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FINITE GROUPS WITH FEW VANISHING ELEMENTS

Jinshan Zhang, Zhencai Shen and Jiangtao Shi
Sichuan University of Science and Engineering, China Agricultural University and Yantai University, P. R. China

Abstract. Let G be a finite group, and $\mathrm{Irr}(G)$ the set of irreducible complex characters of G. We say that an element $g \in G$ is a *vanishing element* of G if there exists χ in $\mathrm{Irr}(G)$ such that $\chi(g) = 0$. Let $\mathrm{Van}(G)$ denote the set of vanishing elements of G, that is, $\mathrm{Van}(G) = \{g \in G \,|\, \chi(g) = 0 \text{ for some } \chi \in \mathrm{Irr}(G)\}$. In this paper, we investigate the finite groups G with the following property: $\mathrm{Van}(G)$ contains at most four conjugacy classes of G.

1. Introduction

Let G be a finite group and $\upsilon(\chi) := \{g \in G \,|\, \chi(g) = 0\}$, where χ is an irreducible complex character of G. A classical theorem of Burnside asserts that $\upsilon(\chi)$ is non-empty for all $\chi \in \mathrm{Irr}_1(G)$, where $\mathrm{Irr}_1(G)$ denotes the set of non-linear irreducible complex characters of G. Our aim in this paper is to analyze a particular subset of $\{x^G : x \in G\}$, which we denote by $\mathrm{Van}(G)$ and which encodes information coming from the set $\mathrm{Irr}(G)$ of irreducible complex characters of G. We say that an element $g \in G$ is a *vanishing element* of G if there exists χ in $\mathrm{Irr}(G)$ such that $\chi(g) = 0$ and otherwise we call x a non-vanishing element. Let $\mathrm{Van}(G)$ denote the set of vanishing elements of G, that is

$$\mathrm{Van}(G) = \{g \in G \,|\, \chi(g) = 0 \text{ for some } \chi \in \mathrm{Irr}(G)\}.$$

Clearly, $\mathrm{Van}(G)$ is a proper normal subset of G. We denote $k_G(N)$ the number of conjugacy classes of G contained in N, where N is a normal subset of G. In this paper, we investigate the finite groups G with the following property: $\mathrm{Van}(G)$ contains at most four conjugacy classes of G. Generally, we define

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DEFINITION 1.1. *A group G is called a VnC-group if* $\mathrm{Van}(G)$ *contains at most n conjugacy classes of G.*

Clearly, if G is a VnC-group, then, in particular every irreducible character of G vanishes on at most n conjugacy classes of G. However, it turns out (see [20] and Theorem 1.4 below) that for $n = 3$ the two classes, in the case of solvable groups, in fact coincide. The aim of this paper is the classification of $V4C$-groups and the main results are the following.

THEOREM 1.2. *Let G be a finite non-abelian and solvable group. If G is a $V4C$-group but not a $V3C$-group, then one of the following is true:*

(1) *G is a Frobenius group with complement Q_8,*
(2) *$G = G'P$, where G' is a normal abelian 2-complement of G, $P \in \mathrm{Syl}_2(G)$, $|P| = 8$, $|Z(G)| = 4$, and $G/Z(G)$ is a Frobenius group with kernel $(G/Z(G))' \cong G'$ and complement $P/Z(G)$ of order 2,*
(3) *$G = (G'\langle u\rangle) \times \langle t\rangle$, where t is an involution and $G'\langle u\rangle$ is a Frobenius group with kernel G' and complement of order 3,*
(4) *$G = G'P$, where G' is a normal abelian 2-complement of G, and $P \in \mathrm{Syl}_2(G)$, $|P| = 4$, $F(G) = G'$, $G \backslash G'$ is a union $x^G \cup y^G \cup z^G \cup h^G$ of four conjugacy classes satisfying $|C_G(x)| = 4$, $|C_G(y)| = 4$, $|C_G(z)| = 6$ and $|C_G(h)| = 12$,*
(5) *G is a Frobenius group with abelian kernel G' and complement of order 5.*

THEOREM 1.3. *Let G be finite non-solvable group. If G is a $V4C$-group, then G is isomorphic to A_5.*

Clearly, A_5 is not a $V3C$-group. Hence by [20, Theorem A] and Theorem 1.3 above we easily get the following result.

THEOREM 1.4. *Let G be a finite non-abelian group. Then G is a $V3C$-group if and only if G is one of the following groups:*

(1) *G is a Frobenius group with abelian kernel G' and complement of order 2*
(2) *G is a Frobenius group with abelian kernel G' and complement of order 3,*
(3) *$G \cong D_8$ or Q_8,*
(4) *G is a Frobenius group with kernel G' and cyclic complement of order 4,*
(5) *$G = G'P$, where G' is a normal and abelian 2-complement of G, $P \in \mathrm{Syl}_2(G)$, $|P| = 4$, $|Z(G)| = 2$, and $G/Z(G)$ is a Frobenius group with kernel $(G/Z(G))' \cong G'$ and complement $P/Z(G)$ of order 2,*
(6) *$G \cong S_4$,*
(7) *$G = (G'\langle t\rangle) \times \langle u\rangle$, where $\langle u\rangle$ is a cyclic group of order 3 and $G'\langle t\rangle$ is a Frobenius group with kernel G' and complement of order 2.*

In this paper, G always denotes a finite group. Notation is standard and taken from [8]. In particular, denote $\pi_e(G)$ the set of all element orders of G, $\mathrm{C}(n)$ a cyclic group of order n, (H, N) a Frobenius group with a complement H and kernel N. we write $G = [F]H$ to denote a semidirect product of a normal subgroup F and a subgroup H of G. For $N \lhd G$, set $\mathrm{Irr}(G|N) = \mathrm{Irr}(G) \setminus \mathrm{Irr}(G/N)$.

2. NILPOTENT $V4C$-GROUPS

In this paper, we shall freely use the following facts. Let $N \lhd G$ and write $\overline{G} = G/N$.

