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Erratum to "Total domination number of Cartesian products" [Math. Commun. 9(2004), 35–44]*

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Abstract. We correct a partial mistake for the total domination number of $\gamma_t(P_6 \square P_k)$ presented in the article "Total domination number of Cartesian products" [Math. Commun. **9**(2004), 35–44].

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1. Introduction and preliminaries

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We use the standard notations $N_G(v)$ for the *open neighborhood* $\{u : uv \in E(G)\}$ and $N_G[v]$ for the *closed neighborhood* $N_G(v) \cup \{v\}$ of a graph G. Throughout the article we only consider simple graphs.

The *Cartesian product* $G \square H$ of the graphs G and H is a graph with vertex set $V(G \square H) = V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent in $G \square H$ whenever ($gg' \in E(G)$ and $h = h'$) or ($g = g'$ and $hh' \in E(H)$). The Cartesian product is commutative and associative (see [5]). For a fixed $h \in V(H)$ we call $G^h = \{(g,h) \in V(G \times H) : g \in V(G)\}$ a *G-layer* in $G \times H$. An *H-layer* ${}^gH$ for a fixed $g \in V(G)$ is defined symmetrically. Any subgraph of $G \square H$ induced by G^h or ${}^gH$ is isomorphic to G or H, respectively. A Cartesian product graph is called a *grid* if both factors are isomorphic to paths. Since here we are only interested in grid graphs, more precisely in $P_6 \square P_k$, we use the following notation for vertices of $P_6 \square P_k$:

$$V(P_6 \square P_k) = \{(i,j) : i \in \{1,\dots,6\}, j \in \{1,\dots,k\}\}.$$

The domination number $\gamma(G)$ of a graph G is one of the classical invariants in graph theory. It is given by the minimum cardinality of a set S for which the union of

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closed neighborhoods centered in the vertices of S covers the whole vertex set of G. Such a set S is called a *dominating set* of G. Hence, each vertex of G is either in S or adjacent to a vertex in S. In other words, we can say that vertices of S control each vertex outside of S. A classical question in such a situation is: what controls the vertices of S? One possible solution to this dilemma is the total domination. A set $D \subseteq V(G)$ is a *total dominating set* of G if every vertex of G is adjacent to a vertex of D. (Hence, also vertices of D are controlled by D.) The *total domination number* of a graph G is the minimum cardinality of a total dominating set of G and it is denoted by $\gamma_t(G)$. A total dominating set D of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-*set*. For more information about total domination in graphs we suggest the recent monograph [6].

Several graph products have been investigated in the last decades and a rich theory involving the structure and recognition of classes of these graphs has emerged, cf. [5]. Probably the most studied graph product is the Cartesian product, which is also the most problematic for domination related problems. We just mention the famous Vizing's conjecture: $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$, which is probably the most challenging problem in the area of domination (see the latest survey on Vizing's conjecture [1]).

Closely related to the problem of domination in grid graphs, recently solved in [4], also the total domination number of grid graphs attracted some attention in the past decade. For instance, in [3], the value of $\gamma_t(P_r \Box P_t)$ was computed for $r \in \{1, 2, 3, 4\}$. This work was continued in [7] for $r \in \{5, 6\}$. Unfortunately, there is a partial mistake in the value of the total domination number of $P_6 \Box P_t$ given in [7], which we correct in the next section.

2. The grid $P_6 \Box P_k$

The following formula appeared in [7] for $k \geq 6$:

$$\gamma_t(P_6 \Box P_k) = \left\lfloor \frac{12k + 21}{7} \right\rfloor. \tag{1}$$

However, this formula is not correct in some cases. As we will show below, this result is not correct when $k \equiv x \pmod 7$ for $x \in \{0, 4, 5, 6\}$. The mistake is due to the facts that, on one hand, not all optimal patterns have been considered in [7] and, on the other hand, the number stated in Equation 1 is incorrect (for $x = 4$). To do so, we need to introduce some terminology. A graph G is an *efficient open domination graph* if there exists a set D, called an *efficient open dominating set*, for which

$$\bigcup_{v \in D} N_G(v) = V(G) \text{ and } N_G(u) \cap N_G(v) = \emptyset$$

for every pair u and v of distinct vertices of D (see [2]). The following result from [8] is useful to prove our results. (We also state the proof to make the present work self contained.)

