GLASNIK MATEMATIČKI
Vol. 49(69)(2014), 1 – 12

THE EDGE WIENER INDEX OF SUSPENSIONS, BOTTLENECKS, AND THORNY GRAPHS

Yaser Alizadeh, Ali Iranmanesh*, Tomislav Došlić and Mahdieh Azari

Hakim Sabzevari University, Iran, Tarbiat Modares University, Iran, University of Zagreb, Croatia and Kazerun Branch, Islamic Azad University, Iran

"In memory of Professor Ante Graovac"

Abstract. Let G be a simple connected graph. The distance between the edges g and $f \in E(G)$ is defined as the distance between the corresponding vertices g and f in the line graph of G. The edge-Wiener index of G is defined as the sum of such distances between all pairs of edges of the graph. Let $G_1 + G_2$ and $G_1 \circ G_2$ be the join and the corona of graphs G_1 and G_2, respectively. In this paper, we present explicit formulas for the edge-Wiener index for these graphs. Then we apply our results to compute the edge-Wiener index of suspensions, bottlenecks, and thorny graphs.

1. Introduction

It is a common phenomenon that a graph-theoretic invariant defined in terms of contributions of vertices of a graph soon obtains a counterpart defined in an analogous way using the edges, and *vice versa*. As recent examples, we mention the Szeged and the PI index. The attempts to carry out such generalizations for distance-based invariants have been hampered by the fact that there are several ways to define a distance between two edges in a simple connected graph. In a recent article [11], four such ways have been investigated, each of them giving rise to a corresponding edge version of Wiener index. It turned out that the most natural and the most useful way is the one based on the distance between the corresponding vertices in the line graph of the considered graph. In this paper, we continue that line of research by computing that edge version of Wiener index for two classes of graphs that arise via graph operations of join and corona.

*Corresponding author.

2010 *Mathematics Subject Classification.* 05C12, 05C76, 92E10.
Key words and phrases. Distance, edge-Wiener index, join, corona.

2. Definitions and preliminaries

Let G be a simple connected graph with the vertex set $V(G)$ and the edge set $E(G)$, respectively. The Wiener index of G is defined by:

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d(u,v),$$

where $d(u,v)$ is the shortest path distance between vertices u and v in G ([20]). The literature on the Wiener index is vast. We refer the reader to a number of papers concerned with computing Wiener indices of several classes of graphs ([7, 8, 10, 12, 15, 17–19]). Let $g = u_1v_1$ and $f = u_2v_2$ be two edges of G. The distance between g and f is denoted by $d_{e|G}(g,f)$ and defined as the distance between the corresponding vertices g and f in the line graph of G. If $g \neq f$, this distance is equal ([11]) to

$$\min\{d(u_1,u_2), d(u_1,v_2), d(v_1,u_2), d(v_1,v_2)\} + 1.$$

Let G be a connected graph. Then the edge-Wiener index ([11]) of G is defined as the sum of the distances (in the line graph) between all pairs of edges of G, i.e.

$$W_e(G) = \sum_{\{g,f\} \subseteq E(G)} d_{e|G}(g,f).$$

In view of the above definition, the edge-Wiener index of a graph equals the ordinary Wiener index of its line graph. We refer the reader to [1–3, 6, 13] for more information on the edge-Wiener index. Distance 1 means that the edges share a vertex; distance 2 means that at least two of the four end vertices of the two edges are adjacent. If the distance between e and f is greater than two, we say that e and f form a pair of distant edges.

Let $N(u)$ denote the neighborhood of a vertex u in G, i.e. the set of all vertices of G adjacent with u. Let uv be an edge of G and z a vertex of G that is not adjacent to u or v. Then all edges connecting z with vertices outside $N(u) \cup N(v)$ are distant from uv. Obviously, there are $|N(z) \backslash (N(u) \cup N(v))|$ such edges for a given vertex z. Now, by summing such contributions over all vertices $z \in V(G) \backslash (N(u) \cup N(v))$, and then over all edges $uv \in E(G)$, we obtain the quantity

$$N(G) = \sum_{uv \in E(G)} \sum_{z \in V(G) \backslash (N(u) \cup N(v))} |N(z) \backslash (N(u) \cup N(v))|.$$

It is easy to see that each pair of distant edges in G is counted exactly four times by $N(G)$; hence, the number of pairs of distant edges of G is equal to $\frac{1}{4}N(G)$. We will find this fact convenient in situations where all pairs of distant edges are at the same distance.

It can be verified by direct calculation that $N(P_n) = N(C_n) = 0$ for $n < 5$ and that $\frac{1}{4}N(P_n) = \binom{n-3}{2}$ and $\frac{1}{4}N(C_n) = \frac{1}{2}n(n-5)$ for $n \geq 5$.

