GLASNIK MATEMATIČKI
Vol. 49(69)(2014), 25 – 36

THE EXTENDIBILITY OF DIOPHANTINE PAIRS I: THE GENERAL CASE

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Abstract. Let a and b be positive integers with $a < b$, such that $ab+1$ is a perfect square. In this paper we give an upper bound for the minimal positive integer c such that $\{a, b, c, d\}$ is the set of positive integers which has the property that the product of any two of its elements increased by 1 is a perfect square and $d \neq a+b+c+2(abc \pm \sqrt{(ab+1)(ac+1)(bc+1)})$.

1. Introduction

A set $\{a_1, a_2, \ldots, a_m\}$ of m positive integers is called a *Diophantine m-tuple* if $a_ia_j + 1$ is a perfect square for all i, j with $1 \leq i < j \leq m$. A folklore conjecture says that there does not exist a Diophantine quintuple.

Arkin, Hoggatt and Strauss [1] found that any Diophantine triple can be extended to a Diophantine quadruple. More precisely, if $\{a, b, c\}$ is a Diophantine triple, then $\{a, b, c, d_+\}$ is a Diophantine quadruple, where

$$d_+ = a + b + c + 2abc + 2rst \tag{1.1}$$

and r, s, t are the positive integers satisfying

$$ab + 1 = r^2, \ \ ac + 1 = s^2, \ \ bc + 1 = t^2.$$

We call such a Diophantine quadruple *regular*. Recently, Dujella ([6]) proved that there does not exist a Diophantine sextuple and that there exist only finitely many Diophantine quintuples. The most recent results concerning the problems with Diophantine m-tuples and also the rich history can be found on Dujella's webpage [4]. The following is a strong version of the folklore conjecture.

2010 *Mathematics Subject Classification.* 11D09, 11J68.
Key words and phrases. Diophantine tuples, simultaneous Diophantine equations.

Conjecture 1.1. *Any Diophantine quadruple is regular.*

For a Diophantine triple $\{a, b, c\}$ with $a < b < c$, put

$$d_- = a + b + c + 2abc - 2rst. \tag{1.2}$$

Then, $d_- > 0$ if and only if $c > a + b + 2r$ and in this case $\{a, b, c, d_-\}$ is a Diophantine quadruple which is also regular with c the fourth (largest) element. The aim of this paper is to give an upper bound for the third element c which is "minimal" in some sense. More precisely, for a fixed Diophantine pair $\{a, b\}$ with $a < b$, we give an upper bound for minimal c such that $\{a, b, c, d\}$ is an irregular Diophantine quadruple with $b < c < d$.

Theorem 1.2. *Let $\{a, b, c\}$ be a Diophantine triple with $a < b$. Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $d > d_+$ and that $\{a, b, c', c\}$ is not a Diophantine quadruple for any c' with $0 < c' < d_-$, where d_+ and d_- are defined by* (1.1) *and* (1.2)*, respectively.*

(1) *If $b < 2a$, then $c < b^6$.*
(2) *If $2a \le b \le 8a$, then $c < 9.5b^4$.*
(3) *If $b > 8a$, then $c < b^5$.*

The proof of Theorem 1.2 needs an improvement of Rickert's theorem (see [10, Theorem 2.5] and Section 3) and the reduction method of Baker and Davenport (see [2], [7, Lemma 5] and Proof of Theorem 1.2 in Section 4). Note that if $\{a, b, c, d\}$ is a Diophantine quadruple with $a < b < c < d_+ < d$, then $b \ge 8$ by [3], [7], and [9]. Theorem 1.2 implies that in order to see whether Conjecture 1.1 holds for a fixed Diophantine pair $\{a, b\}$, one can check the extendibility of Diophantine triples $\{a, b, c\}$ only for small c. In particular, one may expect that Conjecture 1.1 can be shown to hold for some parametric families a, b with $a < b \le 8a$, since, then, the possibilities for the third element c are completely determined (see Lemma 4.1). For example, it is not difficult to see that the non-extendibility of the Diophantine pair $\{k^2 - 1, k^2 + 2k\}$ with an integer $k \ge 2$ or $\{F_{2j}, F_{2j+2}\}$ with a positive integer j (where F_ν denotes the ν-th Fibonacci number) can be reduced to that of the Diophantine triple $\{k^2 - 1, k^2 + 2k, c\}$ or $\{F_{2j}, F_{2j+2}, c\}$ with $c \le c_3^+$, respectively (see Section 4 for the definition of c_3^+). Various parametric families containing these examples will be treated in a subsequent paper. Those families can be solved using the known methods, but it will save arguments. In addition, maybe more interestingly, Theorem 1.2 will be used in our subsequent paper to prove the uniqueness of the extension of Diophantine triple $\{a, b, c\}$, where $a < b < a + 4\sqrt{a}$. The organization of this paper is as follows. In Section 2, we prove some results that help to obtain lower bounds for the solutions of the problem according to three cases: $b < 2a$, $2a \le b \le 8a$, and $b > 8a$. In fact, we first transform the problem into a system of Diophantine equations with the condition $c \ge \min\{9.5b^4, b^5\}$ and then we obtain some lower bounds of the index n of the corresponding sequences. In Section 3, we show a key result (Proposition 3.3) by determining some upper bounds of b in the order 10^5. The upper bounds obtained are so low that we can use the reduction method to completely prove Theorem 1.2. This is done in Section 4 by the means of a program written in `Mathematica`.