Lemma 1 (see [8])**.** *If G is an efficient open domination graph with an efficient open dominating set D, then $\gamma_t(G) = |D|$.*

Proof. If D is an efficient open dominating set of G, then D is also a total dominating set of G and $\gamma_t(G) \leq |D|$ follows. On the other hand, every vertex of D has at least one neighbor in every $\gamma_t(G)$-set D', since

$$\bigcup_{v \in D'} N_G(v) = V(G).$$

Moreover, these neighbors must be different, since

$$\bigcup_{v \in D} N_G(v)$$

forms a partition of $V(G)$. Hence $\gamma_t(G) \geq |D|$ and the equality follows. □

Theorem 1. *Let $k \geq 6$. Then*

$$\gamma_t(P_6 \Box P_k) = \begin{cases} \frac{12k+14}{7}, & \text{if } k \equiv 0 \pmod 7, \\ \frac{12k+16}{7}, & \text{if } k \equiv 1 \pmod 7, \\ \frac{12k+18}{7}, & \text{if } k \equiv 2 \pmod 7, \\ \frac{12k+20}{7}, & \text{if } k \equiv 3 \pmod 7, \\ \frac{12k+8}{7}, & \text{if } k \equiv 4 \pmod 7, \\ \frac{12k+24}{7}, & \text{if } k \equiv 5 \pmod 7, \\ \frac{12k+12}{7}, & \text{if } k \equiv 6 \pmod 7. \end{cases}$$

Proof. First we try to find a total dominating set D for $G = P_6 \Box P_k$ where every vertex is totally dominated exactly once. Notice that if every vertex of G is totally dominated by D exactly once, then G is an efficient open domination graph. Thus, by Lemma 1 we have that $\gamma_t(G) = |D|$. We have only three options, up to the symmetry, to totally dominate each vertex exactly once in the first layer P_6^1 of G, see Figure 1. Moreover, each of these three possibilities expands to the whole G in a unique way (the pattern is forced by the starting position in the first layer P_6^1), again see Figure 1. Double doted lines in each graph of Figure 1 show the positions in which the pattern can stop to obtain a total dominating set for G where every vertex is totally dominated exactly once. This is done when $k \equiv x (\mathrm{mod}\, 7)$ for $x \in \{1, 4, 6\}$. Hence, if $x \in \{1, 4, 6\}$, then D is a $\gamma_t(G)$-set and we only need to know the cardinality of the set D. If we split G into consecutive blocks isomorphic to $P_6 \Box P_7$ and the remainder $P_6 \Box P_x$, $x \in \{1, 4, 6\}$, then it is easy to see that each block contains twelve vertices of D. In the remainder $P_6 \Box P_1$ we get additional four vertices (see Figure 1 b)), in $P_6 \Box P_4$ additional eight vertices (see Figure 1 a)) and in $P_6 \Box P_6$ additional twelve vertices (see Figure 1 b)). For $k \equiv 1 (\mathrm{mod}\, 7)$ we have $k = 7n + 1$ and

$$\gamma_t(P_6 \Box P_k) = 12n + 4 = \frac{12k + 16}{7}$$

by Lemma 1. By doing a similar computation we obtain that $\gamma_t(P_6 \Box P_k) = (12k + 8)/7$ for $k \equiv 4 \pmod 7$ and $\gamma_t(P_6 \Box P_k) = (12k + 12)/7$ for $k \equiv 6 \pmod 7$.

Let now $k \equiv x (\mathrm{mod}\, 7)$, $x \in \{0, 2, 3, 5\}$. Notice that in Figure 1 a), for $x \in \{0, 3\}$, and in Figure 1 b), for $x \in \{2, 5\}$, vertices $(3, k)$ and $(4, k)$ are only not totally dominated vertices in these patterns. (Notice also that in all three patterns we get two or more vertices which are not totally dominated by D.) Hence $D' = D \cup \{(3,k),(4,k)\}$ is a total dominating set of $P_6 \Box P_k$.