Let G_1 and G_2 be two simple connected graphs and n_i and e_i denote the numbers of vertices and edges of G_i, respectively. The *join* of these graphs is denoted by $G_1 + G_2$ and is defined as the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set

$$E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup S,$$

where $S = \{u_1 u_2 \,|u_1 \in V(G_1), u_2 \in V(G_2)\}$. The join of two graphs is also known as their *sum*. Its definition can be extended inductively to more than two graphs in a straightforward manner. It is a commutative operation and hence both its components will appear symmetrically in any formula including distance-based invariants.

We will be interested in the special case when one of the components of a join is a single vertex. For a given graph G we call the graph $K_1 + G$ the *suspension* of G ([21]).

The *corona* of two graphs G_1 and G_1, is denoted by $G_1 \circ G_2$ and is obtained by taking one copy of G_1 and n_1 copies of G_2, and joining all vertices of the i-th copy of G_2 to the i-th vertex of G_1 for $i = 1, 2, \cdots, n_1$. Unlike join, corona is a non-commutative operation, and its component graphs appear in markedly asymmetric roles. Coronas sometimes appear in chemical literature as plerographs of the usual hydrogen-suppressed molecular graphs known as kenographs; see [16] for definitions and more information. The *k-thorny graph* of a given graph G is obtained as $G \circ \overline{K}_k$, where $\overline{K}_k$ denotes the empty graph on k vertices ([14]). Interesting classes of graphs can also be obtained by specializing the first component in the corona product. For example, for a graph G, the graph $K_2 \circ G$ is called its *bottleneck graph*.

The first and the second Zagreb indices ([9]) of a graph G are defined as follows:

$$M_1(G) = \sum_{u \in V(G)} \delta(u)^2,$$
$$M_2(G) = \sum_{uv \in E(G)} \delta(u)\delta(v).$$

Here $\delta(u)$ denotes the degree of vertex u. The Zagreb indices will help us to formulate our results in a more compact way. Similarly, we will also find handy the following hybrid quantities, denoted by $D(x|G)$ and W_{ve}. Let x be a vertex of G and $g = uv$ be an edge of G. Then $D(x, g) = \min\{d(x, u), d(x, v)\}$. $D(x|G)$ is obtained by summing such contributions over all edges of G, $D(x|G) = \sum_{g \in E(G)} D(x, g)$, and $W_{ve}(G)$ is obtained by summing $D(x|G)$ over all vertices $x \in V(G)$,

$$W_{ve}(G) = \sum_{x \in V(G)} D(x|G).$$

One might imagine $W_{ve}(G)$ as some kind of "mixed" Wiener index of G. That quantity was studied in more details in references $[1, 4, 5]$ under the name $Min(G)$. We refer the reader to the above references for explicit formulas for $W_{ve}(G)$ of several classes of graphs.

3. Main results

3.1. *Join.*

Theorem 3.1. *Let G_1 and G_2 be two connected graphs. Then*

$$W_e(G_1 + G_2) = 2\binom{e_1 + e_2 + n_1n_2}{2} - \frac{1}{2}\left(M_1(G_1) + M_1(G_2)\right) + \frac{1}{4}\left(N(G_1) + N(G_2)\right) - s,$$

where $s = (2n_2 - 1)e_1 + (2n_1 - 1)e_2 + \frac{1}{2}n_1n_2(n_1 + n_2 - 2)$.

Proof. All distinct vertices of $G_1 + G_2$ are either at distance 1 or 2. The vertices at distance 2 are precisely those of G_1 that are not adjacent in G_1, and those of G_2 that are not adjacent in G_2. So all distinct edges of $G_1 + G_2$ are either at distance 1, 2 or 3.

Let Q be the set of all pairs of edges of $G_1 + G_2$. We partition Q into six disjoint sets as follows

$$\begin{aligned}
Q_1 &= \{\{g, f\} | g, f \in E(G_1)\},\\
Q_2 &= \{\{g, f\} | g, f \in E(G_2)\},\\
Q_3 &= \{\{g, f\} | g \in E(G_1), f \in E(G_2)\},\\
Q_4 &= \{\{g, f\} | g \in E(G_1), f \in S\},\\
Q_5 &= \{\{g, f\} | g \in E(G_2), f \in S\},\\
Q_6 &= \{\{g, f\} | g, f \in S\}.
\end{aligned}$$

The edge-Wiener index of $G_1 + G_2$ is obtained by summing the contributions of all pairs of edges over those six sets. We proceed to evaluate their contributions in order of increasing complexity.

