GLASNIK MATEMATIČKI
Vol. 49(69)(2014), 13 – 23

REGULAR OPEN ARITHMETIC PROGRESSIONS IN CONNECTED TOPOLOGICAL SPACES ON THE SET OF POSITIVE INTEGERS

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Abstract. In this paper we characterize regular open arithmetic progressions in four connected topological spaces on the set of positive integers with bases consisting of some arithmetic progressions and we examine which of these spaces are semiregular.

1. Preliminaries

The letters $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$, and $\mathcal{P}$ denote the sets of integers, positive integers, non-negative integers, and primes, respectively. For each set A we use the symbols $\operatorname{cl} A$ and $\operatorname{int} A$ to denote the closure and the interior of A, respectively. The symbol $\Theta(a)$ denotes the set of all prime factors of $a \in \mathbb{N}$. For all $a, b \in \mathbb{N}$, we use the symbols (a, b) and $\operatorname{lcm}(a, b)$ to denote the greatest common divisor of a and b and the least common multiple of a and b, respectively. Moreover, for all $a, b \in \mathbb{N}$, the symbols $\{an+b\}$ and $\{an\}$ stand for the infinite arithmetic progressions:

$$\{an + b\} := a \cdot \mathbb{N}_0 + b \quad \text{and} \quad \{an\} := a \cdot \mathbb{N}.$$

Clearly $\{an\} = \{an + a\}$. Let X be a topological space. A subset $A \subset X$ is called a regular open set, if $\operatorname{int} \operatorname{cl} A = A$. Clearly each regular open set is open and if both sets A and B are regular open, then $A \cap B$ is regular open, too. We say that a topological space X is semiregular, if regular open sets form a base of X.

2010 *Mathematics Subject Classification.* 11B25, 54A05, 11A41.
Key words and phrases. Furstenberg's topology, Golomb's topology, Kirch's topology, the division topology, arithmetic progressions, regular open set, semiregular space.

We use standard notation. For the basic results and notions concerning topology and number theory we refer the reader to the monographs of J. L. Kelley ([4]) and W. J. LeVeque ([6]), respectively.

2. Introduction

In 1955 H. Furstenberg ([2]) defined the base of a topology on $\mathbb{Z}$ by means of all arithmetic progressions and gave an elegant topological proof of the infinitude of primes. In 1959 S. Golomb ([3]) presented a similar proof of the infinitude of primes using a topology $\mathcal{D}$ on $\mathbb{N}$ with the base

$$\mathcal{B}_{\mathcal{D}} = \big\{\{an+b\} : (a,b) = 1\big\},$$

defined in 1953 by M. Brown ([1]). Ten years later A. M. Kirch ([5]) defined a topology $\mathcal{D}'$ on $\mathbb{N}$, weaker than Golomb's topology $\mathcal{D}$, with the base

$$\mathcal{B}_{\mathcal{D}'} = \big\{\{an+b\} : (a,b) = 1 \ \text{ and } \ a - \text{square-free}\big\}.$$

Both topologies $\mathcal{D}$ and $\mathcal{D}'$ are Hausdorff, the set $\mathbb{N}$ is connected in these topologies and locally connected in the topology $\mathcal{D}'$, but it is not locally connected in the topology $\mathcal{D}$ (see [3] and [5]). Recently I showed that the arithmetic progression $\{an + b\}$ is connected in the topology $\mathcal{D}$ if and only if $\Theta(a) \subset \Theta(b)$ (see [9, Theorem 3.3]). Moreover, I proved that all arithmetic progressions are connected in the topology $\mathcal{D}'$ ([9, Theorem 3.5]). Some properties of topologies $\mathcal{D}$ and $\mathcal{D}'$ were also described in [8, p. 82–84]. Topologies $\mathcal{D}$ and $\mathcal{D}'$ were called relatively prime integer topology and prime integer topology, respectively. However, I call the topology $\mathcal{D}$ *Golomb's topology* and the topology $\mathcal{D}'$ *Kirch's topology*.

In 1993 G. B. Rizza ([7]) introduced the division topology $\mathcal{T}'$ on $\mathbb{N}$ as follows: for $X \subset \mathbb{N}$ he put

$$g(X) = \operatorname{cl} X = \bigcup_{x\in X} D(x), \ \text{ where } D(x) = \{y \in \mathbb{N} : y \mid x\}.$$

The mapping g forms a topology $\mathcal{T}'$ on $\mathbb{N}$. It is easy to see that the family

$$\mathcal{B}_{\mathcal{T}'} = \big\{\{an\}\big\}$$

is a basis for this topology. In the paper [10] I defined a topology $\mathcal{T}$ on $\mathbb{N}$, stronger than the division topology $\mathcal{T}'$, with the base

$$\mathcal{B}_{\mathcal{T}} = \big\{\{an+b\} : \Theta(a) \subset \Theta(b)\big\}.$$

Both topologies $\mathcal{T}$ and $\mathcal{T}'$ are T_0–topology and they are not a T_1–topology, the set $\mathbb{N}$ is connected in these topologies and locally connected in the topology $\mathcal{T}'$, but it is not locally connected in the topology $\mathcal{T}$ (see [7] and [10]). Moreover I proved that the arithmetic progression $\{an+b\}$ is connected in the topology $\mathcal{T}$ if and only if $(a,b) = 1$ ([10, Theorem 3.4]) and I showed that all arithmetic progressions are connected in the topology $\mathcal{T}'$ ([10, Theorem 4.1]).

