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IDEALIZATION AND POLYNOMIAL IDENTITIES

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Abstract. Let R be a commutative ring, let M be an R-module, let $f(X_1,\dots,X_n)$ be a polynomial (with coefficients from R or $\mathbb{Z}$) and let k be a positive integer. We show that if R satisfies the polynomial identity

$$\prod_{i=1}^{k} f(X_{1i},\dots,X_{ni}) = 0,$$

then the idealization $R(+)M$ satisfies

$$\prod_{i=1}^{k+1} f(X_{1i},\dots,X_{ni}) = 0.$$

1. Introduction

Throughout all rings will be commutative, but not necessarily with identity. For rings with an identity, all modules are assumed to be unital. When considering polynomials over a ring, it is useful for the ring to have an identity. So we define the ring R^1 as follows. If R has an identity, $R^1 = R$. If R does not have an identity, let R^1 be the Dorroh extension of R with $\operatorname{char} R^1 = \operatorname{char} R$ (so $R^1 = R \oplus \mathbb{Z}$ if $\operatorname{char} R = 0$ while $R^1 = R \oplus \mathbb{Z}_n$ if $\operatorname{char} R = n > 0$ with product $(r_1, n_1)(r_2, n_2) = (r_1r_2 + n_2r_1 + n_1r_2, n_1n_2)$). Suppose that R and S are rings so that S is an R-module and n is a positive integer. Recall that for a polynomial $f \in R[X_1,\dots,X_n]$ we say that S satisfies a polynomial identity $f(X_1,\dots,X_n) = 0$ if $f(s_1,\dots,s_n) = 0$ for any $s_1,\dots,s_n \in S$.

Let R be a ring and M an R-module. The idealization or trivial extension $R(+)M$ of R and M is the ring with additive group $R\oplus M$ and multiplication

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given by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$. Here $(0 \oplus M)^2 = 0$, so the nilradical of $R(+)M$ is $\mathrm{nil}(R(+)M) = \mathrm{nil}(R) \oplus M$. For results on idealization, the reader is referred to [2].

Now let R be a ring, M an R-module and let k be a positive integer. The main result of this note (Theorem 2.2) is the following. If there is a polynomial f in n variables so that R satisfies a polynomial identity

$$f(X_{11}, \ldots, X_{n1}) \cdots f(X_{1k}, \ldots, X_{nk}) = 0$$

(resp., $(f(X_1, \ldots, X_n))^k = 0$), then the idealization $R(+)M$ of R and M satisfies the polynomial identity

$$f(X_{11}, \ldots, X_{n1}) \cdots f(X_{1k}, \ldots, X_{nk}) f(X_{1k+1}, \ldots, X_{nk+1}) = 0$$

(resp., $(f(X_1, \ldots, X_n))^{k+1} = 0$). In Section 3 we give a number of applications of our result and examples to show the sharpness of the result.

2. Main Result

Now let $(r_1, m_1), \ldots, (r_n, m_n) \in R(+)M$. For later use first observe that

$$\prod_{i=1}^{n} (r_i, m_i) = \left(\prod_{i=1}^{n} r_i, \sum_{i=1}^{n} \left(\prod_{\substack{j=1 \\ j \neq i}}^{n} r_j \right) m_i \right).$$

Proposition 2.1. *Let R be a ring and M an R-module. Let f be a polynomial and elements $(r_1, m_1), \ldots, (r_n, m_n)$ be as above. Then*

$$f((r_1, m_1), \ldots, (r_n, m_n)) = \left(f(r_1, \ldots, r_n), \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(r_1, \ldots, r_n) m_i \right).$$

In particular, for the case $n = 1$ we have

$$f(r, m) = (f(r), f'(r)m).$$

Proof. Writing f as a linear combination of primitive polynomials, it suffices to prove the result for $f(X_1, \ldots, X_n) = X_1^{s_1} \cdots X_n^{s_n}$. Now for $r \in R$ and $m \in M$, it is easily proved by induction that $(r, m)^k = (r^k, kr^{k-1}m)$. Hence

$$\begin{aligned} f((r_1, m_1), \ldots, (r_n, m_n)) &= (r_1, m_1)^{s_1} \cdots (r_n, m_n)^{s_n} \\ &= (r_1^{s_1}, s_1 r_1^{s_1 - 1} m_1) \cdots (r_n^{s_n}, s_n r_n^{s_n - 1} m_n) \\ &= \left(r_1^{s_1} \cdots r_n^{s_n}, \sum_{i=1}^{n} s_i r_1^{s_1} \cdots r_{i-1}^{s_{i-1}} r_i^{s_i - 1} r_{i+1}^{s_{i+1}} \cdots r_n^{s_n} m_i \right) \\ &= \left(f(r_1, \ldots, r_n), \sum_{i=1}^{n} \frac{\partial f}{\partial X_i}(r_1, \ldots, r_n) m_i \right). \end{aligned}$$

□

Now we give our main result.