The case of Q_3 is the simplest. Let $\{g, f\} \in Q_3$, where $g = u_1v_1 \in E(G_1)$ and $f = u_2v_2 \in E(G_2)$. Then

$$\begin{aligned}
d_{e|G_1+G_2}(g, f) &= \min\{d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2)\} + 1\\
&= \min\{1, 1, 1, 1\} + 1 = 2.
\end{aligned}$$

There are e_1e_2 such pairs of edges in Q_3 and each of them contributes 2 to the edge-Wiener index. Hence, the total contribution of pairs from Q_3 is equal to $2e_1e_2$.

The set Q_6 contains pairs of edges from S. Let $\{g, f\} \in Q_6$ and $g = u_1u_2$, $f = v_1v_2$, where $u_1, v_1 \in V(G_1)$, $u_2, v_2 \in V(G_2)$. Then

$$\begin{aligned} d_{e|G_1+G_2}(g, f) &= \min\{d(u_1, v_1), d(u_1, v_2), d(u_2, v_1), d(u_2, v_2)\} + 1 \\ &= \min\{d(u_1, v_1), 1, 1, d(u_2, v_2)\} + 1. \end{aligned}$$

If the edges g and f share a vertex then $d(u_1, v_1) = 0$ or $d(u_2, v_2) = 0$, so $d_{e|G_1+G_2}(g, f) = 1$, and if they are not adjacent in $G_1 + G_2$, then the distances $d(u_1, v_1)$ and $d(u_2, v_2)$ are either equal to 1 or 2, so $d_{e|G_1+G_2}(g, f) = 2$.

The total number of such pairs of edges in Q_6 is equal to $\binom{n_1n_2}{2}$. Among them there are $n_1\binom{n_2}{2} + n_2\binom{n_1}{2}$ pairs sharing a vertex. Such pairs contribute 1, and all other pairs contribute 2. Hence the total contribution of pairs from Q_6 is equal to

$$2\binom{n_1n_2}{2} - n_1\binom{n_2}{2} - n_2\binom{n_1}{2}.$$

Now, we compute the contribution of pairs from Q_4. Let $g = u_1v_1 \in E(G_1)$ and $f = z_1u_2 \in S$ be a pair of nonadjacent edges in Q_4, where $u_1, v_1, z_1 \in V(G_1)$, $u_2 \in V(G_2)$. Then

$$\begin{aligned} d_{e|G_1+G_2}(g, f) &= \min\{d(u_1, z_1), d(u_1, u_2), d(v_1, z_1), d(v_1, u_2)\} + 1 \\ &= \min\{d(u_1, z_1), 1, d(v_1, z_1), 1\} + 1. \end{aligned}$$

Since g and f are not adjacent in $G_1 + G_2$, so the distances $d(u_1, z_1)$ and $d(v_1, z_1)$ are either equal to 1 or 2. Therefore $d_{e|G_1+G_2}(g, f) = 2$.

The total number of pairs from Q_4 is equal to $e_1n_1n_2$. The adjacent pairs share a vertex in G_1; hence there are $2e_1n_2$ such pairs, and their contribution is given by $2e_1n_2$. All other pairs from Q_4 contribute 2, and the total contribution of Q_4 is equal to $2e_1n_2(n_1 - 1)$.

By symmetry, the total contribution of pairs from Q_5 is equal to $2e_2n_1(n_2 - 1)$.

It remains to compute the contributions of Q_1 and Q_2. Let $\{g, f\} \in Q_1$, where $g = u_1v_1$, $f = z_1t_1$. Then

$$d_{e|G_1+G_2}(g, f) = \min\{d(u_1, z_1), d(u_1, t_1), d(v_1, z_1), d(v_1, t_1)\} + 1.$$

By definition of $G_1 + G_2$, the distances $d(u_1, z_1)$, $d(u_1, t_1)$, $d(v_1, z_1)$ and $d(v_1, t_1)$ are equal to 0, 1 or 2. If the edges g and f share a vertex then exactly one of these four distances is equal to 0. So $d_{e|G_1+G_2}(g, f) = 1$. If g and f are not adjacent in $G_1 + G_2$ and at least one of the four distances is equal to 1, then $d_{e|G_1+G_2}(g, f) = 2$, and if all of them are equal to 2, then $d_{e|G_1+G_2}(g, f) = 3$. Note that the last case happens when the end vertices of g are not adjacent to the end vertices of f in G_1, i.e. when

$$d_{G_1}(u_1, z_1), d_{G_1}(u_1, t_1), d_{G_1}(v_1, z_1), d_{G_1}(v_1, t_1) > 1.$$

Therefore $d_{e|G_1}(g, f) > 2$, which implies that g and f form a pair of distant edges of G_1.