In this paper we characterize regular open arithmetic progressions in four mentioned topologies on the set of positive integers. Moreover we show that the set $\mathbb{N}$ is semiregular in stronger topologies $\mathcal{D}$ and $\mathcal{T}$ and it is not semiregular in weaker topologies $\mathcal{D}'$ and $\mathcal{T}'$. The motivation to our studies were results from [8] concerning regular open arithmetic progressions in Golomb's topology $\mathcal{D}$ on $\mathbb{N}$ and semiregularity of the space $(\mathbb{N}, \mathcal{D})$ (see [8, pages 82-83, Properties 10-12]). Unfortunately proofs of these results are not correct because of a mistake in the proof of Property 10. The authors did not take into account that if $b > p^k$, (where $p \in \mathcal{P}$, $k \in \mathbb{N}$, and $(p^k, b) = 1$), then $\mathrm{cl}\{p^k n + b\} \neq \{p^k n + b\} \cup \{pn\}$ (see [8, p. 83]). Indeed, e.g., $\mathrm{cl}\{3n+4\} = \{3n+1\} \cup \{3n\} \neq \{3n+4\} \cup \{3n\}$. For this reason properties from [8] must be proven in this paper.

3. Auxiliary lemmas

Lemma 3.1. *Let $a = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be the prime factorization of a and assume that $b \leq a$. Then there are numbers $b_1, \dots, b_k \in \mathbb{N}$ such that $b_i \leq p_i^{\alpha_i}$ for each $i \in \{1, \dots, k\}$ and*

$$\{an+b\} = \bigcap_{i \in \{1,\dots,k\}} \{p_i^{\alpha_i} n + b_i\}.$$

In particular, if $(a, b) = 1$, then $b_i < p_i^{\alpha_i}$ for each $i \in \{1, \dots, k\}$.

Proof. It easy to see that if $a = p_1^{\alpha_1} \dots p_k^{\alpha_k}$, where $p_1, \dots p_k \in \mathcal{P}$, then

$$\{an+b\} = \bigcap_{i \in \{1,\dots,k\}} \{p_i^{\alpha_i} n + b\}. \tag{3.1}$$

Observe that for each $i \in \{1, \dots, k\}$ there are $b_i \leq p_i^{\alpha_i}$ and $k_i \in \mathbb{N}_0$ such that $b = k_i p_i^{\alpha_i} + b_i$. So, $b \in \{p_i^{\alpha_i} n + b_i\}$ for each $i \in \{1, \dots, k\}$ and, by assumption, $1 \leq b \leq a$. Moreover, by the Chinese Remainder Theorem (CRT), there is exactly one number x such that $1 \leq x \leq a$ and

$$x \in \bigcap_{i \in \{1,\dots,k\}} \{p_i^{\alpha_i} n + b_i\}.$$

Consequently $x = b$, and by (3.1),

$$\{an+b\} = \bigcap_{i \in \{1,\dots,k\}} \{p_i^{\alpha_i} n + b_i\}.$$

Obviously, if $(a, b) = 1$, then $(p_i, b_i) = 1$, which shows that $b_i < p_i^{\alpha_i}$ for each $i \in \{1, \dots, k\}$. ◻

Lemma 3.2. *Assume that $\mathfrak{T} \in \{\mathcal{D}, \mathcal{D}', \mathcal{T}\}$ and U is $\mathfrak{T}$–open set. If $c \in U$, then there is an arithmetic progression $\{an+c\} \in \mathcal{B}_{\mathfrak{T}}$ such that $\{an+c\} \subset U$.*

Proof. Let $\mathfrak{T} \in \{\mathcal{D}, \mathcal{D}', \mathcal{T}\}$ and $c \in U$. Since the set U is $\mathfrak{T}$–open, there is an arithmetic progression $\{an+b\} \in \mathcal{B}_{\mathfrak{T}}$, such that $c \in \{an+b\} \subset U$. So, $\{an+c\} \subset \{an+b\} \subset U$. Observe that if $(a,b)=1$, then $(a,c)=1$, and if $\Theta(a) \subset \Theta(b)$, then $\Theta(a) \subset \Theta(c)$. Moreover, if $\mathfrak{T} = \mathcal{D}'$, then a is square-free. This implies that $\{an+c\} \in \mathcal{B}_{\mathfrak{T}}$. □

Lemma 3.3. *Assume that* $b_1 \equiv b (\operatorname{mod} a)$ *and* $b_1 \leq a$. *If* $\{an+b\} \in \mathcal{B}_{\mathfrak{T}}$, *then* $\{an+b_1\} \subset \operatorname{cl}\{an+b\}$ *in* $(\mathbb{N}, \mathfrak{T})$ *for* $\mathfrak{T} \in \{\mathcal{D}, \mathcal{D}', \mathcal{T}\}$.