(1) For any $x \in G$, $\overline{x}^{\overline{G}}$ (when viewed as a subset of G, that is, the set $\bigcup_{g \in G} x^g N$) is a union of conjugacy classes of G. Note that $|C_G(x)| = |C_{\overline{G}}(\overline{x})| + \sum\{|\chi(x)|^2 \mid \chi \in \mathrm{Irr}(G|N)\}$ and that $k_G(\overline{x}^{\overline{G}}) = 1$ if and only if $|C_G(x)| = |C_{\overline{G}}(\overline{x})|$, and then $k_G(\overline{x}^{\overline{G}}) = 1$ if and only if $\chi(x) = 0$ for all $\chi \in \mathrm{Irr}(G|N)$.
(2) If G is a VnC-group, then so is G/N.
(3) Any character of $\overline{G}$ can be viewed, by inflation, as a character of G. In particular, if $xN \in \mathrm{Van}(\overline{G})$, then $xN \subseteq \mathrm{Van}(G)$.
(4) Let M be a normal subgroup of G contained in N. If $\psi \in \mathrm{Irr}(N)$ vanishes on $N \setminus M$, then by Clifford's theorem, every irreducible constituent of ψ^G also vanishes on $N \setminus M$. In particular, $N \setminus M \subseteq \mathrm{Van}(G)$.

We will use frequently the following lemma (see [16, Theorem 2.1]).

LEMMA 2.1. *Let G be non-abelian, and let $\chi \in \mathrm{Irr}_1(G)$. Assume that N is a normal subgroup of G such that $G' \leq N < G$. If χ_N is not irreducible, then the following two statements hold:*

(1) *There exists a normal subgroup H of G such that $N \leq H < G$ and $G - H \subseteq \upsilon(\chi)$.*
(2) *If $(G \setminus G') \cap \upsilon(\chi)$ consists of n conjugacy classes of G, then $[H : G']([G : H] - 1) \leq n$.*

The following result gives a lower bound on the number of conjugacy classes of zeros of irreducible characters of p-groups.

LEMMA 2.2. *[13, Theorem C] Let χ be a non-linear irreducible character of a p-group P of degree p^n. Then $k_G(\upsilon(\chi))$ is a multiple of $p-1$ bigger than or equal to $(p+n)(p-1)$. In particular, $k_G(\upsilon(\chi)) \geq p^2 - 1$.*

Next, we classify the non-abelian nilpotent groups in which every irreducible character vanishes on at most four conjugacy classes of G.

THEOREM 2.3. *Suppose that G is a non-abelian nilpotent group. If every irreducible character of G vanishes on at most four conjugacy classes of G, then G is one of the following groups:*

(1) $G \cong D_8$ *or* Q_8,
(2) $G \cong \langle a, b \mid a^8 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$,
(3) $G \cong \langle a, b \mid a^8 = 1, b^2 = a^4, b^{-1}ab = a^{-1} \rangle$,
(4) $G \cong \langle a, b \mid a^8 = b^2 = 1, b^{-1}ab = a^3 \rangle$.

PROOF. Take $\varphi \in \mathrm{Irr}_1(G)$ such that $\varphi_{G'}$ is not irreducible. It follows from the hypothesis and Lemma 2.1 that G has a proper subgroup H such that $G' \leq H < G$, $G - H \subseteq \upsilon(\varphi)$ and $[H : G']\,([G : H] - 1) \leq 4$.

Since G is nilpotent, it follows by Lemma 2.1 that G is a 2-group. Note that $[H : G']\,([G : H] - 1) \leq 4$ and that $|G/G'| \geq 2^2$; then we obtain that $|G/G'| = 4$ or 8. Suppose that $|G/G'| = 8$. Let N be normal in G with $[G' : N] = 2$. Now consider the group G/N of order 16. Note that $[G/N : (G/N)'] = 8$, by the hypothesis, we easily conclude that it is impossible (see [10, P. 300]). Hence we may suppose that $[G : G'] = 4$. If $|G'| = 2$, then G satisfies (1) of the Theorem. So we may assume that $|G'| \geq 4$. Now let N be normal in G with $[G' : N] = 4$. Consider the group $\overline{G} := G/N$ of order 16. Applying [10, P. 300], G/N satisfies (2), (3) or (4) of the Theorem.

Now we show that $N = 1$. Suppose that $N > 1$. Set $M/N = Z(G/N)$. Take the irreducible character ξ of $\overline{G}$ with $k_{\overline{G}}(\upsilon(\xi)) = 4$. Thus the hypothesis yields that $k_{\overline{G}}(\upsilon(\xi)) = k_G(\upsilon(\xi)) = 4$, and so χ vanishes only on $\upsilon(\xi)$ for every $\chi \in \mathrm{Irr}(G|N)$. Hence it follows by [9, Theorem A] that $M = Z(G)$. Consequently, $[G : Z(G)] = 8$.

Take an irreducible character ρ of $\overline{G}$ with $k_{\overline{G}}(\upsilon(\rho)) = 3$. Choose $g \in G$ such that $\overline{g} = \upsilon(\rho) \backslash \upsilon(\xi)$. We easily see that $\overline{g}^{\overline{G}}$ consists of two conjugacy class of G and that $\overline{g}^{\overline{G}} = G' - Z(G)$. Hence $k_G(G \backslash Z(G)) = 6$.

On the other hand, set $G \backslash Z(G) = n_1^G + \cdots + n_s^G$. Then, we get

$$|G \backslash Z(G)| = \frac{|G|}{|C_G(n_1)|} + \cdots + \frac{|G|}{|C_G(n_s)|}.$$

It follows that

$$\frac{1}{|C_G(n_1)|} + \cdots + \frac{1}{|C_G(n_s)|} = 1 - \frac{1}{|G : Z(G)|} = \frac{7}{8}.$$

Recall that all the centralizers of the elements in $G \backslash Z(G)$ have order greater than 4, then $G \backslash Z(G)$ has at least 7 conjugacy classes, a contradiction. The proof is complete. ∎

REMARK 2.4. Clearly, a $V4C$-group is such a group whose irreducible characters vanish on at most four conjugacy classes. But the converse is not true (see the types (2), (3) or (4) in Theorem 2.3).

COROLLARY 2.5. *Suppose that G is a non-abelian nilpotent group. If G is a $V4C$-group, then G is isomorphic to D_8 or Q_8.*

3. Vanishing elements and the nilpotent normal subgroup

In the following, we consider the situation in which a vanishing element of a group G lies in a nilpotent normal subgroup.