2. Lower bounds for solutions

Let $\{a, b, c\}$ be a Diophantine triple, and r, s, t positive integers satisfying $ab+1 = r^2$, $ac+1 = s^2$, $bc+1 = t^2$. Suppose that $\{a, b, c, d\}$ is a Diophantine quadruple with $d_+ < d$. Then, there exist positive integers x, y, z such that $ad + 1 = x^2$, $bd + 1 = y^2$, $cd + 1 = z^2$, from which we obtain

$$az^2 - cx^2 = a - c, \tag{2.1}$$

$$bz^2 - cy^2 = b - c. \tag{2.2}$$

The positive solutions of Diophantine equations (2.1) and (2.2) respectively verify:

$$z\sqrt{a} + x\sqrt{c} = (z_0\sqrt{a} + x_0\sqrt{c})(s + \sqrt{ac})^m, \tag{2.3}$$

$$z\sqrt{b} + y\sqrt{c} = (z_1\sqrt{b} + y_1\sqrt{c})(t + \sqrt{bc})^n, \tag{2.4}$$

where m, n are non-negative integers, and (z_0, x_0), (z_1, y_1) are solutions of (2.1), (2.2), respectively satisfying

$$1 \le x_0 < \sqrt{\frac{s+1}{2}}, \qquad 1 \le |z_0| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{a}}},$$

$$1 \le y_1 < \sqrt{\frac{t+1}{2}}, \qquad 1 \le |z_1| < \sqrt{\frac{c\sqrt{c}}{2\sqrt{b}}}$$

(see [6, Lemma 1]). Thus, we have $z = v_m = w_n$, where

$$v_0 = z_0, \;\; v_1 = sz_0 + cx_0, \;\; v_{m+2} = 2sv_{m+1} - v_m,$$

$$w_0 = z_1, \;\; w_1 = tz_1 + cy_1, \;\; v_{n+2} = 2tw_{n+1} - w_n. \tag{2.5}$$

In what follows, we assume that

(2.6)

$$\{a, b, c', c\} \text{ is not a Diophantine quadruple for any } c' \text{ with } 0 < c' < d_-,$$

where $d_- = a+b+c+2abc-2rst$, in order to narrow the possibilities for the fundamental solutions (z_0, x_0) and (z_1, y_1).

Lemma 2.1. *Assume* (2.6) *and* $c \ge \min\{9.5b^4, b^5\}$. *Then,* $v_{2m+1} \ne w_{2n}$ *and* $v_{2m} \ne w_{2n+1}$. *Moreover, we obtain the following:*

(i) *if* $v_{2m} = w_{2n}$, *then* $z_0 = z_1$ *and* $|z_0| = |z_1| = 1$,

(ii) *if* $v_{2m+1} = w_{2n+1}$, *then* $|z_0| = t$, $|z_1| = s$ *and* $z_0z_1 > 0$.

PROOF. The first assertions $v_{2m+1} \neq w_{2n}$ and $v_{2m} \neq w_{2n+1}$ follow from the same argument as the proof of [6, Lemma 8, (2) and (3)] (see also the proof of [10, Lemma 18]). (i) If $v_{2m} = w_{2n}$, then we have $z_0 = z_1$ by [5, Lemma 3]. Put $d_0 = (z_0^2 - 1)/c$ and suppose that $|z_0| > 1$. Then $\{a, b, c, d_0\}$ is a Diophantine quadruple. If $|z_0| = cr - st$, then we will arrive at a contradiction in the same way as the proof of [6, Lemma 10] ([6, (1.2), p. 198]). If $\{a, b, c, d_0\}$ is an irregular Diophantine quadruple, then since $d_0 < c$, this contradicts assumption (2.6). Hence, we obtain $|z_0| = |z_1| = 1$. (ii) This is exactly [6, Lemma 8, (4)]. ◻