Hence, we partition Q_1 into three sets, Q_1', Q_1'' and Q_1''', made of the pairs of edges at distance 1, 2 and 3 in $G_1 + G_2$, respectively. Then the total contribution of pairs from Q_1 to the edge-Wiener index of $G_1 + G_2$ is given by $|Q_1'| + 2|Q_1''| + 3|Q_1'''|$. The total number of pairs in Q_1 is equal to $\binom{e_1}{2}$. Since Q_1''' is the set of pairs of distant edges of G_1, so $|Q_1'''| = \frac{1}{4}N(G_1)$. Further,

$$|Q_1'| = \sum_{u \in V(G_1)} \binom{\delta(u)}{2} = \frac{1}{2}M_1(G_1) - e_1.$$

From here it immediately follows that the total contribution of Q_1 is given by

$$e_1^2 - \frac{1}{2}M_1(G_1) + \frac{1}{4}N(G_1).$$

Again, the total contribution of Q_2 follows by the symmetry, and the formula from the Theorem follows by adding the contributions of $Q_1, \ldots, Q_6$ and simplifying the resulting expression. ∎

As expected, G_1 and G_2 appear symmetrically in the above formula. It is interesting to note that the formula does not depend on the connectivity of G_1 and G_2. That allows us to compute the edge-Wiener index of joins of graphs that are not themselves connected. In this way, we could reproduce the results from [11] concerning the complete bipartite graphs.

3.2. *Corona.*

THEOREM 3.2. *Let G_1 and G_2 be two simple connected graphs. Then*

$$\begin{aligned} W_e(G_1 \circ G_2) = {} & W_e(G_1) + (n_2 + e_2)^2 W(G_1) - \frac{n_1}{2} M_1(G_2) + \frac{n_1}{4} N(G_2) \\ & + e_2^2 \left[3\binom{n_1}{2} + n_1 \right] + n_1 \binom{n_2}{2} + n_2^2 \binom{n_1}{2} + n_1 e_1 (n_2 + 2e_2) \\ & + 2n_1 e_2 (n_2 - 1) + 2n_1 n_2 e_2 (n_1 - 1) + (n_2 + e_2) W_{ve}(G_1). \end{aligned}$$

PROOF. We denote the copy of G_2 related to the vertex $x \in V(G_1)$ by $G_{2,x}$ and the edge set of $G_{2,x}$ by $S_{2,x}$. By definition of $G_1 \circ G_2$, the distance between two distinct vertices $u, v \in V(G_1 \circ G_2)$ is given by

$$d_{G_1 \circ G_2}(u,v) = \begin{cases} d_{G_1}(u,v) & u, v \in V(G_1) \\ d_{G_1}(u,x) + 1 & u \in V(G_1),\ v \in V(G_{2,x}) \\ 1 & uv \in S_{2,x} \\ 2 & u, v \in V(G_{2,x}),\ uv \notin S_{2,x} \\ d_{G_1}(x,y) + 2 & u \in V(G_{2,x}),\ v \in V(G_{2,y}),\ x \neq y. \end{cases}$$

It is obvious that the graph $G_1 \circ G_2$ has $e_1 + n_1 e_2 + n_1 n_2$ edges. We partition the edge set of $G_1 \circ G_2$ into three sets. The first one is the edge set of G_1, $S_1 = E(G_1)$, the second one contains all edges in all copies of G_2, $S_2 = \bigcup_{x \in V(G_1)} S_{2,x}$, and the third one contains all edges with one end in G_1 and the other end in some copy of G_2, $S_3 = \bigcup_{x\in G_1} S_{3,x}$, where $S_{3,x} = \{e|e = ux, u \in V(G_{2,x})\}$.

Now we start to compute the distances between the edges of these three sets. There are 6 cases:

CASE 1. $\{g, f\} \subseteq S_1$.
It is obvious that $d_{e|G_1\circ G_2}(g, f) = d_{e|G_1}(g, f)$, so

$$W_1 = \sum_{\{g,f\}\subseteq S_1} d_{e|G_1\circ G_2}(g, f) = W_e(G_1).$$

CASE 2. $\{g, f\} \subseteq S_2$, $g \in S_{2,x}$ and $f \in S_{2,y}$.
First, we consider the case $x = y$ and let $g = u_{2,x}v_{2,x}, f = z_{2,x}t_{2,x} \in S_{2,x}$. Then

$$d_{e|G_1\circ G_2}(g, f) = \min\{d(u_{2,x}, z_{2,x}), d(u_{2,x}, t_{2,x}), d(v_{2,x}, z_{2,x}), d(v_{2,x}, t_{2,x})\} + 1.$$