Lemma 3.1. *Let N be a nilpotent subgroup of G such that $|G:N|=2$. Assume that G is non-nilpotent. If N is non-abelian, then $k_G(N\cap \mathrm{Van}(G)) \geq 3$.*

Proof. Consider $\theta \in \mathrm{Irr}_1(N)$. Note that $|G:N|=2$, thus there exists $\chi \in \mathrm{Irr}(G)$ such that either $\chi_N = \theta$ or $\chi = \theta^G$. If $\chi_N = \theta$, then it follows by the nilpotency of N and Lemma 2.1 that $k_N(\upsilon(\theta)) \geq 6$, and so $k_G(\upsilon(\theta)) \geq 3$. Thus $k_G(N \cap \mathrm{Van}(G)) \geq 3$. Hence we may assume that $\chi = \theta^G$.

Now let M be a normal subgroup of N maximal with respect to N/M being non-abelian. Then N/M is a q-group for some prime q and $(N/M)'$ is the unique minimal normal subgroup of N/M. Let $Z > M$ with $Z/M = Z(N/M)$. Consider $\phi \in \mathrm{Irr}(N/M)$ with $\phi(1) = q^n > 1$. The choice of M implies that ϕ is a faithful irreducible character of N/M, and thus ϕ satisfies $\phi(1)^2 = |N/M : Z/M|$ by [8, Theorem 2.31]. Applying [8, Corollary 2.30], ϕ vanishes on $N\backslash Z$.

From the first paragraph of the proof, we have that ϕ^G is an irreducible character of G. Take $\psi = \phi^G$. Observe that ψ vanishes on $N\backslash(Z \cup Z^x)$, where $x \in G\backslash N$. Set $N\backslash(Z\cup Z^x) = n_1^G + \cdots + n_s^G$. Then, we get

$$|N\backslash(Z\cup Z^x)| = \frac{|G|}{|C_G(n_1)|} + \cdots + \frac{|G|}{|C_G(n_s)|},$$

and

$$|N\backslash(Z\cup Z^x)| > |N| - 2|Z|.$$

It follows that

$$\frac{1}{|C_G(n_1)|} + \cdots + \frac{1}{|C_G(n_s)|} > \frac{1}{2} - \frac{1}{q^{2n}}.$$

Assume that $q = 2$. Then $2 \,|\, |N|$. Recall that N is not a 2-group; thus $|C_G(n_i)| \geq 12$. Note that

$$\frac{1}{|C_G(n_1)|} + \cdots + \frac{1}{|C_G(n_s)|} > \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Hence, $s \geq 4$. Now, we assume that q is odd.

Since q is odd, we get have $|C_G(n_i)| \geq 9$. On the other hand, we have

$$\frac{1}{|C_G(n_1)|} + \cdots + \frac{1}{|C_G(n_s)|} > \frac{1}{2} - \frac{1}{9} = \frac{7}{18}.$$

Therefore, $s \geq 4$ and we are done. ∎

Arguing as in Lemma 3.1 we easily see that the following two results.

LEMMA 3.2. *Let N be a nilpotent normal subgroup of G such that $|G : N| = 3$. Assume that G is non-nilpotent. If every element of N is a non-vanishing element of G, then N is abelian.*

LEMMA 3.3. *Assume that $G = NP$, where N is a nilpotent normal 2-complement of G and $P \in \mathrm{Syl}_2(G)$. Assume that $|P| = 4$. If every element of N is a non-vanishing element of G, then N is abelian.*

THEOREM 3.4. *Let G be Frobenius group with kernel N and complement of order 3. If N is non-abelian, then $k_G(N \cap \mathrm{Van}(G)) \geq 3$.*

PROOF. Let G be a minimal counter-example. Then N is a q-group for some prime q, also N' is minimal normal in G and $N' \leq Z(N)$. Let $C = \langle x \rangle$ be a complement of N in G.

Observe that any irreducible C-invariant subgroup of N is of order at most q^3, also that if $q = 2$ then the bound is 2^2. In particular, $|N'| = q^e \leq q^3$.

Let $\psi \in \mathrm{Irr}(N)$ be of maximal degree. Observe that there exists a subgroup Z of N such that ψ vanishes on $N \backslash Z$ and that $Z \geq N'$, $|N/Z| = q^{2m}$.

Suppose that $q > 2$ or $m > 1$. Then there exists an irreducible character χ of G such that χ vanishes on $G\backslash\Delta$, where $\Delta := Z \cup Z^x \cup Z^{x^2}$. Write $|Z/N'| = q^f$. Observe that the centralizer of any element in $N\backslash N'$ has order at least $|N/N'|$. Now

$$\begin{aligned} k_G(G\backslash\Delta) - 2 &= k_G(N\backslash\Delta) \\ &> (q^{2m+f+e} - 3q^{f+e})/3q^e \\ &= q^f(q^{2m} - 3)/3 > 3, \end{aligned}$$

hence, we obtain a contradiction.

Suppose that $q = 2$ and $m = 1$. As ψ is of maximal degree, $\mathrm{cd}(N) = \{1, 2\}$. It follows that either $|N : Z(N)| = 8$ or N has an abelian subgroup of index 2. Observe that if $N/Z(N)$ has order 8, then $N/Z(N)$ has a C-invariant subgroup of order 2, which is impossible because G is a Frobenius group. Assume now that N has an abelian subgroup E of index 2 and set $D = \cap_{g\in G} E^g$. Then all non-linear $\psi \in \mathrm{Irr}(N)$ vanishes on $N\backslash E$. Take $\chi \in \mathrm{Irr}(G)$ of degree 6. We see that χ vanishes on $N\backslash E$, and it follows that χ vanishes on every element of $N\backslash D$. It follows by [9, Theorem A] that $Z(N) \leq D$. Then we get

$$N' \leq Z(N) \leq D \leq N.$$

Observe that $|N/D| \geq 4$, thus we conclude

$$4 \leq |N'| \leq |Z(N)| \leq |D| \leq \frac{|N|}{4}.$$

If all are equalities above, then N is of order 16, $N' = Z(N)$ and $|N : N'| = 4$. However, it is impossible (see [10, P. 300]). Hence $|N| \geq 64$. Set $N\backslash D =$

$n_1^G + \cdots + n_s^G$. Thus, we have

$$|N \backslash D| = \frac{|G|}{|C_G(n_1)|} + \cdots + \frac{|G|}{|C_G(n_s))|}.$$

It follows that

$$(*) \qquad \frac{1}{|C_G(n_1)|} + \cdots + \frac{1}{|C_G(n_s)|} = \frac{1}{3} - \frac{1}{3|N:D|} \geq \frac{1}{4}.$$

Observe that all the centralizers of the elements in $N \backslash D$ have order greater than 4, also that the centralizer of some element, say g, has order at least 32. So we conclude by the above equality (*) that $N \backslash D$ consists of at least 3 conjugacy classes, a contradiction. □

Arguing as in Theorem 3.4, we easily see that the following lemma.