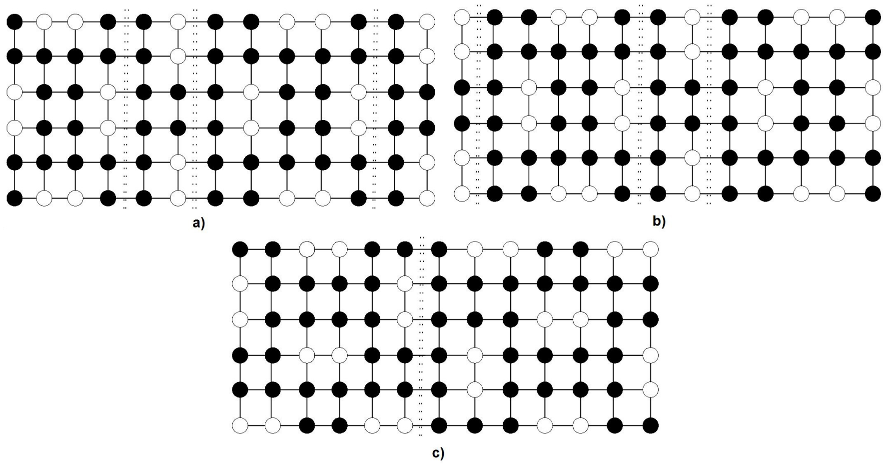

Figure 1: *The efficient open dominating set is given by the white vertices*

For $x = 0$, we have $k = 7n$ and there is no remainder, but two additional vertices in D'. Thus $\gamma_t(P_6 \Box P_k) \leq 12n + 2 = (12k + 14)/7$.

For $x = 2$, we have $k = 7n+2$ and in the remainder $P_6 \Box P_2$ there are six additional vertices in D'. Thus $\gamma_t(P_6 \Box P_k) \leq 12n + 6 = (12k + 18)/7$.

For $x = 3$, we have $k = 7n + 3$ and in the remainder $P_6 \Box P_3$ there are eight additional vertices in D'. Thus $\gamma_t(P_6 \Box P_k) \leq 12n + 8 = (12k + 20)/7$.

For $x = 5$, we have $k = 7n + 5$ and in the remainder $P_6 \Box P_5$ there are twelve additional vertices in D'. Thus $\gamma_t(P_6 \Box P_k) \leq 12n + 12 = (12k + 24)/7$.

We still need to show the lower bounds for $x \in \{0, 2, 3, 5\}$. Let $k = 7n+r$ for some integers $n \geq 1$ and $r \in \{0, 2, 3, 5\}$. Notice that, considering the symmetry, $P_6 \Box P_k$ can be partitioned into $n - 1$ consecutive blocks B_i, $i \in \{1, \ldots, n - 1\}$, isomorphic to $P_6 \Box P_7$ and one final block Y isomorphic to $P_6 \Box P_{7+r}$, with $r \in \{0, 2, 3, 5\}$. Let D be a $\gamma_t(P_6 \Box P_k)$-set and for every block B_i, let B'_i be the subset of B_i obtained from B_i by deleting its last P_6-layer. We denote by L_i this last P_6-layer of B_i. We will show that there are at least twelve vertices in $D \cap B_i$ to totally dominate each B'_i.

Let $i = 1$. It is not hard to see that B'_1 can be totally dominated in B_1 by twelve vertices, only if all these twelve vertices lie in B'_1. Clearly, we need at least four vertices in the first two P_6-layers to totally dominate P_6^1. We have three possibilities in Figure 1 and there are two additional possibilities, where we have exactly four vertices of D in the first two P_6-layers. These two are:

$$A_1 = \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),(5,1),(5,2),(5,3),(5,4),(5,5),(5,6)\},$$
$$A_2 = \{(2,1),(2,2),(5,1),(5,2),(1,4),(3,4),(4,4),(6,4),(1,5),(6,5),(3,6),(4,6)\}.$$

