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Existence and multiplicity of solutions for a class of fractional Kirchhoff-type problem*

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Abstract. In this paper, we establish the existence and multiplicity of solutions to the following fractional Kirchhoff-type problem

$$M(\|u\|^2)(-\Delta)^s u = f(x, u(x)), \text{ in } \Omega \;\; u = 0 \text{ in } \mathbb{R}^N \backslash \Omega,$$

where $N > 2s$ with $s \in (0, 1)$, Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, M and f are two continuous functions, and $(-\Delta)^s$ is a fractional Laplace operator. Our main tools are based on critical point theorems and the truncation technique.

AMS subject classifications: 34C25, 58E30

Key words: fractional Kirchhoff-type problem, integro-differential operator, truncation technique

1. Introduction

In this paper, we are concerned with the existence and multiplicity of solutions for a class of fractional Kirchhoff-type problem

$$\begin{cases} M\left(\displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy\right)(-\Delta)^s u = f(x, u(x)), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1}$$

where $N > 2s$ with $s \in (0, 1)$, Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, M and f are two continuous functions whose properties will be stated later, and the fractional Laplace operator $-(-\Delta)^s$ which, up to normalization factors, may be defined as

$$-(-\Delta)^s u(x) := \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \quad x \in \mathbb{R}^N. \tag{2}$$

As we know, the multidimensional Kirchhoff equation is

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$$\frac{\partial^2 u}{\partial t^2} - (1 + \int_\Omega |\nabla u|^2 dx)\Delta u = 0, \tag{3}$$

where $\Omega \subset \mathbb{R}^N$ and $u : \Omega \to \mathbb{R}$ satisfies some initial or boundary conditions. It arises from the following nonlinear generalization of the well known d'Alembert equation

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{p_0}{h} + \frac{E}{2L}\int_0^L (\frac{\partial u}{\partial x})^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0. \tag{4}$$

This model (4) was proposed by Kirchhoff [12], so the equation of this class is called a Kirchhoff-type equation. Equation (4) describes a vibrating string, taking into account the changes in the length of the string during vibration. Here L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and p_0 is the initial tension. In [13], the hyperbolic problem was proposed by

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f, & \text{in } \Omega \times (0,T), \\ u = 0, & \text{in } \partial\Omega \times (0,T), \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \tag{5}$$

where $M : [0, +\infty) \to \mathbb{R}$ is a continuous function such that $M(s) \geq c > 0$ for any $s \geq 0$, and Ω is a bounded set of $\mathbb{R}^N$ with smooth boundary. This hyperbolic problem has an elliptic version when we look for stationary solutions. In [28], a class of problems was considered among which the following elliptic Kirchhoff-type equation was included

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \tag{6}$$

where Ω is an open subset of $\mathbb{R}^N$.

According to the original formulation of the equation given by Kirchhoff, if there exist two positive constants a and b such that $M : \mathbb{R}^+ \to \mathbb{R}$ can be written in the form $M(s) = a + bs$, then we say that M is a Kirchhoff function.

Our motivation is that we replace the energy $\widetilde{M}(\int_\Omega |\nabla u|^2 dx)$ by

$$\widetilde{M}\left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy\right),$$

where $\widetilde{M}$ is a primitive of M. Then the classical elliptic Kirchhoff-type problem becomes a nonlocal Kirchhoff type problem

$$\frac{\partial^2 u}{\partial t^2} + M\left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy\right)(-\Delta)^s u = 0.$$

The form of its static state may be written as the following form

$$M\left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}} dxdy\right)(-\Delta)^s u = 0 \tag{7}$$

Accordingly, this nonlocal model for the vibrating string may be obtained from (5), by considering the above tension $M(\cdot)$ and replacing the local spatial second derivative with the nonlocal operator $-(-\Delta)^s u$. In this way we obtain a higher dimensional nonlocal Kirchhoff equation (7). It is not artificial to obtain this Kirchhoff equation (7), the detailed deduction of this nonlocal model can be referred to Fiscella and Valdinoci [10]. On the other hand, as the lecture notes of Silvestre [25] say, good understanding of nonlocal equations can ultimately provide better understanding of the limit PDE case.