Lemma 3.5. *[1, Proposition 4.1] Let G be Frobenius group with kernel N and complement of order p, where $p \leq 5$. If $N \cap$ Van$(G) = \emptyset$, then N is abelian.*

4. Conjugacy classes outside a normal subgroup

We will use frequently the following result.

Lemma 4.1. *Let G be non-abelian. Assume that N is a normal subgroup of G such that $G' \leq N < G$ and that $G \backslash N \subseteq$ Van(G). If Van(G) consists of at most n conjugacy classes of G, then $|N : G'|\,(|G : N| - 1) \leq n$.*

Proof. Assume that $|N : G'| = m$ and $|G : N| = r$. Then we have

$$G = N + Nx_1 + \ldots + Nx_{r-1},\ x_i \notin N,$$

and

$$N = G' + G'y_1 + \ldots + G'y_{m-1},\ y_j \notin G'.$$

It follows that

$$(*) \qquad G \backslash N = \sum_{i=1}^{r-1} \sum_{j=1}^{m-1} G'y_jx_i + \sum_{i=1}^{r-1} G'x_i.$$

For $x \notin G'$, $G'x$ is a normal subset of G, and so we conclude by the above equality $(*)$ that $G \backslash N$ consists of at least $m(r-1)$ conjugacy classes. Bearing in mind that $G \backslash N \subseteq$ Van(G), by the hypothesis we obtain that $m(r-1) \leq n$, that is $|N : G'|\,(|G : N| - 1) \leq n$. □

We will also make use of the following result, which is [9, Theorem 4.3]. As usual, we denote by $F(G)$ the Fitting subgroup of a group G.

Lemma 4.2. *Let G be a solvable group and let x be an element of G such that $\chi(x) \neq 0$ for every $\chi \in$ Irr(G). Then the image of x modulo $F(G)$ has 2-power order.*

LEMMA 4.3. *[4, Lemma 2.6] Let G be a solvable group, and N a normal subgroup of G. If $N/F(N)$ is abelian, then $N\backslash F(N) \subseteq \mathrm{Van}(G)$.*

The following lemma is the key to the proof of Theorem 1.2.

LEMMA 4.4. *Suppose that $G = KP$, where K is a normal 2-complement of G and $P \in \mathrm{Syl}_2(G)$. If G is a $V4C$-group and P is non-abelian, then one of the following statements holds:*

(1) *$K = 1$ and $G \cong D_8$ or Q_8.*
(2) *G is a Frobenius group with complement Q_8.*

PROOF. By Corollary 2.5, we may assume that $K > 1$. Since P is non-abelian, it follows by Corollary 2.5 that $G/K \cong P \cong \mathrm{Q}_8$ or D_8. Hence we have that $|G/K : G'K/K| = 4$ and that $G\backslash G'K \subseteq \mathrm{Van}(G/K)$. We then in a position to apply Lemma 4.1 to G, with $G'K$ playing the role of N, obtaining that $K \leq G'$ and so $G\backslash G' \subseteq \mathrm{Van}(G)$. Observe that $k_G(G\backslash G') = 3$ or 4.

Assume that $k_G(G\backslash G') = 3$. Set $G\backslash G' = xG' + yG' + zG'$. Thus, we get

$$|G\backslash G'| = \frac{G}{|C_G(x)|} + \frac{G}{|C_G(y)|} + \frac{G}{|C_G(z)|}.$$

Then we conclude

$$|C_G(x)| = |C_G(y)| = |C_G(z)| = 4.$$

Hence every element in $P\backslash P'$ acts fixed point freely on K. Therefore, since P is a 2-group of class 2, we conclude by [11, Lemma 19.1] that $P \cong Q_8$ and G is a Frobenius group with complement Q_8.

Assume that $k_G(G\backslash G') = 4$. The hypothesis implies that $K \cap \mathrm{Van}(G) = \emptyset$. Since K is of odd order, it follows by Lemma 4.2 that $K \leq F(G)$. Consequently, $F(G) = K \times O_2(G)$. If $O_2(G) = 1$, then, as $F(G') = F(G) = K$ and $|G' : K| = 2$, an application of Lemma 4.3 yields that $G'\backslash K \subseteq \mathrm{Van}(G)$. Thus we obtain a contradiction. Therefore, $O_2(G) > 1$.

Since $O_2(G) > 1$, we get that G' is nilpotent, and also that $Z(G) = O_2(G)$. Set $G' = Q$. We now show that Q is abelian. Otherwise, Suppose that Q is non-abelian. Consider $\theta \in \mathrm{Irr}_1(Q)$. Let $\chi \in \mathrm{Irr}(\theta^G)$. The hypothesis implies that χ_Q is not irreducible. Then by Clifford's theorem, $\chi_Q = e(\theta^{x_1} + \cdots + \theta^{x_t})$, where $x_1, \cdots, x_t$ is a transversal of $\mathrm{I}_G(\theta)$ in G and $e = [\chi_Q, \theta]$. Clearly, $t = 2$ or 4. Let M be a normal subgroup of Q maximal with respect to Q/M being non-abelian. Then Q/M is a q-group for some prime q and $(Q/M)'$ is the unique minimal normal subgroup of Q/M. Let $Z > M$ with $Z/M = Z(Q/M)$. Consider $\theta \in \mathrm{Irr}(Q/M)$ with $\theta(1) = q^n > 1$. Then, by [8, Corollary 2.30 and Theorem 2.31], θ vanishes on $Q\backslash Z$ and $|Q/Z| = q^{2n}$. Notice that χ_Q vanishes on $Q\backslash(Z^{x_1} \cup \cdots \cup Z^{x_t})$. The hypothesis yields that $Q = Z^{x_1} \cup \cdots \cup Z^{x_t}$ and thus $|Q| \leq 4|Z|$. Hence $|Q/Z| \leq 4$, thus we have reached a contradiction (note that q is odd), and so G' is abelian.