LEMMA 2.2. *Assume that* (2.6) *holds.*

(i) *If* $v_{2m} = w_{2n}$, *then* $m < \begin{cases} 1.17n & \text{if } c \geq b^6, \\ 1.25n & \text{if } c \geq 9.5b^4, \\ 1.2n & \text{if } c \geq b^5. \end{cases}$

(ii) *If* $v_{2m+1} = w_{2n+1}$, *then* $m < \begin{cases} 1.17n + 0.17 & \text{if } c \geq b^6, \\ 1.25n + 0.25 & \text{if } c \geq 9.5b^4, \\ 1.2n + 0.2 & \text{if } c \geq b^5. \end{cases}$

PROOF. The proof proceeds along the same lines as that of [6, Lemma 4]. (i) Suppose that $v_{2m} = w_{2n}$. Since

$$v_{2m} > v_1(2s-1)^{2m-1} \geq (c-s)(2s-1)^{2m-1},$$
$$w_{2n} < w_1(2t)^{2n-1} \leq (c+t)(2t)^{2n-1},$$

and $c \geq \min\{9.5b^4, b^5\}$ in any case, we have $(2s-1)^{2m-1} < 1.1(2t)^{2n-1}$. We now have

$$1.1(2t)^{2n-1} = 1.1 \cdot 2^{2n-1}(bc+1)^{n-1/2} < 2.001^{2n}(bc)^{n-1/2}$$

and

$$\begin{aligned}(2s-1)^{2m-1} &= \left(\frac{2\sqrt{ac+1}-1}{\sqrt{ac}}\right)^{2m-1} (ac)^{m-1/2} \\ &> \begin{cases} 1.998^{2m-1}(ac)^{m-1/2} & \text{if } c \geq b^6, \\ 1.994^{2m-1}(ac)^{m-1/2} & \text{if } c \geq \min\{9.5b^4, b^5\}. \end{cases}\end{aligned}$$

In the case of $c \geq b^6$, we have $1.998^{2m-1}c^m < 2.001^{2n}c^{(14n-1)/12}$, which implies that either $m < (14n-1)/12$ or $1.998^{2m-1} < 2.001^{2n}$ holds, that is, $m < \max\{7n/6 - 1/12, 1.003n + 0.5\}$. If $n = 2$, then $m < \max\{2.25, 2.506\} = 2.506$, yielding $m \leq 2 = n$; if $n \geq 3$, then $m < \max\{7n/6, (1.003 + 0.5/3)n\} < 1.17n$, which gives the desired upper bound for m. Similarly, in the case where $c \geq 9.5b^4$ or $c \geq b^5$, we have

$$1.994^{2m-1}c^m < \begin{cases} 1.511^{2n+1}c^{(10n-1)/8} & \text{if } c \geq 9.5b^4, \\ 2.001^{2n}c^{(12n-1)/10} & \text{if } c \geq b^5, \end{cases}$$

and obtain the assertion. (ii) Suppose that $v_{2m+1} = w_{2n+1}$. Since

$$\begin{aligned} v_{2m+1} &> v_1(2s-1)^{2m} \geq (cr-st)(2s-1)^{2m}, \\ w_{2n+1} &< w_1(2t)^{2n} \leq (cr+st)(2t)^{2n}, \end{aligned}$$

and

$$\begin{aligned} \frac{cr+st}{cr-st} &= \frac{(2ab+1)c^2+(a+b)c+2crst+1}{c^2-(a+b)c-1} \\ &< \frac{4.248}{0.999}ab < 4.253ab, \end{aligned}$$

we have $(2s-1)^{2m} < 4.253ab(2t)^{2n}$. In the same way as (i), we have

$$\begin{aligned} 1.994^{2m}c^m &< 4.253 \cdot 2.001^{2n+1}b^{n+1}c^n \\ &< \begin{cases} 2.001^{2n+3.1}c^{(7n+1)/6} & \text{if } c \geq b^6, \\ 1.511^{2n+3.9}c^{(5n+1)/4} & \text{if } c \geq 9.5b^4, \\ 2.001^{2n+3.1}c^{(6n+1)/5} & \text{if } c \geq b^5. \end{cases} \end{aligned}$$

The assertion now follows immediately from these inequalities. □

LEMMA 2.3. *Assume* (2.6).