In a recent paper, in [10], Fiscella and Valdinoci studied the following Kirchhoff-type problem involving an integrodifferential operator

$$\begin{cases} -M\left(\displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N} |u(x)-u(y)|^2 K(x-y)dxdy\right)\mathcal{L}_K u = f(x,u(x)), & \text{in } \Omega, \\ u=0, & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases} \tag{8}$$

where $\mathcal{L}_K$ is an integrodifferential operator with a singular symmetric kernel K defined by

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^N} \left(u(x+y)+u(x-y)-2u(x)\right)K(y)dy, \quad x\in\mathbb{R}^N, \tag{9}$$

where $K:\mathbb{R}^n\setminus\{0\}\to(0,+\infty)$ is a singular symmetric kernel function satisfying the property that

(K) there exist $\theta>0$ and $s\in(0,1)$ such that

$$\theta|x|^{-(N+2s)} \le K(x) \le \theta^{-1}|x|^{-(N+2s)} \text{ for any } x\in\mathbb{R}^N\setminus\{0\}.$$

Clearly, a typical model for K is given by the singular kernel $K(x)=|x|^{-(N+2s)}$ which gives rise to the fractional Laplace operator $-(-\Delta)^s$. As a result, problem (8) reduces to our problem (1).

In recent years, nonlinear equations involving fractional powers of the Laplace operator have played an increasingly important role in physics, probability and finance, see for instance [14, 15, 29, 30], and so on. Meanwhile, increasing research of elliptic equations involving fractional powers of the Laplace operator has been interesting to many people, such as [2, 3, 7, 6, 10, 20, 21, 24, 25, 5, 26, 27] and references therein. Among them, papers [2, 5, 26] studied the different fractional operator which is another type of a nonlocal operator, for details see [23]. Our main interest is to investigate the Kirchhoff-type problem involving fractional powers of the Laplace operator, to the best of our knowledge, using variational techniques, there are intensive results in the document which deal with the elliptic Kirchhoff equation (6) (see [1, 8, 9, 18, 28]) and its generalization forms such as p-Laplacian type, $p(x)$-Laplacian type and so on, but there are only few papers that study this new Kirchhoff-type problem involving fractional powers of the Laplace operator called "fractional Kirchhoff equation".

Inspired by the above articles, in this paper, we would like to investigate the existence and multiplicity of solutions for problem (1) by using the mountain pass theorem and the symmetric mountain theorem together with truncation techniques.

The paper is organized as follows. In Section 2, we give some preliminary facts and provide some basic properties which are needed later and present our main results. Sections 3 and 4 are devoted to the proof of our results.

2. Preliminaries and main results

In this section, we shall give some preliminaries and then present our main results.

For studying problem (1) in a variational framework, in the sequel we denote by $H^s(\mathbb{R}^N)$ the usual fractional Sobolev space endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \|u\|_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}} dxdy,$$

while $X_0^s(\Omega)$ is the function space defined as

$$X_0^s(\Omega) = \big\{u \in H^s(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\big\}.$$

The general definition of $X_0^s(\Omega)$ and its properties can be seen in [20, 21]. We define the inner product $\langle \cdot, \cdot \rangle_{X_0^s(\Omega)}$ on $X_0^s(\Omega)$ as follows

$$\langle u, v\rangle_{X_0^s(\Omega)} = \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)}{|x-y|^{N+2s}} dxdy, \quad \forall u, v \in X_0^s(\Omega).$$

Then the space $X_0^s(\Omega)$ is a Hilbert space endowed with the corresponding norm

$$\|u\| = \left(\int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dxdy\right)^{\frac{1}{2}}. \tag{10}$$

Moreover, $X_0^s(\Omega)$ is compactly embedded into $L^p(\mathbb{R}^N)$ for any $p \in [1, 2_s^*)$, where $2_s^* = \frac{2N}{N-2s}$ (see [20]).

Next, we make the assumptions of M and the nonlinearities term $f(x,u)$ as follows:

(M_0) $M : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing and continuous function;

(M_1) There exists $m_0 > 0$ such that $M(t) \geq m_0 = M(0)$ for any $t \in \mathbb{R}^+$;

(f_0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying

$$|f(x,t)| \leq c(1+|t|^{p-1}), \text{ for all } x \in \Omega \text{ and } t \in \mathbb{R},$$

where $c > 0$, $2 < p < 2_s^*$;

(f_1) $\lim\limits_{t\to 0} \frac{f(x,t)}{t} = 0$, uniformly in $x \in \Omega$;

(f_2) There exist $\mu > 2$ and $R > 0$ such that

$$0 < \mu F(x,t) \leq t f(x,t), \text{ for all } x \in \Omega \text{ and all } t \geq R,$$

where $F(x,t) = \int_0^t f(x,s)ds$.