Set $G\backslash G' = xG' + yG' + zG'$, $x, y, z \in G\backslash G'$. Since $k_G(G\backslash G') = 4$, we may assume that both $xG' \cup yG'$ and zG' are a union of two conjugacy classes of G, respectively. Set $zG' = z^G + h^G$. Recall now that $G/G' \cong P/P'$ is an elementary abelian group of order 4, there exists a normal subgroup N of G such that $G' < N < G$, also that $|G : N| = 2$ and $G\backslash N = xG' + yG'$. Thus we have

$$|G\backslash G'| = \frac{|G|}{|C_G(x)|} + \frac{|G|}{|C_G(y)|} + \frac{|G|}{|C_G(z)|} + \frac{|G|}{|C_G(h)|},$$

and

$$|C_G(x)| = |C_G(y)| = 4.$$

It follows that

$$\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{|C_G(z)|} + \frac{1}{|C_G(h)|}.$$

Therefore, we easily see that $|C_G(z)| = |C_G(h)| = 8$ or $|C_G(z)| = 6$ and $|C_G(h)| = 12$.

Suppose that $|C_G(z)| = 6$ and $|C_G(h)| = 12$. As $[G : G'] = 4$, the order of z is even. Note that $|Z(G)| = 2$, thus it is easy to see that 4 divides $|C_G(z)|$, a contradiction. Hence $|C_G(z)| = |C_G(h)| = 8$, then every element in $P\backslash P'$ acts fixed point freely on K. Observe that G is a Frobenius group with kernel K and complement Q_8, and thus $k_G(G\backslash G') = 3$, a contradiction. The contradiction completes the proof. ∎

The following result, which appears as [15, Theorem 2.2], will turn out to be useful in handling the case that $k_G(G\backslash N) = 2$ and that $G\backslash N \subseteq \mathrm{Van}(G)$, where N is normal in G.

LEMMA 4.5. *Let N be a normal subgroup of a non-abelian solvable group G. Then $k_G(G\backslash N) = 2$ if and only if G is one of the following solvable groups.*

(1) *$N = 1$ and $G \cong S_3$.*
(2) *$|G/N| = 3$ and G is a Frobenius group with kernel N.*
(3) *$|G/N| = 2$ and $|C_G(x)| = 4$ for all $x \in G\backslash N$. In particular, $P \in \mathrm{Syl}_2(G)$ has a cyclic subgroup of order $|P|/2$; furthermore, one of following holds:*
(3.a) *G has a normal and abelian 2-complement.*
(3.b) *G has a normal 2-complement and P is a quaternion group.*
(3.c) *G has an abelian 2-complement and $P \cong D_8$, the dihedral group of order 8.*

PROPOSITION 4.6. *Let N be a normal subgroup of a non-abelian solvable group G. Assume that $k_G(G\backslash N) = 2$ and that $G\backslash N \subseteq \mathrm{Van}(G)$. If G is a $V4C$-group, then G is one of the following solvable groups:*

(1) *G is a Frobenius group with kernel N and complement of order 3,*
(2) *$G \cong D_8$ or Q_8,*
(3) *G is a Frobenius group with complement Q_8,*

(4) *$G = G'P$, where $G' < N$ is a normal 2-complement of G, and $P \in \mathrm{Syl}_2(G)$, $|P| = 4$,*
(5) *$G' = N$, $|G : G'| = 2$ and $G/O_{2'}(G) \cong S_4$.*

Proof. By the hypothesis and Lemma 4.5, G satisfies the condition of Lemma 4.1. Then we conclude by Lemma 4.1 that one of the following three cases occurs: (i) $|G : G'| = 3$, $G' = N$; (ii) $|N : G'| = 2$, $|G : N| = 2$; (iii) $G' = N$, $|G : G'| = 2$.

Assume that $|G : G'| = 3$, $G' = N$. Then by Lemma 4.5, we have that G is a Frobenius group with kernel G' and complement of order 3.

Assume that $|N : G'| = 2$, $|G : N| = 2$. Then it follows by Lemma 4.5 that $G = KP$, where $K \leq G'$ is a normal 2-complement of G and $P \in \mathrm{Syl}_2(G)$. If P is non-abelian, then by Lemma 4.4 we conclude that G satisfies (2) or (3) of the Proposition. Hence we may assume that P is abelian, then we easily see that $G = G'P$, where $G' < N$ is a normal 2-complement of G, and $P \in \mathrm{Syl}_2(G)$, $|P| = 4$.

Assume that $G' = N$, $|G : G'| = 2$. Recall that $|C_G(g)| = 4$ for any $g \in G\backslash G'$. Take $y \in G$ with $T = C_G(y)$ of order 4. Clearly $T \subseteq C_G(T) \subseteq C_G(y) = T$. Let $O_{2'}(G)$ be the largest normal subgroup of odd order in G. Notice that G is solvable, we use [19, Theorem 1, 2] to conclude that $G/O_{2'}(G) \cong \mathrm{S}_4$. The proof is complete. □

The following proposition, which comes from [15, Theorem 3.6] and [17, Theorem 1.1], will turn out to be useful in handling the case that G has a normal subgroup N such that $k_G(G\backslash N) = 3$ and that $G\backslash N \subseteq \mathrm{Van}(G)$, where N is normal in G.