Proof. Let $\mathfrak{T} \in \{\mathcal{D}, \mathcal{D}', \mathcal{T}\}$, $\{an+b\} \in \mathcal{B}_{\mathfrak{T}}$, and $x \in \{an+b_1\}$. We will show that $x \in \operatorname{cl}\{an+b\}$. Since $b_1 \equiv b (\operatorname{mod} a)$ and $b_1 \leq a$, we have $\{an+b\} \subset \{an+b_1\}$. Hence it is sufficient to assume that $x \in \{an+b_1\} \setminus \{an+b\}$. It follows that $\{an+b\} \subset \{an+x\}$. Moreover, if $(a,b)=1$, then $(a,x)=1$, and if $\Theta(a) \subset \Theta(b)$, then $\Theta(a) \subset \Theta(x)$.

Fix the set $U \in \mathfrak{T}$ such that $x \in U$. By Lemma 3.2, there is an arithmetic progression $\{cn+x\} \in \mathcal{B}_{\mathfrak{T}}$ with $\{cn+x\} \subset U$. Since progressions $\{an+x\}$ and $\{cn+x\}$ are $\mathfrak{T}$–open and $x \in \{an+x\} \cap \{cn+x\}$, using again Lemma 3.2, we obtain that there is an arithmetic progression $\{dn+x\} \in \mathcal{B}_{\mathfrak{T}}$ with $\{dn+x\} \subset \{an+x\} \cap \{cn+x\}$. Taking into account that the progression $\{dn+x\}$ is infinite and the set $\{an+x\} \setminus \{an+b\}$ is finite, we can conclude that $\{dn+x\} \cap \{an+b\} \neq \emptyset$. Hence $U \cap \{an+b\} \neq \emptyset$, which proves that $x \in \operatorname{cl}\{an+b\}$ in $(\mathbb{N}, \mathfrak{T})$. □

Lemma 3.4. *Let* $\mathfrak{T} \in \{\mathcal{D}, \mathcal{D}', \mathcal{T}\}$. *If the arithmetic progression* $\{an+b\}$ *is regular open in* $(\mathbb{N}, \mathfrak{T})$, *then* $\{an+b\} \in \mathcal{B}_{\mathfrak{T}}$ *and* $b \leq a$.

Proof. Assume that $\mathfrak{T} \in \{\mathcal{D}, \mathcal{D}', \mathcal{T}\}$ and $\operatorname{int}\operatorname{cl}\{an+b\} = \{an+b\}$ in $(\mathbb{N}, \mathfrak{T})$. Then $\{an+b\}$ is $\mathfrak{T}$–open. Since all $\mathfrak{T}$–open arithmetic progressions belong to the base $\mathcal{B}_{\mathfrak{T}}$, we have $\{an+b\} \in \mathcal{B}_{\mathfrak{T}}$. Now let $b_1 \equiv b (\operatorname{mod} a)$ and $b_1 \leq a$. By Lemma 3.3, $\{an+b_1\} \subset \operatorname{cl}\{an+b\}$. Moreover observe that, if $(a,b)=1$, then $(a,b_1)=1$, and if $\Theta(a) \subset \Theta(b)$, then $\Theta(a) \subset \Theta(b_1)$. So, the progression $\{an+b_1\} \in \mathfrak{T}$. Therefore

$$\{an+b_1\} = \operatorname{int}\{an+b_1\} \subset \operatorname{int}\operatorname{cl}\{an+b\} = \{an+b\},$$

which proves $b = b_1 \leq a$. □

Lemma 3.5. *Assume that* a *and* b *are odd. If* $\{an+b\} \in \mathcal{B}_{\mathfrak{T}}$, *then* $\operatorname{cl}\{an+b\} = \operatorname{cl}\{2an+b\}$ *in* $(\mathbb{N}, \mathfrak{T})$ *for* $\mathfrak{T} \in \{\mathcal{D}, \mathcal{D}'\}$.

Proof. Since $\operatorname{cl}\{2an+b\} \subset \operatorname{cl}\{an+b\}$, it is sufficient to show the opposite inclusion. Let $x \in \operatorname{cl}\{an+b\}$. Fix the set $U \in \mathfrak{T}$ such that $x \in U$. By Lemma 3.2, there is an arithmetic progression $\{cn+x\} \in \mathcal{B}_{\mathfrak{T}}$ with $\{cn+x\} \subset U$. Observe that

$$\{an+b\} = \{2an+b\} \cup \{2an+(a+b)\},$$

where progressions $\{2an+b\}$ and $\{2an+(a+b)\}$ are disjoint. Moreover, since a and b are odd, $a+b$ is even. The assumption $x \in \operatorname{cl}\{an+b\}$ implies that