(i) *If* $v_{2m} = w_{2n}$, *then*

$$n > \begin{cases} a^{-1/2}c^{1/8} & \text{if } b < 2a \text{ and } c \geq b^6, \\ 0.131a^{1/2}b^{-1}c^{1/2} & \text{if } b \geq 2a \text{ and } c \geq 9.5b^4, \\ 1.356a^{1/2}b^{-1}c^{1/2} & \text{if } b > 8a \text{ and } c \geq b^5. \end{cases}$$

(ii) *If* $v_{2m+1} = w_{2n+1}$, *then*

$$n > \min\left\{\alpha a^{-1/2}b^{-1/4}c^{1/4}, 0.816b^{-3/4}c^{1/4}\right\},$$

where

$$\alpha = \begin{cases} 0.673 & \text{if } b < 2a \text{ and } c \geq b^6, \\ 0.622 & \text{if } b \geq 2a \text{ and } c \geq 9.5b^4, \\ 0.653 & \text{if } b > 8a \text{ and } c \geq b^5. \end{cases}$$

PROOF. The proof proceeds along the same line as that of [6, Lemma 10]. Note that we may assume that $m \geq n$, $m \geq 2$ and $n \geq 2$ in both cases of (i) and (ii) (see [6, Lemma 3] and [10, Lemma 8] or [6, Lemmas 5, 7]). (i) Suppose that $v_{2m} = w_{2n}$. We see from [6, Lemma 9] with $z_0 = z_1 = \pm 1$ and $x_0 = y_1 = 1$ that

$$\pm am^2 + sm \equiv \pm bn^2 + tn \pmod{4c}. \tag{2.7}$$

Consider first the case where $b < 2a$ and $c \geq b^6$. Suppose that $n \leq a^{-1/2}c^{1/8}$. Since $m < 1.17n$ by Lemma 2.2 and $c \geq b^6 \geq 8^6$, it is easy to see that

$$am^2 < c, \ \ sm < c, \ \ bn^2 < c, \ \ tn < c. \tag{2.8}$$

It follows from (2.7) that

$$\pm am^2 + sm = \pm bn^2 + tn. \tag{2.9}$$

Moreover, squaring both sides of (2.7) twice, we have

$$\left\{(am^2 - bn^2)^2 - (m^2 + n^2)\right\}^2 \equiv 4m^2n^2 \pmod{c}. \tag{2.10}$$

Since

$$\left\{(am^2 - bn^2)^2 - (m^2 + n^2)\right\}^2 < (an^2)^4 < c,$$
$$4m^2n^2 \leq 4 \cdot 1.17^2 a^{-2} c^{1/2} < c,$$

(2.10) is in fact an equation, and hence

$$am^2 - bn^2 = \pm(m \pm n).$$

If $am^2 - bn^2 = \pm(m+n)$, then (2.9) implies $m(s \pm 1) = n(t \mp 1)$, and we have

$$\left\{a\left(\frac{t \mp 1}{s \pm 1}\right)^2 - b\right\} n = \pm\left(\frac{t \mp 1}{s \pm 1} + 1\right),$$

which yields either

$$n = \frac{(s+1)(s+t)}{2(at + bs + b - a)} > \frac{ac}{b(2s+1)} > 0.24a^{-1/2}c^{1/2} > a^{-1/2}c^{1/8}$$

or

$$n = \frac{(s+t)(s-1)}{2(at + bs + a - b)} > \frac{s-1}{2b} > 0.24a^{-1/2}c^{1/2} > a^{-1/2}c^{1/8}.$$

Hence we obtain a contradiction. If $am^2 - bn^2 = \pm(m - n)$, then we obtain a contradiction similarly

$$n = \frac{(t-s)(s \pm 1)}{2(bs - at \pm (b-a))} > \frac{t-s}{2(b-a)} = \frac{c}{2(s+t)} > 0.24b^{-1/2}c^{1/2} > a^{-1/2}c^{1/8}.$$

Therefore, if $b < 2a$ and $c \geq b^6$, then $n > a^{-1/2}c^{1/8}$. Consider secondly the case where $b \geq 2a$ and $c \geq 9.5b^4$. Suppose that $n \leq 0.131a^{1/2}b^{-1}c^{1/2}$. Using Lemma 2.2 and $c \geq 9.5b^4 \geq 9.5 \cdot 8^4$, one may easily check (2.8) and from (2.7) obtain (2.9). If $z_0 = 1$, then $am^2 + sm = bn^2 + tn$. Since $b \geq 2a$ and $m < 1.25n$, we have

$$bn^2 + tn > 2an^2 + 1.414sn,$$
$$am^2 + sm < 1.57an^2 + 1.25sn.$$

This leads to a contradiction. If $z_0 = -1$, then $t/m - s/n = bn/m - am/n < b$, while

$$\frac{t}{m} - \frac{s}{n} > \left(\frac{1}{1.25}\frac{t}{s} - 1\right)\frac{s}{n} > b.$$