Proposition 4.7. *Let N be a normal subgroup of a non-abelian solvable group G. Then $G\backslash N$ is a union $x^G \cup y^G \cup z^G$ of three conjugacy classes satisfying $|x^G| \geq |y^G| \geq |z^G|$ if and only if one of the following is true:*

(1) *$N = 1$ and $G \cong A_4$ or D_{10},*
(2) *$G/N \cong S_3$ and $G \cong S_4$,*
(3) *G is a Frobenius group with kernel N and cyclic complement of order 4,*
(4) *$G \cong D_8$ or Q_8,*
(5) *$|G/N| = 4$ and G is a Frobenius group with complement Q_8,*
(6) *$|G/N| = 2$, $|C_G(x)| = |C_G(y)| = |C_G(z)| = 6$. In this case, N is of odd order and N has a normal and abelian 3-complement,*
(7) *$|G/N| = 2$, $|C_G(x)| = 4$, $|C_G(y)| = 6$ and $|C_G(z)| = 12$ and in this case, either G has a normal 2-complement or $G/O_{2'}(G) \cong S_4$,*
(8) *$|G/N| = 2$, $|C_G(x)| = 4$, $|C_G(y)| = |C_G(z)| = 8$ and in this case, either $G/O_{2'}(G) \cong GL(2,3)$ with abelian $O_{2'}(G)$, or $G/O_{2'}(G)$ is isomorphic to a non-abelian group of order 16.*

LEMMA 4.8 ([20, Lemma 2.7]). *Let G be a meta-abelian group. If $[G : G'] = p$, then G is a Frobenius group with kernel G' and complement of order p.*

The following result will turn out to be useful in handling the case that $G/F(G)$ is abelian.

THEOREM 4.9. *Let G be a non-nilpotent group such that $G/F(G)$ is abelian. Assume that $k_G(G\backslash F(G)) = 4$. If G is a $V4C$-group, then G is one of the following groups:*

(1) *$G = G'P$, where G' is a normal abelian 2-complement of G, $P \in \mathrm{Syl}_2(G)$, $|P| = 8$, $|Z(G)| = 4$, and $G/Z(G)$ is a Frobenius group with kernel $(G/Z(G))' \cong G'$ and complement $P/Z(G)$ of order 2,*
(2) *$G = (G'\langle u\rangle) \times \langle t\rangle$, where t is an involution and $G'\langle u\rangle$ is a Frobenius group with kernel G' and complement of order 3,*
(3) *$G = G'P$, where G' is a normal abelian 2-complement of G, and $P \in \mathrm{Syl}_2(G)$, $|P| = 4$, $F(G) = G'$, $G\backslash G'$ is a union $x^G \cup y^G \cup z^G \cup h^G$ of four conjugacy classes satisfying $|C_G(x)| = 4$, $|C_G(y)| = 4$, $|C_G(z)| = 6$ and $|C_G(h)| = 12$,*
(4) *G is a Frobenius group with abelian kernel G' and complement of order 5.*

PROOF. Since $G/F(G)$ is abelian, it follows the hypothesis and by Lemma 4.3 that $G\backslash F(G) = \mathrm{Van}(G)$, and so $|G : F(G)| \leq 5$.

CASE 1. $|G : F(G)| = 2$.

In this case, the hypothesis together with Lemma 3.1 yields that $F(G)$ is abelian. Notice that $G = KP$, where K is a normal abelian 2-complement of G and $P \in \mathrm{Syl}_2(G)$. Then, as $k_G(G\backslash F(G)) = 4$ and $G/F(G)$ is abelian, it follows by Lemma 4.4 that P is abelian. Consequently, $G' \leq K$. Applying Lemma 3.1, we conclude that $|P| \leq 8$ and that one of the following two cases occurs: (i) $|F(G) : G'| = 1$, 2 or 4; (ii) $|F(G) : G'| = 3$.

CASE 1.1. $|F(G) : G'| = 1$, 2 or 4.

First, assume that $F(G) = G'$. Since G' is abelian, it follows by Lemma 4.8 that G is a Frobenius group with abelian kernel G' and complement of order 2, and so $k_G(G\backslash F(G)) = 1$, a contradiction.

Second, assume that $|F(G) : G'| = 2$. Then $|G : G'| = 4$, and thus $G = G'P$, where G' is a normal 2-complement of G and $|P| = 4$. Observe that $G/O_2(G)$ is a Frobenius group with Frobenius kernel $(G/O_2(G))' \cong G'$ and complement of order 2. Thus $k_G(G\backslash F(G)) = 2$, a contradiction.

Finally, assume that $|F(G) : G'| = 4$. Thus $|G : G'| = 8$ and so $G = G'P$, where G' is a normal 2-complement of G and $|P| = 8$. Clearly, $|O_2(G)| = 4$, and thus $|G/O_2(G) : (G/O_2(G))'| = 2$. Furthermore, applying again Lemma 4.8, we see that $G/O_2(G)$ is a Frobenius group with Frobenius kernel $(G/O_2(G))' \cong G'$ and complement of order 2. Thus G satisfies (1) of the theorem.

CASE 1.2. $|F(G) : G'| = 3$.

Recall now that $G' \leq K \leq F(G)$, then $K = F(G)$ and so $|P| = 2$. Observe that $G\backslash F(G) = xG' + yG' + zG'$, where $x, y, z \in G\backslash F(G)$. Since $k_G(G\backslash F(G)) = 4$, we may assume that both $xG' \cup yG'$ and zG' are a union of two conjugacy classes of G, respectively. Set $zG' = z^G + h^G$. we have

$$|G\backslash N| = \frac{|G|}{|C_G(x)|} + \frac{|G|}{|C_G(y)|} + \frac{|G|}{|C_G(z)|} + \frac{|G|}{|C_G(h)|},$$

and

$$|C_G(x)| = |C_G(y)| = 6.$$

It follows that

$$\frac{1}{2} = \frac{1}{6} + \frac{1}{6} + \frac{1}{|C_G(z)|} + \frac{1}{|C_G(h)|}.$$

Hence $|C_G(z)| = |C_G(h)| = 12$, or $|C_G(z)| = 8$ and $|C_G(h)| = 24$. As $|P| = 2$, we reach a contradiction.

CASE 2. $|G : F(G)| = 3$.