Clearly, the distances $d(u_{2,x}, z_{2,x})$, $d(u_{2,x}, t_{2,x})$, $d(v_{2,x}, z_{2,x})$, and $d(v_{2,x}, t_{2,x})$ are equal to 0, 1 or 2. So $d_{e|G_1\circ G_2}(g, f) = 1$, 2 or 3. In this case, the vertex x and its related copy, $G_{2,x}$, form a copy of $K_1 + G_2$. So, by the same reasoning as in the proof of Theorem 3.1 we obtain

$$\sum_{\{g,f\}\subseteq S_{2,x}} d_{e|G_1\circ G_2}(g, f) = e_2^2 - \frac{1}{2}M_1(G_2) + \frac{1}{4}N(G_2).$$

Now, let $x \neq y$ and $g = u_{2,x}v_{2,x}$ and $f = u_{2,y}v_{2,y}$. Then

$$\begin{aligned} d_{e|G_1\circ G_2}(g, f) &= \min\{d(u_{2,x}, u_{2,y}), d(u_{2,x}, v_{2,y}), d(v_{2,x}, u_{2,y}), d(v_{2,x}, v_{2,y})\} + 1 \\ &= \min\{d_{G_1}(x, y) + 2, d_{G_1}(x, y) + 2, d_{G_1}(x, y) + 2, \\ &\qquad d_{G_1}(x, y) + 2\} + 1 \\ &= d_{G_1}(x, y) + 3. \end{aligned}$$

Now,

$$\begin{aligned} W_2 &= \sum_{\{g,f\}\subseteq S_2} d_{e|G_1\circ G_2}(g, f) \\ &= \sum_{x\in V(G_1)} \sum_{\{g,f\}\subseteq S_{2,x}} d_{e|G_1\circ G_2}(g, f) + \sum_{\{x,y\}\subseteq V(G_1)} \sum_{g\in S_{2,x}} \sum_{f\in S_{2,y}} d_{e|G_1\circ G_2}(g, f) \\ &= n_1 \left(e_2^2 - \frac{1}{2}M_1(G_2) + \frac{1}{4}N(G_2)\right) + \sum_{\{x,y\}\subseteq V(G_1)} (3 + d_{G_1}(x, y))\, e_2^2 \\ &= n_1 \left(e_2^2 - \frac{1}{2}M_1(G_2) + \frac{1}{4}N(G_2)\right) + e_2^2 \left(3\binom{n_1}{2} + W(G_1)\right). \end{aligned}$$

CASE 3. $\{g, f\} \subseteq S_3$, $g \in S_{3,x}$ and $f \in S_{3,y}$.
If $x = y$ then the edges g and f share the vertex x. So $d_{e|G_1\circ G_2}(g, f) = 1$. If

$x \neq y$ and $g = u_{2,x}x \in S_{3,x}$, $f = u_{2,y}y \in S_{3,y}$, then

$$\begin{aligned} d_{e|G_1 \circ G_2}(g,f) &= \min\{d(u_{2,x},u_{2,y}), d(u_{2,x},y), d(x,u_{2,y}), d(x,y)\} + 1 \\ &= \min\{d_{G_1}(x,y)+2, d_{G_1}(x,y)+1, d_{G_1}(x,y)+1, d_{G_1}(x,y)\} + 1 \\ &= d_{G_1}(x,y) + 1. \end{aligned}$$

Now,

$$\begin{aligned} W_3 &= \sum_{\{g,f\}\subseteq S_3} d_{e|G_1\circ G_2}(g,f) \\ &= \sum_{x\in V(G_1)} \sum_{\{g,f\}\subseteq S_{3,x}} d_{e|G_1\circ G_2}(g,f) + \sum_{\{x,y\}\subseteq V(G_1)} \sum_{g\in S_{3,x}} \sum_{f\in S_{3,y}} d_{e|G_1\circ G_2}(g,f) \\ &= \sum_{x\in V(G_1)} \sum_{\{g,f\}\subseteq S_{3,x}} 1 + \sum_{\{x,y\}\subseteq V(G_1)} \sum_{g\in S_{3,x}} \sum_{f\in S_{3,y}} (d_{G_1}(x,y)+1) \\ &= \sum_{x\in V(G_1)} \frac{1}{2} n_2(n_2-1) + \sum_{\{x,y\}\subseteq V(G_1)} \sum_{g\in S_{3,x}} (d_{G_1}(x,y)+1)n_2 \\ &= \frac{1}{2} n_1 n_2 (n_2-1) + n_2^2 W(G_1) + \frac{1}{2} n_2^2 (n_1-1) n_1. \end{aligned}$$