If we have more than four vertices of D in the first two P_6-layers, then this is even easier to see. Also, each of these sets does not totally dominate the whole set B_1 and at least two vertices of B_1 are not totally dominated by them. Notice that we can exchange the last two vertices of A_2 for $\{(2,7),(3,7),(4,7),(5,7)\}$. Hence the whole set B_1 is totally dominated, but then we use fourteen vertices. We will denote such a set by A'_2. If we set $A'_1 = A_1 \cup \{(2,7),(5,7)\}$, then we also get a total dominating set of B_1 with fourteen vertices. Now, if $|B_1 \cap D| = 12$, then $L_1 \cap D = \emptyset$ and, if $|B_1 \cap D| = 14$, then $L_1 \cap D$ contains two vertices in the case $A'_1 \subset D$ or four vertices when $A'_2 \subset D$. The remaining option $|B_1 \cap D| > 14$ leads to a contradiction with D being a $\gamma_t(G)$-set as can be seen later from the context.

Let now $i = 2$. If $L_1 \cap D = \emptyset$, then we have the same arguments as for B_1 and we obtain at least twelve vertices in $B_2 \cap D$. If $|L_1 \cap D| = 2$, then $(2,7),(5,7) \in D$ and $(2,8)$ and $(5,8)$ are already totally dominated. To totally dominate other vertices of P_6^8 we need additional four vertices from P_6^8 or P_6^9 in D. These vertices have no influence on layers P_6^{11}, P_6^{12} and P_6^{13}. Hence, we need six additional vertices to totally dominate these layers to finish B'_2. With this we already have at least ten vertices in $B_2 \cap D$, which gives twelve together with two vertices of $|L_1 \cap D|$. Notice that there is a possibility to totally dominate the whole set B_2, if we have (at least) two additional vertices in $L_2 \cap D$. If $|L_1 \cap D| = 4$, then $A'_2 \cap L_2 \subset D$ and $(2,8),(3,8),(4,8),(5,8)$ are dominated by them. In this case there is only one possibility to totally dominate B'_2 with ten additional vertices and this happens when

$$\{(1,9),(6,9),(1,10),(3,10),(4,10),(6,10),(2,12),(5,12),(2,13),(5,13)\} \subset D.$$

For every other option we need more vertices in $B'_2 \cap D$. So, in this case we have fourteen vertices, together with four vertices of $|L_1 \cap D|$, but from these four vertices of $|L_1 \cap D|$, two of them must be counted for B_1. Thus, there are twelve vertices for B_1 and twelve vertices for B_2. Notice that there is a possibility to totally dominate the whole set B_2, if we have (at least) two additional vertices in $L_2 \cap D$ (these are $(2,14)$ and $(5,14)$).

We continue for $i \in \{3,\dots,n-1\}$ and, by using the same procedure, for every B'_i we can find twelve vertices, which totally dominate B_i and no vertex is counted twice. Hence, $|D| \geq 12(n-1)$ and we still need to check Y. If $L_{n-1} \cap D = \emptyset$, then we are immediately done for Y, since we have

$$\gamma_t(P_6 \Box P_9) = 14, \gamma_t(P_6 \Box P_9) = 18, \gamma_t(P_6 \Box P_{10}) = 20 \text{ and } \gamma_t(P_6 \Box P_{12}) = 24,$$

for $r = 0, r = 2, r = 3$ and $r = 5$, respectively. Altogether, we have

$$|D| \geq 12(n-1) + 14 = 12n + 2 = \frac{12k + 14}{7},$$

for $r = 0$ and $k = 7n$. Similarly, we get other values. Now, if

$$L_{n-1} \cap D = \{(2, 7(n-1)), (5, 7(n-1))\},$$

then these two vertices have not been counted yet. Also, if $|L_{n-1} \cap D| = 4$, then as above, two of these vertices are counted for B_{n-1} and the other two vertices are still not counted. Hence, in these two cases, by applying the same argument as before, for this situation we easily get the desired values:

$$|Y \cap D| \geq 12, |Y \cap D| \geq 16, |Y \cap D| \geq 18 \text{ and } |Y \cap D| \geq 22,$$

for $r = 0, r = 2, r = 3$ and $r = 5$, respectively. By adding the two vertices from $L_{n-1} \cap D$ we get the final solution. □

It is straightforward to observe that for $k \equiv x \pmod 7$, $x \in \{0, 5, 6\}$, the result of Theorem 1 gives smaller values than Equation 1.

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