It follows from [17] that we have the following identity

$$\|u\| = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^N)}.$$

This leads us to establish as a definition that the solutions to our problem are in a variational framework.

Definition 1. *We say $u \in X_0^s(\Omega)$ is weak solution of (1) if for every $v \in X_0^s(\Omega)$ one has*

$$M(\|u\|^2)\int_{\mathbb{R}^N\times\mathbb{R}^N}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}dxdy = \int_\Omega f(x,u(x))v(x)dx \tag{11}$$

In the sequel we will omit the term weak when referring to solutions that satisfy the conditions of Definition 1.

In order to prove our main results, we need to study an auxiliary truncated problem, and such truncation technique was introduced in [10] to deal with the fractional Kirchhoff equation. Given $a \in \mathbb{R}^+$, assume that there exists $t_0 > 0$ such that $M(t_0) = a$. Now, let

$$M_a(t) = \begin{cases} M(t), & \text{if } 0 \le t \le t_0, \\ a, & \text{if } t \ge t_0. \end{cases}$$

We introduce an auxiliary problem

$$\begin{cases} -M_a(\|u\|^2)(-\Delta)^s u = f(x,u(x)), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{12}$$

with f defined as in problem (1).

Our fundamental idea is that we first prove that problem (12) has a solution u in $X_0^s(\Omega)$ and if we can verify that u satisfies $\|u\| \le t_0$, then $M_a(\|u\|^2) = M(\|u\|^2)$, as a result, u is a solution of problem (1).

The Euler functional $\Phi_a : X_0^s(\Omega) \to \mathbb{R}$ corresponding to problem (12) is defined by

$$\Phi_a(u) = \frac{1}{2}\widetilde{M_a}(\|u\|^2) - \int_\Omega F(x,u(x))dx, \quad u \in X_0^s(\Omega),$$

where $\widetilde{M_a}(t) = \int_0^t M_a(s)ds$, under the assumptions $(M_0)-(M_1)$ and $(f_0)-(f_1)$, the functional Φ_a is well defined. By a standard argument, $\Phi_a \in C^1(X_0^s(\Omega),\mathbb{R})$ and

$$\begin{aligned}\langle \Phi_a'(u), v\rangle &= M_a(\|u\|^2)\int_{\mathbb{R}^{2N}}(u(x)-u(y))(v(x)-v(y))K(x-y)dxdy \\ &\quad - \int_\Omega f(x,u(x))v(x)dx\end{aligned}$$

for all $u, v \in X_0^s(\Omega)$. Moreover, critical points of functional Φ_a are weak solutions of problem (12).

The following maximum principle and a priori estimate are crucial to prove our main results.

Proposition 1 (See [24]). *Let $\Omega \subset\subset \mathbb{R}^N$ be an open set, let u be a lower semicontinuous function $\bar{\Omega}$ such that $(-\Delta)^s u \ge 0$ in Ω and $u \ge 0$ in $\mathbb{R}^N \backslash \Omega$. Then $u \ge 0$ in $\mathbb{R}^N$. Moreover, if $u(x) \equiv 0$ for one point x inside Ω, then $u \equiv 0$ in the whole $\mathbb{R}^N$.*

Theorem 1 (See [3]). *Let u be a positive solution to the problem*

$$\begin{cases} (-\Delta)^s u = f(x,u(x)), & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{13}$$

and assume that $|f(x,t)| \leq C(1+|t|^p)$ *for some* $1 \leq p \leq 2_s^* - 1$ *and* $C > 0$. *Then* $u \in L^\infty(\mathbb{R}^N)$.

Observe that if u is a non-negative solution of problem (1), the weak solution is bounded by Theorem 1 and therefore it is a continuous viscosity solution according to Theorem 1 in [22]. By virtue of Proposition 1, we obtain that u is strictly positive in Ω, and therefore u is a positive solution of (1).