This is also a contradiction. Therefore, if $b \geq 2a$ and $c \geq 9.5b^4$, then $n > 0.131a^{1/2}b^{-1}c^{1/2}$. Finally, the case where $b > 8a$ and $c \geq b^5$ can be shown in the same way as above, by noting $t > 2\sqrt{2}s$. (ii) Suppose that $v_{2m+1} = w_{2n+1}$. We see from [6, Lemma 9] with $z_0 = \pm t$, $z_1 = \pm s$ and $x_0 = y_1 = r$ that

$$\pm astm(m+1) + rm \equiv \pm bstn(n+1) + rn \pmod{2c}.$$

Multiplying this congruence by s and by t, respectively, we obtain

$$\pm atm(m+1) + rsm \equiv \pm btn(n+1) + rsn \pmod{2c}, \tag{2.11}$$

$$\pm asm(m+1) + rtm \equiv \pm bsn(n+1) + rtn \pmod{2c}. \tag{2.12}$$

Consider the case where $c \geq b^6$. Suppose that

$$n \leq \min\{0.673a^{-1/2}b^{-1/4}c^{1/4}, 0.816b^{-3/4}c^{1/4}\}.$$

Since $m < 1.17n + 0.17$ by Lemma 2.2 and $c \geq b^6 \geq 8^6$, we have

$$\begin{aligned} atm(m+1) &< \sqrt{1+\frac{1}{bc}}\left(1.17+\frac{0.17}{n}\right)\left(1.17+\frac{1.17}{n}\right)ab^{1/2}c^{1/2}n^2 < c, \\ btn(n+1) &< \sqrt{1+\frac{1}{bc}}\left(1+\frac{1}{n}\right)b^{3/2}c^{1/2}n^2 < c, \\ rtm &< \sqrt{1+\frac{1}{ab}}\sqrt{1+\frac{1}{bc}}\left(1.17+\frac{0.17}{n}\right)a^{1/2}bc^{1/2}n < c. \end{aligned}$$

Hence, congruences (2.11) and (2.12) are in fact equations, and we obtain

$$rm(s^2 - t^2) = rn(s^2 - t^2),$$
$$am(m+1)(t^2 - s^2) = bn(n+1)(t^2 - s^2),$$

that is, $m = n$ and $am(m+1) = bn(n+1)$, which contradict $m > 0$ and $a < b$. Therefore, if $c \geq b^6$, then $n > \min\{0.673a^{-1/2}b^{-1/4}c^{1/4}, 0.816b^{-3/4}c^{1/4}\}$. Similarly, in the case where $c \geq 9.5b^4$ or $c \geq b^5$, we will arrive to a contradiction if we suppose that $n \leq \min\{\alpha a^{-1/2}b^{-1/4}c^{1/4}, 0.816b^{-3/4}c^{1/4}\}$. This completes the proof of Lemma 2.3. ∎

3. Upper bounds for the second elements

First of all, we quote the lemma giving an upper bound for z, which is shown by using an improvement of Rickert's theorem ([8, Theorem 2.5])

Lemma 3.1. ([8, Lemma 2.9]) *Let $\{a,b,c,d\}$ be a Diophantine quadruple with $a < b < c < d$. Assume that $c > 9.5a'b(b-a)^2/a$, where $a' = \max\{a, b-a\}$. Then,*

$$\log z < \frac{4\log(4.001a^{1/2}(a')^{1/2}b^2c)\log(1.299a^{1/2}b^{1/2}(b-a)^{-1}c)}{\log(0.1053a(a')^{-1}b^{-1}(b-a)^{-2}c)}.$$

Lemma 3.2. *Assume that $c \geq \min\{9.5b^4, b^5\}$. If $z = v_{m'} = w_{n'}$ with $(m', n') \in \{(2m, 2n), (2m+1, 2n+1)\}$, then*

$$\log z > \frac{n'}{2}\log(4bc).$$

PROOF. One verifies that $y_1\sqrt{c}-|z_1|\sqrt{b}>2\sqrt{b}$ and $w_{n'}>(t+\sqrt{bc})^{n'}>(4bc)^{n'/2}$ in the same way as the proof of [5, Theorem 3]. □

PROPOSITION 3.3. *Assume* (2.6).