In this case, by Lemma 3.2, we get that $F(G)$ is abelian. It follows by Lemma 4.1 and Lemma 4.8 that $|F(G) : G'| = 2$. Observe that $G\backslash F(G) = x_1G' + x_2G' + x_3G' + x_4G'$, $x_i \in G\backslash F(G)$, where x_1 is a 3-element of G. We easily see that $|C_G(x_i)| = 6$. As x_1 is a 3-element of G, $|C_G(x)| = 6$ implies that $|G|_3 = 3$. Set $P \in \mathrm{Syl}_3(G)$. Then $|P| = 3$. Note that $F(G)$ is abelian. By Fitting lemma, we get $F(G) = C_{F(G)}(P) \times [F(G), P]$. Obviously, $C_{F(G)}(P) = Z(G)$ and $|Z(G)| = 2$ since $|C_G(g)| = 6$ for every $g \in G\backslash F(G)$. So, $G = B \times Z(G)$, where $B = [F(G), P]P$. Observe that B is a Frobenius group with kernel $B' = G' = [F(G), P]$ and complement of order 3. Hence, G satisfies (2) of the theorem.

CASE 3. $|G : F(G)| = 4$.

In this case, applying again Lemma 4.1, we get that $G' = F(G)$. Clearly, $G = KP$, where $K \leq G'$ is a normal 2-complement of G and $P \in \mathrm{Syl}_2(G)$. If P is non-abelian, then by Lemma 4.4, it is impossible (note that if G has the structure described in Lemma 4.4(2), then $K = F(G) < G'$, but we have $G' = F(G)$). In the following, we suppose that P is abelian and so $|P| = 4$. Then it follows from the hypothesis and Lemma 3.3 that G' is abelian.

We may assume that $G\backslash G' = xG' + yG' + zG'$, $x, y, z \in G\backslash G'$. Since $k_G(G\backslash G') = 4$, we may assume that both $xG' \cup yG'$ and zG' are a union of two conjugacy classes of G, respectively. Set $zG' = z^G + h^G$. we have

$$|G\backslash F(G)| = \frac{|G|}{|C_G(x)|} + \frac{|G|}{|C_G(y)|} + \frac{|G|}{|C_G(z)|} + \frac{|G|}{|C_G(h)|},$$

and

$$|C_G(x)| = |C_G(y)| = 4.$$

It follows that

$$\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{|C_G(z)|} + \frac{1}{|C_G(h)|}.$$

Hence $|C_G(z)| = 6$ and $|C_G(h)| = 12$ (note that $|P| = 4$). Thus G satisfies (3) of the theorem.

CASE 4. $|G : F(G)| = 5$.

Clearly, $G' = F(G)$. It follows by Lemma 4.8 that G is a Frobenius group with kernel G' and complement of order 5. Then by Lemma 3.5, we have that G' is abelian, and so G satisfies (4) of the theorem. The proof is completed. ∎

5. PROOF OF THEOREM 1.2

For a finite solvable group G, we define characteristic subgroup $F_i(G)$ by letting $F_1(G) = F(G)$, the unique largest nilpotent normal subgroup of G, and $F_{i+1}(G)/F_i(G) = F(G/F_i(G))$. The nilpotent length (or the Fitting height) of a group G, denoted by $\mathrm{nl}(G)$, is the smallest number l for which $F_l(G) = G$.

The following two results, which can be found in [14], will turn out to be useful in proof of Theorem 1.2.

LEMMA 5.1. *Let G be a solvable group. Then* $\mathrm{nl}(G) \leq (2m(G) + 5)/3$, *where $m(G)$ denotes the maximal number of conjugacy classes of G on which some $\chi \in \mathrm{Irr}(G)$ vanishes.*

LEMMA 5.2. *Suppose that* $\mathrm{nl}(G) \geq 2$. *If $|F_2(G) : F(G)|$ is not prime, then there exists $\chi \in \mathrm{Irr}(G)$ such that χ vanishes on at least two conjugacy classes of G contained in $F_2(G) - F(G)$.*

Now we are ready to prove Theorem 1.2.

PROOF OF THEOREM 1.2. Since G is solvable, the hypothesis together with Lemma 5.1 yield that $\mathrm{nl}(G) \leq 4$. If $\mathrm{nl}(G) = 1$, then by Corollary 2.5, $G \cong \mathrm{D}_8$ or Q_8. Thus G is a $V3C$-group. Hence we may assume that $\mathrm{nl}(G) = 2$, 3 or 4.

CASE 1. $\mathrm{nl}(G) = 2$.

In this case, if $G/F(G)$ is non-abelian, then by Corollary 2.5, $G/F(G) \cong \mathrm{D}_8$ or Q_8. Set $N/F(G) = (G/F(G))'$. We have that $|G : N| = 4$ and that $G \backslash N \subseteq \mathrm{Van}(G)$. Then it follows by Lemma 4.1 that $N = G'$, and thus $G = KP$, where $K < G'$ is a normal 2-complement of G and $P \in \mathrm{Syl}_2(G)$. Thus Lemma 4.4 implies that G is a Frobenius group with complement Q_8 (note that $\mathrm{nl}(G) = 2$). Thus G satisfies (1) of the Theorem. Hence we assume that $G/F(G)$ is abelian.

Since $G/F(G)$ is abelian, it follows by Lemma 4.3 that $G \backslash F(G) \subseteq \mathrm{Van}(G)$, and so $k_G(G \backslash F(G)) \leq 4$. If $k_G(G \backslash F(G)) = 1$, then G is a Frobenius group with kernel G' and complement of order 2, and thus G is a $V3C$-group. Therefore, $k_G(G \backslash F(G)) = 2$, 3 or 4.

CASE 1.1. $k_G(G \backslash F(G)) = 2$.

In the case, note that $G\backslash F(G) \subseteq \mathrm{Van}(G)$, then by the hypothesis and Proposition 4.6, we have to the following three cases.

CASE 1.1.1. G is a Frobenius group with kernel G' and complement of order 3.

In this case, by Theorem 3.4, G' is abelian. Thus G is a $V3C$-group.

CASE 1.1.2. $G = G'P$, where $G' < F(G)$ and G' is a normal 2-complement of G, and $P \in \mathrm{Syl}_2(G)$, $|P| = 4$.

In this case, observe that $F(G) = G' \times O_2(G)$ and $O_2(G) = Z(G)$. Observe that $k_{G/Z(G)}(G/Z(G)\backslash F(G)/Z(G)) = 1$, then $G/Z(G)$ is a Frobenius group with Frobenius kernel $(G/Z(G))' \cong G'$ and complement of order 2. Then G is of type (5) of Theorem 1.4, so a $V3C$-group.