Lemma 1. *Assume that* u *is a positive solution of problem (12), and* $|f(x,t)| \leq C_0|t|^{q-1} + C|t|^{p-1}$ *for all* $x \in \Omega$ *and* $t \in \mathbb{R}$, *where* $1 < q \leq p$, $2 < p \leq 2^*$ *and* $C_0 \geq 0$, $C > 0$. *Then there exists* $\theta > 0$, *independent of* M_a, *such that*

$$\|u\|^2 \leq \max\Big\{M_a(\|u\|^2)^{\frac{2-p+q}{p-1}}, M_a(\|u\|^2)^{\frac{2}{p-1}}\Big\}\theta. \tag{14}$$

Proof. Let u be a positive solution for problem (12), then

$$w = \frac{u}{M_a(\|u\|^2)^{\frac{1}{p-2}}}$$

is a positive solution of

$$\begin{cases} (-\Delta)^s w = g(x, w(x)), & \text{in } \Omega, \\ w = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where

$$g(x,w) = \frac{f(x, M_a(\|u\|^2)^{\frac{1}{p-2}} w)}{M_a(\|u\|^2)^{\frac{p-1}{p-2}}}.$$

It is easy to check that

$$|g(x,s)| \leq C(1+|s|^{p-1}) \quad \text{for all} \quad x \in \Omega \ \text{ and } \ s \in \mathbb{R},$$

where C is dependent on m_0. By Theorem 1, there exists $C_* > 0$ such that $\|w\|_\infty \leq C_*$. Therefore,

$$\|u\|_\infty \leq M_a(\|u\|^2)^{\frac{1}{p-2}} C_*$$

and consequently,

$$\begin{aligned} \|u\|^2 &= M_a(\|u\|^2)^{-1} \int_\Omega f(x,u(x))u(x)dx \\ &\leq CM_a(\|u\|^2)^{-1} \int_\Omega \Big(|u(x)|^q + |u(x)|^p\Big)dx \\ &\leq CM_a(\|u\|^2)^{-1}\Big(\|u\|_\infty^q + \|u\|_\infty^p\Big)|\Omega| \\ &\leq \max\Big\{M_a(\|u\|^2)^{\frac{q-p+2}{p-2}}, M_a(\|u\|^2)^{\frac{2}{p-2}}\Big\}(C_0C_*^q + CC_*^p)|\Omega|. \end{aligned}$$

Therefore, we take $\theta = (C_0C_*^q + CC_*^p)|\Omega|$ and the conclusion follows. $\square$

Finally, we state our main results in the following Theorems.

Theorem 2. *Assume that f satisfies the assumptions $(f_0)-(f_2)$. Suppose that M satisfies $(M_0)-(M_1)$ and there exists $t_0>0$ such that*

$$M(t_0)<\frac{\mu m_0}{2} \quad \textit{and} \quad \max\left\{M(t_0)^{\frac{2-p+q}{p-2}}, M(t_0)^{\frac{2}{p-2}}\right\}\leq \frac{t_0}{\theta}.$$

Then problem (1) has a positive solution.

We can obtain the existence of infinitely many solutions for problem (1) under the following assumption via the symmetric mountain pass Theorem 5 stated later.

(f_3) $\lim\limits_{t\to 0}\frac{f(x,t)}{t}=+\infty$ uniformly in $x\in\Omega$;

(f_4) $f(x,-t)=-f(x,t)$.

The second result is stated as follows:

Theorem 3. *Assume that f satisfies the assumptions (f_0), $(f_3)-(f_4)$, and M satisfies $(M_0)-(M_1)$. Then problem (1) has a sequence of nontrivial weak solutions $\{u_k\}$ for $k\in\mathbb{N}$ large.*

Remark 1. *When $M\equiv 1$, the existence of infinitely many solutions via the symmetric mountain pass Theorem was proved in [4].*

For proving our main results, the following mountain pass Theorem and symmetric mountain pass Theorem are our main tools, which can be found in [19, 11], respectively. We present them as follows.