Case 4. $g \in S_1, f \in S_2$.
Let $g = u_1 v_1 \in S_1$, $f = u_{2,x}v_{2,x} \in S_{2,x}$, for some $x \in V(G_1)$. Then

$$\begin{aligned} d_{e|G_1 \circ G_2}(g,f) &= \min\{d(u_1,u_{2,x}), d(u_1,v_{2,x}), d(v_1,u_{2,x}), d(v_1,v_{2,x})\} + 1 \\ &= \min\{d_{G_1}(u_1,x)+1, d_{G_1}(u_1,x)+1, d_{G_1}(v_1,x)+1, \\ &\qquad d_{G_1}(v_1,x)+1\} + 1 \\ &= \min\{d_{G_1}(u_1,x), d_{G_1}(v_1,x)\} + 2 \\ &= D_{G_1}(x,g) + 2. \end{aligned}$$

Now,

$$\begin{aligned} W_4 &= \sum_{x\in V(G_1)} \sum_{f\in S_{2,x}} \sum_{g\in S_1} d_{e|G_1\circ G_2}(g,f) = \sum_{x\in V(G_1)} \sum_{f\in S_{2,x}} \sum_{g\in S_1} (D_{G_1}(x,g)+2) \\ &= \sum_{x\in V(G_1)} \sum_{f\in S_{2,x}} (D(x|G_1) + 2e_1) = \sum_{x\in V(G_1)} (D(x\,|G_1) + 2e_1\,)\, e_2 \\ &= e_2\left(W_{ve}(G_1) + 2n_1 e_1\right). \end{aligned}$$

Case 5. $g \in S_1, f \in S_3$.
Similar to the above case, for $g \in S_1, f \in S_{3,x}$, $d_{e|G_1\circ G_2}(g,f) = D_{G_1}(x,g)+1$.

So,

$$
\begin{aligned}
W_5 &= \sum_{x\in V(G_1)} \sum_{f\in S_{3,x}} \sum_{g\in S_1} d_{e|G_1\circ G_2}(g,f) = \sum_{x\in V(G_1)} \sum_{f\in S_{3,x}} \sum_{g\in S_1} (D_{G_1}(x,g)+1) \\
&= \sum_{x\in V(G_1)} \sum_{f\in S_{3,x}} (D(x\,|G_1)+e_1) = \sum_{x\in V(G_1)} (D(x|G_1)+e_1)\, n_2 \\
&= n_2\,(W_{ve}(G_1)+n_1e_1)\,.
\end{aligned}
$$

Case 6. $g\in S_2, f\in S_3$.
If $g\in S_{2,x}, f\in S_{3,x}$, for some $x\in V(G_1)$, then $d_{e|G_1\circ G_2}(g,f)=1$ or 2. The edge g is adjacent to two edges of $S_{3,x}$ and its distance to other edges is 2. So,

$$
\begin{aligned}
\sum_{x\in V(G_1)} \sum_{g\in S_{2,x}} \sum_{f\in S_{3,x}} d_{e|G_1\circ G_2}(g,f) &= \sum_{x\in V(G_1)} \sum_{g\in S_{2,x}} (2+2(n_2-2)) \\
&= 2(n_2-1)e_2n_1.
\end{aligned}
$$

If $g=u_{2,x}v_{2,x}\in S_{2,x}, f=u_{2,y}y\in S_{3,y}$, where $x,y\in V(G_1)$ and $x\neq y$, then

$$
\begin{aligned}
d_{e|G_1\circ G_2}(g,f) &= \min\{d(u_{2,x},u_{2,y}), d(u_{2,x},y), d(v_{2,x},u_{2,y}), d(v_{2,x},y)\}+1 \\
&= \min\{d_{G_1}(x,y)+2, d_{G_1}(x,y)+1, d_{G_1}(x,y)+2, \\
&\qquad d_{G_1}(x,y)+1\}+1 \\
&= d_{G_1}(x,y)+2.
\end{aligned}
$$

Now,

$$
\begin{aligned}
W_6 &= \sum_{x,y\in V(G_1)} \sum_{g\in S_{2,x}} \sum_{f\in S_{3,y}} d_{e|G_1\circ G_2}(g,f) \\
&= \sum_{x\in V(G_1)} \sum_{g\in S_{2,x}} \sum_{f\in S_{3,x}} d_{e|G_1\circ G_2}(g,f) \\
&\quad + \sum_{x\neq y\in V(G_1)} \sum_{g\in S_{2,x}} \sum_{f\in S_{3,y}} d_{e|G_1\circ G_2}(g,f) \\
&= (2n_2-2)e_2n_1 + \sum_{x\neq y\in V(G_1)} (d_{G_1}(x,y)+2)n_2e_2 \\
&= 2(n_2-1)n_1e_2 + \left(\sum_{x\neq y\in V(G_1)} d_{G_1}(x,y) + 2n_1(n_1-1)\right) n_2e_2 \\
&= 2(n_2-1)n_1e_2 + 2(W(G_1)+n_1(n_1-1))n_2e_2.
\end{aligned}
$$