$V := \{an+b\} \cap \{cn+x\} \neq \emptyset$. Since $V \in \mathfrak{T}$, there is an arithmetic progression $\{dn+e\} \in \mathcal{B}_{\mathfrak{T}}$ such that $\{dn+e\} \subset V$ and $(d,e) = 1$. If $\{dn+e\} \subset \{2an+(a+b)\}$, then d and e are even, which contradicts $(d,e)=1$. So,

$$\emptyset \neq \{dn+e\} \cap \{2an+b\} \subset U \cap \{2an+b\},$$

which proves that $x \in \mathrm{cl}\{2an+b\}$ in $(\mathbb{N}, \mathfrak{T})$. □

LEMMA 3.6. *Assume that* $\Theta(a) \subset \Theta(b)$. *If there are numbers* $p \in \mathcal{P}$ *and* $s \in \mathbb{N}$ *such that* $a = ps$ *and* $p^2 \nmid a$, *then* $\mathrm{cl}\{an+b\} \supset \{sn+b\}$ *in* $(\mathbb{N}, \mathcal{T})$.

PROOF. Let $x \in \{sn+b\}$. Then there is an $n_0 \in \mathbb{N}_0$ such that $x = sn_0+b$. If $n_0 = 0$, then $x = b \in \mathrm{cl}\{an+b\}$ in $(\mathbb{N}, \mathcal{T})$. So, we can assume that $n_0 \neq 0$. We consider two cases.

<u>*Case 1*</u>. $p \mid n_0$.

Then there is $k_0 \in \mathbb{N}_0$ such that $n_0 = pk_0$. Hence, since $a = ps$, we have

$$x = spk_0 + b = ak_0 + b \in \{an+b\} \subset \mathrm{cl}\{an+b\}$$

in $(\mathbb{N}, \mathcal{T})$.

<u>*Case 2*</u>. $p \nmid n_0$.

Then fix the set $U \in \mathcal{T}$ such that $x \in U$. By Lemma 3.2, there is an arithmetic progression $\{cn+x\} \in \mathcal{B}_{\mathcal{T}}$ with $\{cn+x\} \subset U$. If $p \mid c$, then $p \mid x$. But $x = sn_0+b$, $p \mid b$ and $p \nmid n_0$, whence $p \nmid x$, a contradiction. So, $(p,c) = 1$. Moreover, it is easy to see that

$$\{an+b\} = \{pn+b\} \cap \{sn+b\}$$

and

$$\{sn+x\} \cap \{cn+x\} = \{\mathrm{lcm}(s,c)n+x\}.$$

Since $p^2 \nmid a$ and $a = ps$, we have $(p,s) = 1$. Therefore $(p, \mathrm{lcm}(s,c)) = 1$. By CRT, we obtain that

$$\begin{aligned}\emptyset \neq \{pn+b\} \cap \{\mathrm{lcm}(s,c)n+x\} = \{pn+b\} \cap \{sn+x\} \cap \{cn+x\} \subset \\ \subset \{pn+b\} \cap \{sn+b\} \cap \{cn+x\} \subset \{an+b\} \cap U,\end{aligned}$$

which proves that $x \in \mathrm{cl}\{an+b\}$ in $(\mathbb{N}, \mathcal{T})$. □

LEMMA 3.7. *Assume that* $p \in \mathcal{P}$ *and* $k \in \mathbb{N}$. *If* $b < p^k$ *and* $(b, p^k) = 1$, *then* $\mathrm{cl}\{p^k n+b\} = \{p^k n+b\} \cup \{pn\}$ *in* $(\mathbb{N}, \mathcal{D})$.

PROOF. Let $x \in \mathrm{cl}\{p^k n+b\}$. Observe that

$$\mathbb{N} = \bigcup_{c=1}^{p^k} \{p^k n+c\}, \tag{3.2}$$

where all progressions $\{p^k n+c\}$ are pairwise disjoint. Fix a $c \in \{1, \ldots, p^k\} \setminus \{b\}$ such that $x \in \{p^k n+c\}$. If $(c, p^k) = 1$, then the progression $\{p^k n+c\}$ is $\mathcal{D}$–open and, by (3.2), $\{p^k n+b\} \cap \{p^k n+c\} = \emptyset$, which contradicts the assumption

$x \in \mathrm{cl}\{p^k n + b\}$. And if $(c, p^k) \neq 1$, then $p \mid c$, whence $\{p^k n + c\} \subset \{pn\}$. So, $x \in \{pn\}$ and consequently $\mathrm{cl}\{p^k n + b\} \subset \{p^k n + b\} \cup \{pn\}$ in $(\mathbb{N}, \mathcal{D})$.