Theorem 4 (See [19])**.** *Let X be a real Banach space. Suppose $\Phi\in C^1(X,\mathbb{R})$ satisfies the (PS) condition. Assume that*

(1) $\Phi(0)=0$;

(2) *there exists $\rho>0$ and $\alpha>0$ such that $\Phi(u)\geq\alpha$ for all $\|u\|=\rho$;*

(3) *there exists $e\in X$ such that $\Phi(e)<\alpha$ for $\|e\|>\rho$.*

Then

$$c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\Phi(\gamma(t))\geq\alpha$$

is a critical value, where

$$\Gamma=\{\gamma\in C([0,1],X):\quad \gamma(0)=0,\quad \gamma(1)=e\}.$$

Theorem 5 (See [11])**.** *Let X be an infinite dimensional Banach space. Suppose $\Phi\in C^1(X,\mathbb{R})$ satisfies the following condition:*

(1) *Φ is even, bounded from below, $\Phi(0)=0$ and Φ satisfies the (PS) condition;*

(2) *For each $k\in\mathbb{N}$, there exists $A_k\in\Gamma_k$ such that $\sup_{u\in A_k}\Phi(u)<0$, where*

$$\Gamma_k=\{A: \text{ A closed symmetric subset of } X \text{ and } 0\notin A,\ \gamma(A)\geq k\}$$

and $\gamma(A)$ is a genus of a closed symmetric set A.

Then Φ admits a sequence of critical points $\{u_k\}$ such that $\Phi(u_k)\leq 0$, $u_k\neq 0$ and $\|u_k\|\to 0$ as $k\to+\infty$.

3. Proof of Theorem 2

Since we intend to find a positive solution for problem (1), in this paper, let us assume that

$$f(x,t)=0, \quad \text{for all} \quad x\in\Omega \ \text{ and } \ t\le 0.$$

By the assumption of M, we obviously have

$$a<\frac{\mu m_0}{2}, \quad \text{and} \quad \theta\max\left\{a^{\frac{2-p+q}{p-2}}, a^{\frac{2}{p-2}}\right\}\le t_0. \tag{15}$$

First we show that the functional Φ_a has a structure of mountain pass geometry.

Lemma 2. *Suppose that* $(M_0)-(M_1)$ *and* $(f_0)-(f_2)$ *hold. Then*

(*i*) *there exist two constants* $\rho,\alpha>0$ *independent of* a *such that*

$$\Phi_a(u)\ge\alpha>0,$$

for any $u\in X_0^s(\Omega)$ *with* $\|u\|=\rho$;

(*ii*) *there exists* $e\in X_0^s(\Omega)$ *such that* $\Phi_a(e)<0$ *for* $\|e\|>\rho$.

Proof. (*i*): By hypothesis (f_0) and (f_1), for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|F(x,t)|\le\frac{\varepsilon}{2}|t|^2+C_\varepsilon|t|^p, \quad \text{for all} \quad x\in\Omega \ \text{ and } \ t\in\mathbb{R}. \tag{16}$$

From (16) and the Sobolev equality, we have

$$\begin{aligned}\Phi_a(u)&=\frac{1}{2}\widetilde{M}_a(\|u\|^2)-\int_\Omega F(x,u(x))dx\\&\ge\frac{m_0}{2}\|u\|^2-\frac{\varepsilon}{2}\|u\|_2^2-C_\varepsilon\|u\|_p^p\\&\ge(\frac{m_0}{2}-\frac{\varepsilon}{2}C_1)\|u\|^2-C_2C_\varepsilon\|u\|^p,\end{aligned}$$

where $C_1,C_2>0$ are two constants. Choosing $\varepsilon<\frac{m_0}{2C_1}$, we can find two constants $\alpha,\rho>0$ so that part (*i*) holds.

(*ii*): By hypotheses (f_0) and (f_2), by a standard argument, there exist two constants $C_3,C_4>0$ such that

$$F(x,t)\ge C_3|t|^\mu-C_4, \text{ for all } x\in\Omega \text{ and } t\in\mathbb{R}. \tag{17}$$

Taking $u_0\in X_0^s(\Omega)$ such that $\|u_0\|=1$ and $u_0(x)\ge0$ a.e. in $\mathbb{R}^N$. By (17), we have

$$\begin{aligned}\Phi_a(tu_0)&=\frac{1}{2}\widetilde{M}_a(t^2)-\int_\Omega F(x,tu_0(x))dx\\&\le\frac{a}{2}t^2-C_3t^\mu\int_\Omega|u_0(x)|^\mu dx+C_4|\Omega|.\end{aligned}$$