THEOREM 2.2. *Let R be a ring and M an R-module. Let $f = f(X_1, \ldots, X_n) \in R^1[X_1, \ldots, X_n]$ or $f \in \mathbb{Z}[X_1, \ldots, X_n]$ and let k be a positive integer. Suppose that R satisfies the polynomial identity $\prod_{i=1}^{k} f(X_{1i}, \ldots, X_{ni}) = 0$ (resp., $(f(X_1, \ldots, X_n))^k = 0$). Then $R(+)M$ satisfies the polynomial identity $\prod_{i=1}^{k+1} f(X_{1i}, \ldots, X_{ni}) = 0$ (resp., $(f(X_1, \ldots, X_n))^{k+1} = 0$).*

PROOF. We are given that $\prod_{i=1}^{k} f(r_{1i}, \ldots, r_{ni}) = 0$ for any $\{r_{ij}\}_{\substack{1\le i\le n\\ 1\le j\le k}} \subseteq R$ (resp., $(f(r_1, \ldots, r_n))^k = 0$ for any $r_1, \ldots, r_n \in R$). We need that

$$\prod_{i=1}^{k+1} f((r_{1i}, m_{1i}), \ldots, (r_{ni}, m_{ni})) = 0 \text{ for any } \{(r_{ij}, m_{ij})\}_{\substack{1\le i\le n\\ 1\le j\le k+1}} \subseteq R(+)M$$

(resp., $(f((r_1, m_1), \ldots, (r_n, m_n))^{k+1} = 0$ for any $(r_1, m_1), \ldots, (r_n, m_n) \in R(+)M)$).

We prove the first statement; for the second one we proceed in a similar way. Using Proposition 2.1 and the displayed equality before it we get that

$$\begin{aligned}
&\prod_{i=1}^{k+1} f((r_{1i}, m_{1i}), \ldots, (r_{ni}, m_{ni})) \\
&\quad = \prod_{i=1}^{k+1} \left(f(r_{1i}, \ldots, r_{ni}), \sum_{j=1}^{n} \frac{\partial f}{\partial X_j}(r_{1i}, \ldots, r_{ni}) m_{ji} \right) \\
&\quad = \left(\prod_{i=1}^{k+1} f(r_{1i}, \ldots, r_{ni}), \sum_{i=1}^{k+1} \left(\prod_{\substack{j=1\\ j\neq i}}^{k+1} f(r_{1j}, \ldots, r_{nj}) \right) \sum_{j=1}^{n} \frac{\partial f}{\partial X_j}(r_{1i}, \ldots, r_{ni}) m_{ji} \right) \\
&\quad = (0, 0);
\end{aligned}$$

for the last equality use that

$$\prod_{j=1}^{k+1} f(r_{1ij}, \ldots, r_{ni}) = \prod_{\substack{\ell=1\\ \ell\neq i}}^{k+1} f(r_{1\ell}, \ldots, r_{n\ell}) = 0.$$

∎

3. EXAMPLES

The purpose of this section is to illustrate the sharpness of Theorem 2.2 and to illustrate Theorem 2.2 for certain simple polynomial identities. We begin by noting a dichotomy in the case of whether R is reduced.

Suppose that R is reduced, i.e., $\mathrm{nil}(R) = 0$. If R satisfies $\prod_{i=1}^{k} f(X_{1i}, \ldots, X_{ni}) = 0$ (or $(f(X_1, \ldots, X_n))^k = 0$), then R satisfies $(f(X_1, \ldots, X_n))^k = 0$ and hence $f(X_1, \ldots, X_n) = 0$ because R is reduced. Hence if R satisfies

$\prod_{i=1}^{k} f(X_{1i}, \ldots, X_{ni}) = 0$ or $(f(X_1, \ldots, X_n))^k = 0$ for some $k \geq 1$, then $R(+)M$ satisfies $\prod_{i=1}^{k} f(X_{1i}, \ldots, X_{ni}) = 0$.

However, if R is not reduced, R can satisfy $(f(X_1, \ldots, X_n))^2 = 0$ but not $\prod_{i=1}^{2} f(X_{1i}, \ldots, X_{ni}) = 0$ as our first example shows.

Example 3.1 (R satisfies $(f(X))^2 = 0$ but not $f(X)f(Y) = 0$). Take R to be the maximal ideal $(x, y) = (X, Y)/(X^2, Y^2)$ of the local ring $\mathbb{Z}_2[x, y] = \mathbb{Z}_2[X, Y]/(X^2, Y^2)$, and $f(X) = X$. Then $(f(r))^2 = r^2 = 0$ for each $r \in R$, but $f(x)f(y) = xy \neq 0$. For an example where the ring has an identity we may take $R = \mathbb{Z}_4(+)\mathbb{Z}_4$, and $f(X) = X(1 + X)$. It is easily checked that R satisfies $(f(X))^2 = 0$ but does not satisfy $f(X)f(Y) = 0$.

