm{Coker}\, d, \mathrm{Ker}\, d, k)$$

is $-\psi^* k$. By Theorem 1, the pair $(\psi, 0)$ realizes a braided Ann-functor if and only if $\overline{\psi^* k} = 0$ in $H^3_{ab}(R, \mathrm{Ker}\, d)$. Thus, the first assertion of the theorem follows from Lemmas 4 and 5. The second one follows from Theorems 1 and 6. □

6. Ring extensions of commutative rings

We define the *obstruction* of a regular homomorphism $\psi : R \to P_B$ in the case B, R are commutative rings. For each $r \in R$, choose a bimultiplication $\varphi(r) \in \psi(r)$ with $\varphi(1) = id_B$. The bimultiplication $\varphi(r)$ induces functions $f, g : R^2 \to B$ satisifying the relations

$$\varphi(r) + \varphi(s) = \tau_{f(r,s)} + \varphi(r + s), \tag{36}$$

$$\varphi(r) \circ \varphi(s) = \tau_{g(r,s)} + \varphi(rs). \tag{37}$$

The functions f, g are normalized in the sense that $f(r, 0) = f(0, s) = 0, g(r, 1) = g(1, s) = 0$. The ring structure of M_B induces a family $h = (\xi, \eta, \alpha, \beta, \lambda, \rho)$ by

$$\begin{aligned}
\xi(r,s,t) &= f(r,s+t) + f(s,t) - f(r,s) - f(r+s,t),\\
\eta(r,s) &= f(r,s) - f(s,r),\\
\alpha(r,s,t) &= \varphi(r)g(s,t) - g(rs,t) + g(r,st) - g(r,s)\varphi(t),\\
\beta(r,s) &= g(r,s) - g(s,r),\\
\lambda(r,s,t) &= g(r,s+t) - g(r,s) - g(r,t) + \varphi(r)f(s,t) - f(rs,rt),\\
\rho(r,s,t) &= g(r+s,t) - g(r,t) - g(s,t) + f(r,s)\varphi(t) - f(rt,st).
\end{aligned}$$

According to [10], the family of functions $k = (\xi, \eta, \alpha, \lambda, \rho)$ is a 3-cocycle in the group $Z^3_{MacL}(R, C_B)$. It is easy to verify that $h = (k, \beta)$ is an element in $Z^3_{ab}(R, C_B)$. The cohomology class $\overline{h}$ is called the *obstruction* of the regular homomorphism $\psi : R \to P_A$, denoted by $\mathrm{Obs}(\psi)$. Note that h is of form $\delta(f, g)$, but it is not a 3-coboundary since the functions f, g do not take values in C_B.

Proposition 6. *Let B, R be commutative rings and $\psi : R \to P_B$ a regular homomorphism. Then, the obstruction* $\mathrm{Obs}\,\psi$ *in the sense of MacLane [10] is coincident with the obstruction of the pair $(\mathcal{M}, \psi)$ in the sense of* (20).

Proof. Suppose that the reduced Ann-category of $\mathcal{M}_B$ is $\mathcal{S} = (P_B, C_B, h)$. Let $H = (H, \breve{H}, \widetilde{H}) : \mathcal{S} \to \mathcal{M}_B$ be the canonical Ann-equivalence. We choose $\varphi = H \circ \psi : R \to \mathcal{M}_B$. Then, the pair of functions

$$f = \psi^* \breve{H}, \quad g = \psi^* \widetilde{H},$$

satisfies (36) and (37). By the above determination, $\mathrm{Obs}(\psi) = \overline{\delta(f, g)}$. Besides, the compatibility of H with the constraints implies $\psi^* h = \delta(f, g)$, so that $\overline{\psi^* h} = \mathrm{Obs}(\psi)$. □

Proposition 7. *Each commutative ring extension of B by R is regarded as a ring extension of type $(B, L_B, \tau, j, 0)$, where L_B is a certain commutative subring of M_B.*

Proof. Consider a commutative extension of commutative rings

$$\mathcal{E} : 0 \to B \xrightarrow{i} E \xrightarrow{p} R \to 0.$$

For any $e \in E$, the inner bimultiplication τ_e of E induces a bimultiplication of B. This gives a homomorphism $\tau' : E \to M_B$. Since $q \circ \tau' = 0$, then the universal property of coker implies that there is a ring homomorphism $\psi : R \to P_B$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & R & \longrightarrow & 0 \\ & & \| & & \downarrow{\tau'} & & \downarrow{\psi} & & \\ & & B & \xrightarrow{\tau} & M_B & \xrightarrow{q} & P_B & \longrightarrow & 0. \end{array}$$

Since ψ is a regular homomorphism, then two elements of ring $K = q^{-1}(\mathrm{Im}\psi)$ are permutable. A bimultiplication σ on B is *strong* if $\sigma b = b\sigma$ for all $b \in B$. Denote by L_B the subset of K consisting of strong bimultiplications. Then, L_B is a ring. Indeed, for $\sigma, \sigma' \in L_B, b \in B$, we have

$$\sigma\sigma'(b) = \sigma(\sigma' b) = \sigma(b\sigma') \stackrel{(3)}{=} (\sigma b)\sigma' = (b\sigma)\sigma' = (b)\sigma\sigma'.$$

Moreover, it is commutative

$$\sigma\sigma'(b) = \sigma(\sigma' b) = (\sigma' b)\sigma \stackrel{(3)}{=} \sigma'(b\sigma) = \sigma'(\sigma b) = \sigma'\sigma(b).$$

Since E is commutative , then $\tau' E \subset L_B$. Thus, the above diagram induces the following commutative one

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{j} & E & \xrightarrow{p} & R & \longrightarrow & 0 \\ & & \| & & \downarrow{\tau'} & & \downarrow{\psi} & & \\ & & B & \xrightarrow{\tau} & L_B & \xrightarrow{q} & \mathrm{Coker}\,\tau & \longrightarrow & 0. \end{array}$$

The homomorphism τ defines a strong braided E-system $(B, L_B, \tau, \iota, 0)$, where $\iota : L_B \to M_B$ is the canonical embedding and $\eta = 0$. The embedding $j : B \to E$ defines a strong braided E-system $(B, E, j, \tau', 0)$, where τ' is given by inner bimultiplications and the braiding is 0. Then, $(id_B, \tau') : (B \xrightarrow{i} E) \to (B \xrightarrow{\tau} L_B)$ is a morphism of strong braided E-systems. Thus, each commutative extension $\mathcal{E}$ inducing $\psi : R \to P_B$ is viewed as a ring extension of type $(B, L_M, \tau, \iota, 0)$. □

Denote by $\mathrm{Ext}^{ab}(R, B, \psi)$ the set of equivalence classes of commutative extensions of the ring B by the ring R inducing $\psi : R \to P_B$.

Theorem 8. *Let B, R be commutative rings and $\psi : R \to P_B$ a regular homomorphism. Then the vanishing of* $\mathrm{Obs}(\psi)$ *in $H^3_{ab}(R, C_B)$ is necessary and sufficient for there to exist a commutative extension of the ring B by the ring R. Further, if* $\mathrm{Obs}(\psi)$ *vanishes, there is a bijection*

$$Ext^{ab}(R, B, \psi) \leftrightarrow H^2_{ab}(R, C_B).$$

Proof. It follows from Propositions 6, 7 and Theorem 7. □

This result is a version of Theorems 5 and 7 [10] for commutative rings.

Acknowledgement

The author is much indebted to the referee, whose useful observations greatly improved our exposition.

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