Obviously, $\Phi_a(tu_0)\to-\infty$ as $t\to+\infty$. Therefore, we can choose t_0 large enough such that $e=t_0u_0$ with $\|e\|\ge\rho$ so that part (*ii*) holds. $\square$

Lemma 3. *The functional Φ_a satisfies the (PS) condition, that is, if a sequence $\{u_n\} \subset X_0^s(\Omega)$ satisfies $\Phi_a(u_n)$ bounded and $\Phi_a'(u_n) \to 0$ as $n \to \infty$, then $\{u_n\}$ has a convergence subsequence.*

Proof. Let $\{u_n\}$ be a sequence such that

$$\Phi_a(u_n) \text{ bounded and } \Phi_a'(u_n) \to 0 \tag{18}$$

as $n \to \infty$. By hypothesis (f_2) and (18), we have

$$\begin{aligned}
\mu\Phi_a(u_n) - \langle \Phi_a'(u_n), u_n\rangle &= \frac{\mu}{2}\widetilde{M}_a(\|u_n\|^2) - M_a(\|u_n\|^2)\|u_n\|^2 \\
&\quad + \int_\Omega \Big[f(x,u_n(x))u_n(x) - \mu F(x,u_n(x))\Big]dx \\
&\geq (\frac{\mu}{2}m_0 - a)\|u_n\|^2 + \int_{\{x\in\Omega:|u_n(x)|\geq R\}} \Big[f(x,u_n(x))u_n(x) \\
&\quad -\mu F(x,u_n(x))\Big]dx \\
&\quad + \int_{\{x\in\Omega:|u_n(x)|\leq R\}} \Big[f(x,u_n(x))u_n(x) - \mu F(x,u_n(x))\Big]dx \\
&\geq (\frac{\mu}{2}m_0 - a)\|u_n\|^2 + \int_{\{x\in\Omega:|u_n(x)|\leq R\}} \Big[f(x,u_n(x))u_n(x) \\
&\quad -\mu F(x,u_n(x))\Big]dx \\
&\geq (\frac{\mu}{2}m_0 - a)\|u_n\|^2 - C_5,
\end{aligned}$$

which implies that $\{u_n\}$ is bounded in $X_0^s(\Omega)$ (using (18)). Since $X_0^s(\Omega)$ is reflexive, by Sobolev embedding, up to a subsequence, there exists $u \in X_0^s(\Omega)$ such that $u_n \rightharpoonup u$ in $X_0^s(\Omega)$, $u_n \to u$ in $L^q(\mathbb{R}^N)$ for $q \in [1, 2_s^*)$ and $u_n \to u$ a.e. in $\mathbb{R}^N$. By hypothesis (f_0) and Hölder's inequality, we get

$$\int_\Omega f(x,u_n(x))(u_n(x)-u(x))dx \leq c\Big(\|u_n - u\|_1 + \|u_n - u\|_p^p\Big) \to 0$$

and combing with (18), we deduce that

$$\begin{aligned}
m_0|\langle u_n, u_n - u\rangle_{X_0^s(\Omega)}| \leq M_a(\|u_n\|^2)|\langle u_n, u_n - u\rangle_{X_0^s(\Omega)}| &= \Big|\langle \Phi_a'(u_n), u_n - u\rangle \\
&\quad + \int_\Omega f(x,u_n(x))(u_n(x)-u(x))dx\Big|.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \|u_n\| = \|u\|$ and so

$$\lim_{n\to\infty}\|u_n - u\|^2 = \lim_{n\to\infty}\Big(\|u_n\|^2 + \|u\|^2 - 2\langle u_n, u\rangle_{X_0^s(\Omega)}\Big) = 0.$$

□

Proof of Theorem 2. From Lemma 2 and Lemma 3, applying the mountain pass Theorem 4, we obtain that problem (12) has a non-negative solution $u \in X_0^s(\Omega)$ such that

$$c = \Phi_a(u) = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi_a(\gamma(t)) \geq \alpha > 0.$$