Now the formula for the edge-Wiener index of $G_1\circ G_2$ follows by adding all six contributions and simplifying the resulting expression. ∎

Again, it is interesting to note that the formula of Theorem 3.2 does not include any invariants of G_2 that depend on its connectivity. Hence, it is possible to apply Theorem 3.2 to the cases of $G_1 \circ G_2$ with disconnected G_2. As mentioned before, such cases arise in transitions from kenographs to plerographs, where G_2 is given as an empty graph, i.e., as $\overline{K}_n$ for some positive integer n.

4. Examples and concluding remarks

Now we can obtain explicit formulas for the edge-Wiener indices of some classes of graphs by specializing components in joins and coronas. We start by computing the edge-Wiener index of a suspension of a graph G.

Corollary 4.1. *Let G be a simple graph. Then*

$$W_e(K_1 + G) = W_e(K_1 \circ G) = 2\binom{n+e}{2} - \binom{n}{2} - \frac{1}{2}M_1(G) + \frac{1}{4}N(G) - e.$$

Here and in the rest of this section n and e denote the number of vertices and the number of edges of G, respectively.

Now the formulas for the *wheel graph* $W_n = K_1 + C_n$ and for the *fan graph* $K_1 + P_n$ follow at once. Both graphs allow alternative representations as $K_1 \circ C_n$ and $K_1 \circ P_n$, respectively.

Corollary 4.2. *For $n \geq 5$,*

$$W_e(K_1 + C_n) = 4n^2 - 7n;$$
$$W_e(K_1 + P_n) = 4(n^2 - 3n + 3).$$

Our next example is the *windmill graph* (sometimes also called a flower graph), the corona of K_1 and m copies of K_2. We denote it by $F_m = K_1 \circ (mK_2)$.

Corollary 4.3. $W_e(K_1 \circ (mK_2)) = m(7m - 4)$.

Now we turn our attention toward coronas. For the k-thorny graph of a graph G we obtain the following formula.

Corollary 4.4. *Let G be a simple connected graph. Then*

$$W_e(G \circ \overline{K}_k) = W_e(G) + k^2\left[W(G) + \binom{n}{2}\right] + n\binom{k}{2} + k\left[ne + W_{ve}(G)\right].$$

We present formulas for the *k-thorny cycle* $C_n \circ \overline{K}_k$ and the *k-thorny path* $P_n \circ \overline{K}_k$. We use known results for the edge-Wiener indices of paths and cycles ([11]) and our results on the mixed Wiener indices from the end of Section 2.

Corollary 4.5. *We have*

$$W_e(C_n \circ \overline{K}_k) = \frac{n}{8}[n^2(k+1)^2 + 4nk(k+1) - 4k], \quad \textit{for } n \text{ even},$$

$$W_e(C_n \circ \overline{K}_k) = \frac{n(k+1)}{8}[n^2 + nk(n+4) - k - 1], \quad \textit{for } n \text{ odd},$$

and

$$W_e(P_n \circ \overline{K}_k) = \frac{n}{6}[n^2(k+1)^2 + 3n(k^2-1) - k(k+5) + 2].$$

Finally, we consider the bottleneck graph of a given graph G.

COROLLARY 4.6. *Let G be a simple graph. Then*

$$W_e(K_2 \circ G) = 6e^2 + 10ne + 3n^2 + n - M_1(G) + \frac{1}{2}N(G).$$

The above list of examples is by no means exhaustive. Nevertheless, there are still many classes of chemically interesting and relevant graphs not covered by our approach. It would be interesting to find closed formulas for edge-Wiener indices of various classes of branched and unbranched polymers, of benzenoid graphs, and of nanotubes and various other nanostructures. In order to achieve that goal, further research into properties of edge Wiener index under graph operations such as splices, links, and the rooted product, will be necessary.

ACKNOWLEDGEMENTS.

The authors would like to thank the referee for the careful reading and useful suggestions which led us to improve the paper. Partial support of the Ministry of Science, Education and Sport of the Republic of Croatia (Grants No. 037-0000000-2779 and 177-0000000-0884) is gratefully acknowledged by one of the authors (TD).