(1) *If* $b<2a$ *and* $c\geq b^6$*, then* $b<2.1\cdot 10^4$.
(2) *If* $2a\leq b\leq 8a$ *and* $c\geq 9.5b^4$*, then* $b<1.3\cdot 10^5$.
(3) *If* $b>8a$ *and* $c\geq b^5$*, then* $b<2\cdot 10^3$.

PROOF. Since $c\geq \min\{9.5b^4,b^5\}$ in any case, we have $c>9.5a'b(b-a)^2/a$ (note that $b=8$ or $b\geq 10$) and we may apply Lemma 3.1, together with Lemma 3.2 implies that

$$\frac{n'}{8}<\frac{\log(4.001a^{1/2}(a')^{1/2}b^2c)\log(1.299a^{1/2}b^{1/2}(b-a)^{-1}c)}{\log(4bc)\log(0.1053a(a')^{-1}b^{-1}(b-a)^{-2}c)},\tag{3.1}$$

where $n'\in\{2n,2n+1\}$ and $a'=\max\{a,b-a\}$. (1) Assume that $b<2a$ and $c\geq b^6$. Since $b/2<a'=a<b$ and $7\leq b-a<b/2$ ($b=a+7$ holds only if $\{a,b\}=\{8,15\}$), we have

$$4.001a^{1/2}(a')^{1/2}b^2c<4.001b^{1/2}b^{1/2}b^2c=4.001b^3c,$$
$$1.299a^{1/2}b^{1/2}(b-a)^{-1}c<1.299b^{1/2}b^{1/2}7^{-1}c<0.1856bc,$$
$$0.1053a(a')^{-1}b^{-1}(b-a)^{-2}c>0.1053b^{-1}\left(\frac{b}{2}\right)^{-2}c=0.4212b^{-3}c,$$

which together with (3.1) imply that

$$\frac{n'}{8}<\frac{\log(4.001b^3c)\log(0.1856bc)}{\log(4bc)\log(0.4212b^{-3}c)}=:f(c).$$

Since $f(c)$ is a decreasing function with respect to c, we have $f(c)\leq f(b^6)$ and thus

$$\frac{n'}{8}<\frac{\log(4.001b^9)\log(0.1856b^7)}{\log(4b^7)\log(0.4212b^3)}<\frac{9\cdot 7}{7\cdot 3}f_1(b)=3f_1(b),\tag{3.2}$$

where

$$f_1(b)=\frac{\log(1.167b)\log(0.7862b)}{\log(1.219b)\log(0.7495b)}.$$

(i) If $v_{2m}=w_{2n}$, then (3.2) and Lemma 2.3 together imply that $3f_1(b)>0.25a^{-1/2}c^{1/8}>0.25b^{1/4}$. Since $f_1(b)$ is decreasing, if $b\geq 2.1\cdot 10^4$, then $3f_1(b)\leq 3f_1(2.1\cdot 10^4)<3.002$, which contradicts $0.25b^{1/4}\geq 0.25(2.1\cdot 10^4)^{1/4}>3.009$. Therefore, $b<2.1\cdot 10^4$. (ii) If $v_{2m+1}=w_{2n+1}$, then (3.2) and Lemma 2.3 together imply that

$$\min\left\{0.16825a^{-1/2}b^{-1/4}c^{1/4},0.204b^{-3/4}c^{1/4}\right\}+\frac{1}{8}<3f_1(b).$$

Since

$$0.16825a^{-1/2}b^{-1/4}c^{1/4}>0.16825b^{3/4},\quad 0.204b^{-3/4}c^{1/4}\geq 0.204b^{3/4},$$

we have $b^{3/4} < 17.9f_1(b) - 0.742$. Since $f_1(b)$ is decreasing, we see that $b \le 48$. (2) Assume that $2a \le b \le 8a$ and $c \ge 9.5b^4$. Since $b/2 \le a' = b - a \le 7b/8$ and $b/8 \le a \le b/2$, we have

$$4.001a^{1/2}(a')^{1/2}b^2c \le 4.001\left(\frac{b}{2}\right)^{1/2}\left(\frac{7}{8}b\right)^{1/2}b^2c < 2.647b^3c,$$

$$1.299a^{1/2}b^{1/2}(b-a)^{-1}c \le 1.299\left(\frac{b}{2}\right)^{1/2}b^{1/2}\left(\frac{b}{2}\right)^{-1}c < 1.838c,$$

$$0.1053a(a')^{-1}b^{-1}(b-a)^{-2}c \ge 0.1053\cdot\frac{b}{8}\left(\frac{7}{8}b\right)^{-1}b^{-1}\left(\frac{7}{8}b\right)^{-2}c > 0.0196b^{-3}c,$$

which together with (3.1) imply that

$$\frac{n'}{8} < \frac{\log(2.647b^3c)\log(1.838c)}{\log(4bc)\log(0.0196b^{-3}c)} =: g(c).$$