By Proposition 1, we see that $u > 0$ in Ω, that is, this solution is positive. Now, we prove that $\|u\|^2 \leq t_0$. If not, then $\|u\|^2 > t_0$ and this implies that $M_a(\|u\|^2) = a$. By Lemma 1, we see that

$$\|u\|^2 \leq \max\left\{M_a(\|u\|^2)^{\frac{2-p+q}{p-2}}, M_a(\|u\|^2)^{\frac{2}{p-2}}\right\}\theta.$$

This implies that

$$t_0 < \max\left\{a^{\frac{2-p+q}{p-2}}, a^{\frac{2}{p-2}}\right\}\theta$$

which contradicts with (15). Therefore, $\|u\|^2 \leq t_0$, in this case, we have $M_a(t) = M(t)$, this implies that $u \in X_0^s(\Omega)$ is a positive solution of problem (1). This ends the proof of Theorem 2. □

4. Proof of Theorem 3

For proving Theorem 3, we observe that for given $m_0 < a < \frac{\mu m_0}{2}$, by the hypothesis $(M_0) - (M_1)$, there exists $t_0 > 0$ such that $M(t_0) = a$. Now the truncation function $M_a(t)$ may be well defined.

Let $h \in C^\infty([0,\infty), \mathbb{R})$ such that $0 \leq h(t) \leq 1$ for $t \in [0, +\infty)$ and for above defined $t_0 > 0$, $h(t) = 1$ for $0 \leq t \leq \frac{t_0}{2}$ and $h(t) = 0$ for $t \geq t_0$. Let $\varphi(u) = h(\|u\|)$. We consider the truncation functional

$$I_a(u) = \frac{1}{2}\widetilde{M}_a(\|u\|^2) - \varphi(u)\int_\Omega F(x, u(x))dx.$$

Clearly, $I_a \in C^1(X_0^s(\Omega), \mathbb{R})$.

For $\|u\| \geq t_0$, we have $I_a(u) = \frac{a}{2}\|u\|^2$, which implies that $I_a(u) \to +\infty$ as $\|u\| \to +\infty$. Hence I_a is coercive on $X_0^s(\Omega)$. Thus I_a is bounded from below and satisfies the (PS) condition.

From the hypothesis (f_4), we see that $I_a(u)$ is even and $I_a(0) = 0$. By (f_3), for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(x,t) \geq \frac{1}{2}\varepsilon^{-1}t^2, \text{ for } |t| \leq \delta. \tag{19}$$

Given any $k \in \mathbb{N}$, let $E_k \subset X_0^s(\Omega)$ be a finite dimensional subspace with dimension k, (the existence of such finite subspace can be referred to [21]). Then there exists a constant $\alpha_k > 0$ such that $|u| \leq \alpha_k\|u\|$ for $u \in E_k$. Therefore, for any $u \in E_k$ with $\|u\| = \rho \leq \min\left\{\frac{\delta}{\alpha_k}, \frac{t_0}{2}\right\}$ and ε small enough, we have

$$I_a(u) \leq \frac{a}{2}\rho^2 - \frac{1}{2}\varepsilon^{-1}C_9\rho^2 = (\frac{a}{2} - \frac{1}{2}\varepsilon^{-1}C_9)\rho^2 < 0$$

where $C_9 > 0$ is a constant such that $\|u\|_2 \geq C_9\|u\|$ according to $u \in E_k$. Therefore, we obtain

$$\Big\{u \in E_k : \|u\| = \rho\Big\} \subset \Big\{u \in E_k : I_a(u) < 0\Big\}.$$

Since $\gamma(\Big\{u \in E_k : \|u\| = \rho\Big\}) = k$, hence by the monotonicity of genus $\gamma(A)$, we get

$$\gamma\Big(\{u \in E_k : I_a(u) < 0\}\Big) \geq k.$$

Choosing $A_k = \Big\{u \in E_k : I_a(u) < 0\Big\}$, then $A_k \in \Gamma_k$ and $\sup\limits_{u \in A_k} I_a(u) < 0$. Consequently, all the conditions of Theorem 5 are verified, and as a result, there exists a sequence $\{u_k\}$ such that

$$I_a(u_k) \leq 0, I'_a(u_k) = 0 \text{ and } \|u_k\| \to 0, \text{ as } k \to \infty.$$

Hence, we can take k large enough such that $\|u_k\| \leq \frac{t_0}{2}$, and such $\{u_k\}$ is a solution of problem (1).

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