REFERENCES

[1] M. Azari and A. Iranmanesh, *Computation of the edge Wiener indices of the sum of graphs*, Ars Combin. **100** (2011), 113–128.

[2] M. Azari, A. Iranmanesh and A. Tehranian, *A method for calculating an edge version of the Wiener number of a graph operation*, Util. Math. **87** (2012), 151–164.

[3] M. Azari, A. Iranmanesh and A. Tehranian, *Computation of the first edge Wiener index of a composition of graphs*, Studia Univ. Babes Bolyai Chem. **4** (2010), 183–196.

[4] M. Azari, A. Iranmanesh and A. Tehranian, *Maximum and Minimum polynomials of a composite graph*, Austral. J. Basic Appl. Sci. **5(9)** (2011), 825–830.

[5] M. Azari, A. Iranmanesh and A. Tehranian, *Two topological indices of three chemical structures*, MATCH Commun. Math. Comput. Chem. **69** (2013), 69–86.

[6] P. Dankelmann, I. Gutman, S. Mukwembi and H. C. Swart, *The edge Wiener index of a graph*, Discrete Math. **309** (2009), 3452–3457.

[7] I. Gutman, *A new method for the calculation of the Wiener number of acyclic molecules*, J. Mol. Struct. (Theochem) **285** (1993), 137–142.

[8] I. Gutman, *Calculating the Wiener number: the Doyle–Graver method*, J. Serb. Chem. Soc. **58** (1993), 745–750.

[9] I. Gutman, N. Trinajstić, *Graph theory and molecular orbitals, Total $\pi-$ electron energy of alternant hydrocarbons*, Chem. Phys. Lett. **17** (1972), 535-538.

[10] I. Gutman, Y. N. Yeh, S. L. Lee and Y. L. Luo, *Some recent results in the theory of the Wiener number*, Indian J. Chem. **32A** (1993), 651–661.

[11] A. Iranmanesh, I. Gutman, O. Khormali and A. Mahmiani, *The edge versions of Wiener index*, MATCH Commun. Math. Comput. Chem. **61** (2009), 663–672.

[12] M. Juvan, B. Mohar, A. Graovac, S. Klavžar and J. Žerovnik, *Fast computation of the Wiener index of fasciagraphs and rotagraphs*, J. Chem. Inf. Comput. Sci. **35** (1995), 834–840.

[13] M. H. Khalifeh, H. Yousefi Azari, A. R. Ashrafi and S. G. Wagner, *Some new results on distance-based graph invariants*, European J. Combin. **30** (2009), 1149–1163.

[14] D. J. Klein, T. Došlić and D. Bonchev, *Vertex-weightings for distance moments and thorny graphs*, Discrete Appl. Math. **155** (2007), 2294–2302.

[15] H. Liu and X. F. Pan, *On the Wiener index of trees with fixed diameter*, MATCH Commun. Math. Comput. Chem. **60** (2008), 85–94.

[16] A. Miličević and N. Trinajstić, *Combinatorial enumeration in chemistry*, in Chemical Modelling: Applications and Theory, RSC Publishing, Cambridge, 2006, 405–469.

[17] P. Senn, *The computation of the distance matrix and the Wiener index for graphs of arbitrary complexity with weighted vertices and edges*, Comput. Chem. **12** (1988), 219–227.

[18] D. Stevanović, *Maximizing Wiener index of graphs with fixed maximum degree*, MATCH Commun. Math. Comput. Chem. **60** (2008), 71–83.

[19] S. G. Wagner, *A class of trees and its Wiener index*, Acta Appl. Math. **91** (2006), 119–132.

[20] H. Wiener, *Structural determination of paraffin boiling points*, J. Am. Chem. Soc. **69** (1947), 17–20.

[21] Z. Yarahmadi, T. Došlić and A. R. Ashrafi, *The bipartite edge frustration of composite graphs*, Discrete Appl. Math. **158** (2010), 1551–1558.

Y. Alizadeh
Department of Mathematics
Hakim Sabzevari University
Sabzevar
Iran
E-mail: `y.alizadeh@hsu.ac.ir`

A. Iranmanesh
Department of Mathematics
Tarbiat Modares University
P. O. Box: 14115-137, Tehran
Iran
E-mail: `iranmanesh@modares.ac.ir`

T. Došlić
Faculty of Civil Engineering
University of Zagreb
Kačićeva 26, 10000 Zagreb
Croatia
E-mail: `doslic@grad.hr`

M. Azari
Department of Mathematics
Kazerun Branch, Islamic Azad University
P. O. Box: 73135-168, Kazerun
Iran
E-mail: `azari@kau.ac.ir`

Received: 25.4.2013.
Revised: 30.8.2013.