Since $g(c)$ is decreasing with respect to c, we have $g(c) \le g(9.5b^4)$ and thus

$$\frac{n'}{8} < \frac{\log(25.1465b^7)\log(17.461b^4)}{\log(38b^5)\log(0.1862b)} < \frac{7\cdot 4}{5}g_1(b) = 5.6g_1(b), \tag{3.3}$$

where

$$g_1(b) = \frac{\log(1.586b)\log(2.045b)}{\log(2.069b)\log(0.1862b)}.$$

(i) If $v_{2m} = w_{2n}$, then (3.3) and Lemma 2.3 together show that

$$0.03275a^{1/2}b^{-1}c^{1/2} < 5.6g_1(b).$$

Since

$$0.03275a^{1/2}b^{-1}c^{1/2} \ge 0.03275\left(\frac{b}{8}\right)^{1/2}b^{-1}\left(9.5b^4\right)^{1/2} > 0.03568b^{3/2},$$

we have $b^{3/2} < 157g_1(b)$. Since $g_1(b)$ is decreasing, we see that $b \le 46$. Therefore, if $v_{2m} = w_{2n}$, then $b \le 46$. (ii) If $v_{2m+1} = w_{2n+1}$, then (3.3) and Lemma 2.3 together give

$$\min\left\{0.1555a^{-1/2}b^{-1/4}c^{1/4}, 0.204b^{-3/4}c^{1/4}\right\} + \frac{1}{8} < 5.6g_1(b).$$

Since

$$0.1555a^{-1/2}b^{-1/4}c^{1/4} > 0.386b^{1/4},\ \ 0.204b^{-3/4}c^{1/4} > 0.3581b^{1/4},$$

we see that $b^{1/4} < 15.7g_1(b) - 0.349$ and that $b < 1.3\cdot 10^5$. Therefore, if $v_{2m+1} = w_{2n+1}$, then $b < 1.3\cdot 10^5$. (3) Assume that $b > 8a$ and $c \ge b^5$. Since $7b/8 < a' = b - a < b$ and $1 \le a < b/8$, we see from (3.1) that

$$\frac{n'}{8} < \frac{\log(1.415b^3c)\log(0.5249c)}{\log(4bc)\log(0.1053b^{-4}c)} =: h(c).$$

Since $h(c)$ is decreasing with respect to c and $c \geq b^5$, we have

$$\frac{n'}{8} < \frac{\log(1.415b^8)\log(0.5249b^5)}{\log(4b^6)\log(0.1053b)} < \frac{20}{3} \cdot \frac{\log(1.045b)\log(0.8791b)}{\log(1.259b)\log(0.1053b)}.$$

In the same way as (2), one may prove the following:

(i) If $v_{2m} = w_{2n}$, then $b \leq 18$;

(ii) If $v_{2m+1} = w_{2n+1}$, then $b < 2 \cdot 10^3$.

This completes the proof of Proposition 3.3. ∎

4. Proof of Theorem 1.2

It remains to obtain absolute lower bounds for b which contradict the upper bounds in Proposition 3.3. In order to do that by computer, we start by showing that if $a < b \leq 8a$, then all the possible c's appearing as the third elements can be described explicitly, which makes the programs run faster. Let $\{a, b, c\}$ be a Diophantine triple, and s, t positive integers satisfying $ac + 1 = s^2$, $bc + 1 = t^2$. Then, we have

$$at^2 - bs^2 = a - b. \tag{4.1}$$

If (t, s) belongs to the same class as either of the solutions $(\pm 1, 1)$, then s can be expressed as $s = s_\nu^\pm$, where

$$s_0 = s_0^\pm = 1,\ s_1^\pm = r \pm a,\ s_{\nu+2}^\pm = 2rs_{\nu+1}^\pm - s_\nu^\pm \tag{4.2}$$

with r the positive integer satisfying $ab + 1 = r^2$. Define $c_\nu^\pm = ((s_\nu^\pm)^2 - 1)/a$. By [11, Theorem 8], if $a < b < 4a$, then c has to be of the form $c = c_\nu^\pm$ for some ν and some sign. The following lemma generalizes this result.