Now we will show the opposite inclusion. Clearly $\{p^k n+b\} \subset \mathrm{cl}\{p^k n+b\}$. So, assume that $x \in \{pn\}$. Fix the set $U \in \mathcal{D}$ such that $x \in U$. By Lemma 3.2, there is an arithmetic progression $\{an+x\} \in \mathcal{B}_{\mathcal{D}}$ with $\{an+x\} \subset U$. Therefore $(a, x) = 1$, whence $(a, p^k) = 1$. By CRT, $\{an + x\} \cap \{p^k n + b\} \neq \emptyset$. It proves that $U \cap \{p^k n + b\} \neq \emptyset$, whence $x \in \mathrm{cl}\{p^k n + b\}$ in $(\mathbb{N}, \mathcal{D})$. ∎

Lemma 3.8. *Assume that $p \in \mathcal{P}$. If $b < p$, then* $\mathrm{cl}\{pn + b\} = \{pn + b\} \cup \{pn\}$ *in* $(\mathbb{N}, \mathcal{D}')$.

Proof. Since set $\{pn+b\} \cup \{pn\}$ is $\mathcal{D}'$–closed and $\mathcal{D}' \subset \mathcal{D}$, by Lemma 3.7 applied for $k = 1$, we have $\mathrm{cl}\{pn + b\} = \{pn + b\} \cup \{pn\}$ in $(\mathbb{N}, \mathcal{D}')$. ∎

Lemma 3.9. *Assume that $p \in \mathcal{P}$ and $k \in \mathbb{N}$. If $b \leq p^k$ and $p \mid b$, then*

$$\mathrm{cl}\{p^k n + b\} = \bigcup_{c \in \{1,\ldots,p-1\}} \{pn + c\} \cup \{p^k n + b\}$$

in $(\mathbb{N}, \mathcal{T})$.

Proof. Define

$$A := \bigcup_{c \in \{1,\ldots,p-1\}} \{pn + c\} \cup \{p^k n + b\}.$$

Observe that the set

$$\mathbb{N} \setminus A = \bigcup_{s \in S} \{p^k n + s\}, \quad \text{where} \quad S := \{s \leq p^k : p \mid s \wedge s \neq b\},$$

is $\mathcal{T}$–open and nonempty for $k \neq 1$. Hence A is $\mathcal{T}$–closed and since $\{p^k n+b\} \subset A$, we have $\mathrm{cl}\{p^k n + b\} \subset \mathrm{cl}\, A = A$ in $(\mathbb{N}, \mathcal{T})$.

Now we will show the opposite inclusion. Clearly $\{p^k n+b\} \subset \mathrm{cl}\{p^k n+b\}$. So, we can assume that $x \in A \setminus \{p^k n + b\}$. Observe that $(p, x) = 1$. Fix the set $U \in \mathcal{T}$ such that $x \in U$. By Lemma 3.2, there is an arithmetic progression $\{cn+x\} \subset U$ with $\Theta(c) \subset \Theta(x)$. If $p \mid c$, then $p \mid x$, a contradiction. Therefore $p \nmid c$, whence $(c, p^k) = 1$. So, by CRT, $\{cn + x\} \cap \{p^k n + b\} \neq \emptyset$. It proves that $U \cap \{p^k n + b\} \neq \emptyset$, whence $x \in \mathrm{cl}\{p^k n + b\}$ in $(\mathbb{N}, \mathcal{T})$. ∎

Lemma 3.10. *Assume that $p \in \mathcal{P}$, $k \in \mathbb{N}$, $p^k > 2$, and $b < p^k$. If $(p^k, b) = 1$, then the arithmetic progression $\{p^k n+b\}$ is regular open in* $(\mathbb{N}, \mathcal{D})$.

Proof. We will prove that $\mathrm{int}\,\mathrm{cl}\{p^k n + b\} = \{p^k n + b\}$ in $(\mathbb{N}, \mathcal{D})$. Since

$$\{p^k n + b\} = \mathrm{int}\{p^k n + b\} \subset \mathrm{int}\,\mathrm{cl}\{p^k n + b\},$$

it is sufficient to show the opposite inclusion. Let $x \in \mathrm{int}\,\mathrm{cl}\{p^k n + b\}$. Then there is $\mathcal{D}$–open set $U \subset \mathrm{cl}\{p^k n + b\}$ such that $x \in U$. By Lemma 3.7, we have $U \subset \{p^k n + b\} \cup \{pn\}$. Now suppose that $U \cap \{pn\} \neq \emptyset$. Then there is a $z \in U$ such that $p \mid z$. Since the set U is $\mathcal{D}$–open, by Lemma 3.2, there exists an arithmetic progression $\{tn+z\} \subset U$ with $(t,z)=1$. So, $(p,t)=1$, whence $(p^k,t)=1$. By CRT, $\{tn+z\}\cap\{p^kn+c\}\neq\emptyset$ for each $c\in\{1,\dots,p^k\}$. Consequently, since $\{p^kn\}\subset\{pn\}$, condition $p^k>2$ implies that

$$\big(\mathbb{N}\setminus\big(\{p^kn+b\}\cup\{pn\}\big)\big)\cap\{tn+z\}\subset\big(\mathbb{N}\setminus\big(\{p^kn+b\}\cup\{pn\}\big)\big)\cap U\neq\emptyset,$$

which is impossible. Therefore $x\in U\subset\{p^kn+b\}$, and we have $\operatorname{int}\operatorname{cl}\{p^kn+b\}\subset\{p^kn+b\}$ in $(\mathbb{N},\mathcal{D})$. This completes the proof. ∎