We next illustrate the sharpness of Theorem 2.2. Suppose that the ring R satisfies $\prod_{i=1}^{k} f(X_{1i}, \ldots, X_{ni}) = 0$, but not $\prod_{i=1}^{k-1} f(X_{1i}, \ldots, X_{ni}) = 0$. So for a nonzero R-module M, $R(+)M$ satisfies $\prod_{i=1}^{k+1} f(X_{1i}, \ldots, X_{ni}) = 0$. We next give examples to show that $R(+)M$ may or may not satisfy $\prod_{i=1}^{k} f(X_{1i}, \ldots, X_{ni}) = 0$.

Example 3.2 (R satisfies $\prod_{i=1}^{k} f(X_i) = 0$, $R(+)M$ satisfies $\prod_{i=1}^{k+1} f(X_i) = 0$, but $R(+)M$ does not satisfy $\prod_{i=1}^{k} f(X_i) = 0$). Let $f(X) = X(1 + X)$ and $f_k = f(X_1) \cdots f(X_k)$. Put $R_1 = \mathbb{Z}_2$ and for $k \geq 1$, $R_{k+1} = R_k(+)R_k$. Now R_1 satisfies $f_1 = 0$ so R_k satisfies $f_k = 0$. It is easily checked that R_2 does not satisfy $f_1 = 0$. Suppose that R_k does not satisfy $f_{k-1} = 0$. So there exist $r_1, \ldots, r_{k-1} \in R_k$ with $r_1 \cdots r_{k-1}(1 + r_1) \cdots (1 + r_{k-1}) \neq 0$. Consider $(r_1, 0), \ldots, (r_{k-1}, 0), (0, 1) \in R_k(+)R_k = R_{k+1}$. Then

$$\begin{aligned}
&(r_1, 0) \cdots (r_{k-1}, 0)(0, 1)((1, 0) + (r_1, 0)) \cdots ((1, 0) + (r_{k-1}, 0))((1, 0) + (0, 1)) \\
&= (r_1 \cdots r_{k-1}(1 + r_1) \cdots (1 + r_{k-1}), 0)(0, 1)(1, 1) \\
&= (0, r_1 \cdots r_{k-1}(1 + r_1) \cdots (1 + r_{k-1})) \neq (0, 0).
\end{aligned}$$

So R_{k+1} does not satisfy f_k.

(R and $R(+)M$ satisfy $f = 0$). Let $R = \mathbb{Z}_4, M = \mathbb{Z}_2$ and $f(X) = X^2(1 + X)^2$. Then both $\mathbb{Z}_4$ and $\mathbb{Z}_4(+)\mathbb{Z}_2$ satisfy $f(X) = 0$.

Now we would like to illustrate Theorem 2.2 with some simple polynomial identities. But first observe that if R satisfies a polynomial identity $f(X_1, \ldots, X_n) = 0$, then f must necessarily have the constant term 0. The interpretation of our theorem for the simplest cases $f(X) = X$ and $f(X) = rX$, $r \in R$, are left to the reader. Let us consider the quadratic polynomial $f(X) = aX^2 + bX = X(aX + b)$. Assume that R has an identity; so $a + b = 0$ and then $f(X) = aX(X - 1) = 0$. Putting $X = 2$ gives $2a = 0$. The simplest case is when $\operatorname{char} R = 2$ and $a = 1$; that is, $f(X) = X(X + 1)$, which is covered in the next example.

Example 3.3. Let R be a ring and $f(X) = X(1 + X)$. Now R satisfies $f(X) = 0$ if and only if R is Boolean. A ring R is said to be n-Boolean ([1]) if

$\operatorname{char} R = 2$ and R satisfies $f(X_1)\cdots f(X_n) = 0$. So R is 1-Boolean if and only if R is Boolean. A 2-Boolean ring is called a *Boolean-like ring* ([3, 4]). By Theorem 2.2 if R satisfies $\prod_{i=1}^{n} f(X_i) = 0$, then for an R-module M, $R(+)M$ satisfies $\prod_{i=1}^{n+1} f(X_i) = 0$. Since $\operatorname{char} R(+)M = \operatorname{char} R$, this gives that if R is n-Boolean, then $R(+)M$ is $(n+1)$-Boolean. Hence if R is a Boolean ring, then $R(+)M$ is a Boolean-like ring. In [1, Theorem 8] it is shown that R is n-Boolean if and only if $R/\operatorname{nil}(R)$ is Boolean, $\operatorname{char} R = 2$, and $\operatorname{nil}(R)^n = 0$. Using this characterization, it is shown in [1, Theorem 9] that R is n-Boolean implies $R(+)M$ is $(n+1)$-Boolean. We remark that [1, Theorem 10] says that every Boolean-like ring has the form $R(+)M$ for some Boolean ring R and R-module M. However, a 3-Boolean ring need not have the form $R(+)M$ where R is 2-Boolean ([1, page 74]). Note that throughout this example we could replace the polynomial $f(X) = X(1+X)$ by $g(X) = X(1-X)$ since $g(X) = -f(-X)$.

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