Lemma 4.1. *Let $\{a, b, c\}$ be a Diophantine triple. Assume that $a < b \leq 8a$. Then, $c = c_\nu^\pm$ for some ν and some sign.*

Proof. Define s', t' by $s' = rs - at$, $t' = rt - bs$. Then, (t, s) and (t', s') belong to the same class of solutions of Diophantine equation (4.1). Put $c' = ((s')^2 - 1)/a$. We may assume that $c' < r^2 \leq c$. If $c' > b$, then [11, Theorem 2] implies that $c' = a + b + 2r = c_1^+$ and hence $c = c_2^+$. If $c' = b$, then $a+b+c+2abc-2rst = b$ shows that $c^2-2(a+2b+2ab^2)c+a^2-4ab-4 = 0$. Since the discriminant of this equation with respect to c is $16(ab+1)^2(b^2+1)$, we must have $b = 0$, which is a contradiction. Suppose that $c' < b$. If $c' = 0$, then $s' = rs - at = 1$, and $c = c_1^-$ or c_1^+. Let $r' = s'r - at'$ and $b' = ((r')^2 - 1)/a$. Then, $b' = a + b + c' + 2abc' - 2rs't'$, and thus, [11, Lemma 4] and $b \leq 8a$ together imply that

$$b' < \frac{b}{4ac'} \leq \frac{8a}{4ac'} = \frac{2}{c'}.$$

If $c' \geq 2$, then $b' = 0$ and $b = a + c' + 2s'$. If $c' = 1$, then, since $b' + 1 = b'c' + 1$ must be a square, $b' = 0$ and $b = a + c' + 2s'$. In either case, we obtain

$c' = a + b - 2r = c_1^-$ and hence $c = c_2^-$. This completes the proof of Lemma 4.1. □

We are now ready to complete the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. Using programs written in `Mathematica`, we are able to finish the proof of Theorem 1.2. We separately consider the cases that appear in the Proposition 3.3. It remains to show the uniqueness of the extension of Diophantine triple to a quadruple depending on the parity of indices. In the case $b \leq 8a$, we use Lemma 4.1 which implies that we know all possible values of c that extend a Diophantine pair $\{a, b\}$ to a Diophantine triple $\{a, b, c\}$. In the case $b > 8a$, we first find the fundamental solutions of equation (4.1) which give the sequences in which c can be. We do that by finding all possible values of s_0 for a fixed a and b. We have an estimate $s_0 < \sqrt{\frac{r+1}{2}}$. When we get s_0 we easily compute t_0 from (4.1). Moreover, it gives us $s_1 = rs_0 + at_0$ and we have the recurrence relation $s_{\nu+2} = 2rs_{\nu+1} - s_\nu$. Then, we get $c_\nu = (s_\nu^2 - 1)/a$. It is interesting that in most of the cases, c is given by $c = c_\nu^\pm$ corresponding to the fundamental solutions $(s_0, t_0) = (1, \pm 1)$. We prove Conjecture 1.1 for all pairs $\{a, b\}$ with $a < b$ and b bounded by $2.1 \cdot 10^4$ if $b < 2a$, by $1.3 \cdot 10^5$ if $2a \leq b \leq 8a$, and by $2 \cdot 10^3$ if $b > 8a$. Let us also mention that because we are able to prove the uniqueness of the extension of Diophantine triple $\{a, b, c\}$ starting with the smallest possible c, there was no loss of generality in assuming that Diophantine triple $\{a, b, c\}$ cannot be extended to a quadruple $\{a, b, c, d\}$ with $d < d_-$, which gives us the exact values of fundamental solution z_0 and z_1 that we use in reduction.

For the remaining values, we will need the computational method of reduction of Baker-Davenport. In the programs, we start with a, then for all possible values of b (notice that we know an upper bound for it) we check if $r = \sqrt{ab+1}$ is an integer using the command `IntegerQ`. Then for fixed a and b, we find all possible values of c using the upper bounds for c in terms of a and b obtained from [6, Proposition 5]. Notice that in all cases c grows exponentially. Then, for fixed a, b and c, we apply Baker-Davenport reduction (see [7, Lemma 5]). `Mathematica` easily helps us to compute the corresponding continued fractions and convergents. To run all programs and to finish our proof, it roughly took 300 hours on 2.80 GHz Intel Core 2 Duo 2.98GB. □

ACKNOWLEDGEMENTS.

The first author was supported by the Ministry of Science, Education and Sports (grant 037-0372781-2821) of Republic of Croatia. The second author was partially supported by JSPS KAKENHI Grant Number 25400025. The third author was supported by Purdue University North Central.

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Received: 12.12.2012.
Revised: 2.4.2013. & 8.7.2013.