Lemma 3.11. *Let $p\in\mathcal{P}\setminus\{2\}$. If $b<p$, then the arithmetic progression $\{pn+b\}$ is regular open in $(\mathbb{N},\mathcal{D}')$.*

Proof. By Lemma 3.8, we have $\operatorname{int}\operatorname{cl}\{pn+b\}=\operatorname{int}\big(\{pn+b\}\cup\{pn\}\big)$ in $(\mathbb{N},\mathcal{D}')$. Since $\mathcal{D}'\subset\mathcal{D}$ and $\{pn+b\}\in\mathcal{D}'$, using Lemmas 3.10 and 3.7, we obtain that $\operatorname{int}\big(\{pn+b\}\cup\{pn\}\big)=\{pn+b\}$ in $(\mathbb{N},\mathcal{D}')$. It proves that the arithmetic progression $\{pn+b\}$ is regular open in $(\mathbb{N},\mathcal{D}')$. ∎

Lemma 3.12. *Assume that $p\in\mathcal{P}$, $k\in\mathbb{N}\setminus\{1\}$, and $b\leq p^k$. If $p\mid b$, then the arithmetic progression $\{p^kn+b\}$ is regular open in $(\mathbb{N},\mathcal{T})$.*

Proof. We will prove that $\operatorname{int}\operatorname{cl}\{p^kn+b\}=\{p^kn+b\}$ in $(\mathbb{N},\mathcal{T})$. Since

$$\{p^kn+b\}=\operatorname{int}\{p^kn+b\}\subset\operatorname{int}\operatorname{cl}\{p^kn+b\},$$

it is sufficient to show the opposite inclusion. Let $x\in\operatorname{int}\operatorname{cl}\{p^kn+b\}$. Then there is $\mathcal{T}$–open set $U\subset\operatorname{cl}\{p^kn+b\}$ such that $x\in U$. By Lemma 3.9, we have

$$U\subset\bigcup_{c\in\{1,\dots,p-1\}}\{pn+c\}\cup\{p^kn+b\}.$$

Now suppose that $U\cap\{pn+c\}\neq\emptyset$ for some $c\in\{1,\dots,p-1\}$. Then there is a $z\in U\cap\{pn+c\}$. Since $z\in U$, by Lemma 3.2, there is an arithmetic progression $\{\alpha n+z\}\subset U$ with with $\Theta(\alpha)\subset\Theta(z)$, and since $z\in\{pn+c\}$, we have $(p,z)=1$. If $p\mid\alpha$, then $p\mid z$, a contradiction. Therefore $p\nmid\alpha$, whence $(\alpha,p)=1$. So, by CRT, $\{\alpha n+z\}\cap\{p^kn+t\}\neq\emptyset$ for each $t\in\mathbb{N}$. It proves that

$$U\cap\Big(\mathbb{N}\setminus\bigcup_{c\in\{1,\dots,p-1\}}\{pn+c\}\cup\{p^kn+b\}\Big)\neq\emptyset,$$

which is impossible. Therefore $U\subset\{p^kn+b\}$, whence $x\in\{p^kn+b\}$. ∎

4. Main results

Theorem 4.1. *The arithmetic progression $\{an+b\}$ is regular open in $(\mathbb{N},\mathcal{D})$ if and only if $(a,b)=1$, $b<a$, and if a is even, then $4\mid a$.*

Proof. First assume that the arithmetic progression $\{an+b\}$ is regular open in $(\mathbb{N},\mathcal{D})$. By Lemma 3.4, $\{an+b\}\in\mathcal{B}_{\mathcal{D}}$ and $b\leq a$. Hence $(a,b)=1$ and $b<a$. Now suppose that a is even and $4\nmid a$. Then there exists an $a_1\in\mathbb{N}$ such that $a=2a_1$. Moreover, numbers a_1 and b are odd and $(a_1,b)=1$.

So, by Lemma 3.5, $\operatorname{cl}\{a_1n+b\} = \operatorname{cl}\{an+b\}$. Since $\{a_1n+b\} \in \mathcal{D}$ and $\operatorname{int}\operatorname{cl}\{an+b\} = \{an+b\}$, we have

$$\begin{aligned}\{an+b\} \subsetneq \{a_1n+b\} = \operatorname{int}\{a_1n+b\} \subset \operatorname{int}\operatorname{cl}\{a_1n+b\} = \operatorname{int}\operatorname{cl}\{an+b\}\\ = \{an+b\},\end{aligned}$$

a contradiction. Consequently, if a is even, then $4 \mid a$.

Now assume that $(a,b)=1$, $b<a$, and if a is even, then $4 \mid a$. Let $a = p_1^{\alpha_1}\dots p_k^{\alpha_k}$ be the prime factorization of a. By Lemma 3.1, there are numbers $b_1,\dots,b_k \in \mathbb{N}$ such that $b_i < p_i^{\alpha_i}$ for each $i \in \{1,\dots,k\}$ and

$$\{an+b\} = \bigcap_{i\in\{1,\dots,k\}} \{p_i^{\alpha_i}n+b_i\}. \tag{4.1}$$

Condition $(a,b)=1$ implies that $(p_i^{\alpha_i},b)=1$ for each $i \in \{1,\dots,k\}$, whence $(p_i^{\alpha_i},b_i)=1$ for each $i \in \{1,\dots,k\}$. Moreover, we know that if a is even, then $4 \mid a$. So, all $p_i^{\alpha_i} > 2$. By Lemma 3.10, the arithmetic progression $\{p_i^{\alpha_i}n+b_i\}$ is regular open in $(\mathbb{N},\mathcal{D})$ for each $i \in \{1,\dots,k\}$. Consequently, using condition (4.1), we obtain that the arithmetic progression $\{an+b\}$ is regular open in $(\mathbb{N},\mathcal{D})$, too. ∎

Corollary 4.2. *The space $(\mathbb{N},\mathcal{D})$ is semiregular.*

Proof. It is easy to see that $\mathcal{B}'_{\mathcal{D}} = \big\{\{an+b\} : (a,b)=1,\ \ b<a\big\}$ is the base of Golomb's topology $\mathcal{D}$ on $\mathbb{N}$, too. Moreover, each arithmetic progression $\{2cn+b\}$ such that $2 \nmid c$ and $(2c,b)=1$ may be written as

$$\{2cn+b\} = \{4cn+b\} \cup \{4cn+(2c+b)\}.$$

Since $(2c,b)=1$, we have $(4c,b)=1$ and $(4c,2c+b)=1$. Hence the family $\mathcal{B}''_{\mathcal{D}} = \mathcal{B}'_{\mathcal{D}} \setminus \big\{\{2cn+b\} : 2 \nmid c\big\}$ is the another base of $(\mathbb{N},\mathcal{D})$. So, by Theorem 4.1, the base $\mathcal{B}''_{\mathcal{D}}$ consists of regular open sets, which proves that $(\mathbb{N},\mathcal{D})$ is semiregular. ∎

Theorem 4.3. *The arithmetic progression $\{an+b\}$ is regular open in $(\mathbb{N},\mathcal{D}')$ if and only if $(a,b)=1$, $b<a$, and a is odd and square-free.*

Proof. First assume that the arithmetic progression $\{an+b\}$ is regular open in $(\mathbb{N},\mathcal{D}')$. By Lemma 3.4, $\{an+b\} \in \mathcal{B}_{\mathcal{D}'}$ and $b \le a$. Hence a is square-free, $(a,b)=1$, and $b<a$. Now suppose that a is even. Then there is a square-free number $a_1 \in \mathbb{N}$ such that $a = 2a_1$. Moreover, numbers a_1 and b are odd and $(a_1,b)=1$. So, by Lemma 3.5, $\operatorname{cl}\{a_1n+b\} = \operatorname{cl}\{an+b\}$. Since $\{a_1n+b\} \in \mathcal{D}'$ and $\operatorname{int}\operatorname{cl}\{an+b\} = \{an+b\}$, we have

$$\begin{aligned}\{an+b\} \subsetneq \{a_1n+b\} = \operatorname{int}\{a_1n+b\} \subset \operatorname{int}\operatorname{cl}\{a_1n+b\} = \operatorname{int}\operatorname{cl}\{an+b\}\\ = \{an+b\},\end{aligned}$$

a contradiction. So, a is odd.

Now assume that $(a, b) = 1$, $b < a$, and a is odd and square-free. Let $a = p_1 \dots p_k$ be the prime factorization of a. By Lemma 3.1, there are numbers $b_1, \dots, b_k \in \mathbb{N}$ such that $b_i < p_i$ for each $i \in \{1, \dots, k\}$ and

$$\{an + b\} = \bigcap_{i \in \{1,\dots,k\}} \{p_i n + b_i\}. \tag{4.2}$$

Condition $(a, b) = 1$ implies that $(p_i, b) = 1$ for each $i \in \{1, \dots, k\}$, whence $(p_i, b_i) = 1$ for each $i \in \{1, \dots, k\}$. Since a is odd, all $p_i > 2$. By Lemma 3.11, the arithmetic progression $\{p_i n + b_i\}$ is regular open in $(\mathbb{N}, \mathcal{D}')$ for each $i \in \{1, \dots, k\}$. Consequently, using condition (4.2), we obtain that the arithmetic progression $\{an + b\}$ is regular open in $(\mathbb{N}, \mathcal{D}')$, too. ∎

COROLLARY 4.4. *The space* $(\mathbb{N}, \mathcal{D}')$ *is not semiregular.*

PROOF. By Lemma 3.8, the arithmetic progression $\{2n+1\}$ is not regular open. Moreover, observe that $\{2n + 1\}$ consists of odd numbers only and $1 \in \{2n+1\}$. But all other arithmetic progressions, which are regular open in $(\mathbb{N}, \mathcal{D}')$ and to which 1 belongs, consist of both odd and even numbers. Hence there is no base of Kirch's topology $\mathcal{D}'$ on $\mathbb{N}$ consisting of regular open sets. This proves that the space $(\mathbb{N}, \mathcal{D}')$ is not semiregular. ∎

THEOREM 4.5. *The arithmetic progression* $\{an + b\}$ *is regular open in* $(\mathbb{N}, \mathcal{T})$ *if and only if* $\Theta(a) \subset \Theta(b)$, $b \le a$, *and if* $p \mid a$, *then* $p^2 \mid a$ *for each* $p \in \mathcal{P}$.

PROOF. First assume that the arithmetic progression $\{an + b\}$ is regular open in $(\mathbb{N}, \mathcal{T})$. By Lemma 3.4, $\{an + b\} \in \mathcal{B}_{\mathcal{T}}$ and $b \le a$. So, $\Theta(a) \subset \Theta(b)$. Now suppose that there are numbers $p \in \mathcal{P}$ and $s \in \mathbb{N}$ such that $a = ps$ and $p^2 \nmid a$. By Lemma 3.6, $\mathrm{cl}\{an + b\} \supset \{sn + b\}$ in $(\mathbb{N}, \mathcal{T})$. Since $\Theta(s) \subset \Theta(a) \subset \Theta(b)$, the arithmetic progression $\{sn + b\}$ is $\mathcal{T}$–open. Therefore

$$\{an + b\} \subsetneq \{sn + b\} = \mathrm{int}\{sn + b\} \subset \mathrm{int}\,\mathrm{cl}\{an + b\},$$

a contradiction. Consequently, if $p \mid a$, then $p^2 \mid a$ for each $p \in \mathcal{P}$.

Now assume that $\Theta(a) \subset \Theta(b)$, $b \le a$, and if $p \mid a$, then $p^2 \mid a$ for each $p \in \mathcal{P}$. Let $a = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ be the prime factorization of a. By Lemma 3.1, there are numbers $b_1, \dots, b_k \in \mathbb{N}$ such that $b_i \le p_i^{\alpha_i}$ for each $i \in \{1, \dots, k\}$ and

$$\{an + b\} = \bigcap_{i \in \{1,\dots,k\}} \{p_i^{\alpha_i} n + b_i\}. \tag{4.3}$$

Condition $\Theta(a) \subset \Theta(b)$ implies that $\Theta(p_i^{\alpha_i}) \subset \Theta(b)$ for each $i \in \{1, \dots, k\}$, whence $p_i \mid b_i$ for each $i \in \{1, \dots, k\}$. Moreover, we know that if $p \mid a$, then $p^2 \mid a$ for each $p \in \mathcal{P}$. So, all $\alpha_i > 1$. By Lemma 3.12, the arithmetic progression $\{p_i^{\alpha_i} n + b_i\}$ is regular open in $(\mathbb{N}, \mathcal{T})$ for each $i \in \{1, \dots, k\}$. Consequently, using condition (4.3), we obtain that the arithmetic progression $\{an + b\}$ is regular open in $(\mathbb{N}, \mathcal{T})$, too. ∎

Corollary 4.6. *The space* $(\mathbb{N}, \mathcal{T})$ *is semiregular.*

Proof. It is easy to see that $\mathcal{B}'_{\mathcal{T}} = \big\{\{an+b\} : \Theta(a) \subset \Theta(b),\ \ b \le a\big\}$ is the base of topology $\mathcal{T}$ on $\mathbb{N}$, too. Moreover, each arithmetic progression $\{an+b\}$ may be written as

$$\{an+b\} = \bigcup_{k=0}^{a-1} \{a^2 n + (ka+b)\}.$$

Clearly $\Theta(a) \subset \Theta(ka+b)$, for each $k \in \{0, \ldots, a-1\}$. Hence the family

$$\mathcal{B}''_{\mathcal{T}} = \mathcal{B}'_{\mathcal{T}} \setminus \big\{\{an+b\} : \exists p \in \mathcal{P} \text{ such that } p \mid a \wedge p^2 \nmid a\big\}$$

is the another base of $(\mathbb{N}, \mathcal{T})$. So, by Theorem 4.5, the base $\mathcal{B}''_{\mathcal{T}}$ consists of regular open sets, which proves that $(\mathbb{N}, \mathcal{T})$ is semiregular. $\square$

Theorem 4.7. *Every arithmetic progression is not regular open in* $(\mathbb{N}, \mathcal{T}')$.

Proof. Since the arithmetic progression $\{an+b\}$ is $\mathcal{T}'$–open if and only if $b = a$, and since $\mathrm{cl}\{an\} = \mathbb{N}$ in $(\mathbb{N}, \mathcal{T}')$, there is not regular open arithmetic progression in the division topology $(\mathbb{N}, \mathcal{T}')$. $\square$

An immediate consequence of Theorem 4.7 is the following corollary.

Corollary 4.8. *The space* $(\mathbb{N}, \mathcal{T}')$ *is not semiregular.*

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Received: 8.1.2013.
Revised: 23.5.2013.