CASE 1.1.3. $G = KP$, where $K < G'$ is a normal abelian 2-complement of G, $P \in \mathrm{Syl}_2(G)$, $P \cong \mathrm{D}_8$ or Q_8, $|G : G'| = 4$, and G' is abelian, $G\backslash G'$ is a union $x^G \cup y^G \cup z^G \cup h^G$ of four conjugacy classes satisfying $|C_G(x)| = 4$, $|C_G(y)| = 4$, $|C_G(z)| = 6$ and $|C_G(h)| = 12$.

As $k_G(G\backslash F(G)) = 2$, we get that $|G : F(G)| = 2$ and $|F(G) : G'| = 2$. Clearly, $F(G) = K \times O_2(G)$ and so $F(G)$ is abelian. Note that $12 \mid |F(G)|$ and hence $12 \mid |C_G(g)|$ for any $g \in F(G)$ yielding a contradiction.

CASE 1.2. $k_G(G\backslash F(G)) = 3$. Thus $G\backslash F(G)$ is a union $x^G \cup y^G \cup z^G$ of three conjugacy classes of G.

Assume that G is of type (8) of Proposition 4.7. Recall that $\mathrm{Van}(\mathrm{GL}(2,3))$ contains 6 conjugacy classes of $\mathrm{GL}(2,3)$ (see [6, P.161]), hence we may assume that $G/O_{2'}(G)$ is isomorphic to a non-abelian group of order 16. It follows by Corollary 2.5 that the case also does not occur.

Assume that G is of type (5) of Proposition 4.7. Then $|G : F(G)| = 4$ and G is a Frobenius group with complement Q_8, which is impossible (as $F(G)$ is the Frobenuis kernel). From the hypothesis and Proposition 4.7, we only need to consider the following three cases:

CASE 1.2.1. G is a Frobenius group with abelian kernel G' and cyclic complement of order 4.

In this case, G is of type (4) of Theorem 1.4, so a $V3C$-group.

CASE 1.2.2. $|G : F(G)| = 2$, $|C_G(x)| = |C_G(y)| = |C_G(z)| = 6$, and $F(G)$ is of odd order and has an abelian 3-complement.

In this case, applying the hypothesis and Lemma 3.1, we get that $F(G)$ is abelian. If $G' = F(G)$, then Lemma 4.8 yields that G is a Frobenius group with kernel G' and cyclic complement of order 2, and thus G is of type (1) of Theorem 1.4, so a $V3C$-group. Hence $G' < F(G)$. Furthermore, applying again by Lemma 4.1, we have that $|F(G) : G'| = 3$.

Recall that $G\backslash F(G) = xG' + yG' + zG'$, where $x, y, z \in G\backslash F(G)$. Then by the second orthogonality relation we have

$$6 = |C_G(g)| = |G/G'| + \sum\{|\chi(g)|^2 \mid \chi \in \mathrm{Irr}_1(G)\},$$

for all $g \in G\backslash F(G)$. Hence $\chi(g) = 0$ for all $g \in G\backslash F(G)$ and all $\chi \in \mathrm{Irr}_1(G)$.

Set $P \in \mathrm{Syl}_2(G)$. Note that $|G : F(G)| = 2$ and that $F(G)$ is of odd order. Let $P = \langle t \rangle$, where t is an involution. By Fitting Lemma, we have $F(G) = C_{F(G)}(P) \times [F(G), P]$. Obviously, $C_{F(G)}(t) = C_{F(G)}(P) = Z(G)$. Since $|C_G(g)| = 6$ for every $g \in G \backslash F(G)$, we conclude that $|Z(G)| = 3$. So, $G = B \times Z(G)$, where $B = [F(G), P]P$. Observe that B is a Frobenius group with kernel $B' = G' = [F(G), P]G'$ and complement of order 2. Then G is of type (7) of Theorem 1.4, so a $V3C$-group.

CASE 1.2.3. $|G : F(G)| = 2$, $|C_G(x)| = 4$, $|C_G(y)| = 6$ and $|C_G(z)| = 12$.

In this case, it follows by Lemma 3.1 that $F(G)$ is abelian. Recall now that $G = KP$, where $K \leq G'$ is a normal 2-complement of G and $P \in \mathrm{Syl}_2(G)$. It follows by Lemma 4.4 that P is abelian (note that $|G : F(G)| = 2$), and so $K = G'$ and $|P| = 4$.

Notice that $|G : F(G)| = 2$ and $F(G)$ is abelian. It easily follows that $F(G) = G' \times Z(G)$ and that $Z(G)$ is of order 2. Then we have that $[G/Z(G) : (G/Z(G))'] = [G/Z(G) : F(G)/Z(G)] = 2$, and thus it follows by Lemma 4.8 that G is of type (5) of Theorem 1.4, so a $V3C$-group.

CASE 1.3. $k_G(G \backslash F(G)) = 4$.

Note that $G/F(G)$ is abelian and $G \backslash F(G) = \mathrm{Van}(G)$. Then applying Theorem 4.9, G satisfies (2), (3), (4) or (5) of Theorem 1.2.

CASE 2. $\mathrm{nl}(G) = 3$.

Form the proof of the case when $\mathrm{nl}(G) = 2$, we see that if $\mathrm{nl}(G) = 2$ then $G \backslash F(G) \subseteq \mathrm{Van}(G)$ and $F(G)$ is abelian. So, if $\mathrm{nl}(G) = 3$, then $G/F(G) \backslash F_2(G)/F(G) \subseteq \mathrm{Van}(G/F(G))$ and $F_2(G)/F(G)$ is abelian. It follows by Lemma 4.3 that $F_2(G) \backslash F(G) \subseteq \mathrm{Van}(G)$, and thus $G \backslash F(G) \subseteq \mathrm{Van}(G)$ (note that $G \backslash F_2(G) \subseteq \mathrm{Van}(G)$). Hence the hypothesis yields that $k_{G/F(G)}(G/F(G)) \leq 5$. Since $\mathrm{nl}(G/F(G)) = 2$, it follows by [18] that $G/F(G)$ is isomorphic to one of the following groups: S_3, D_{10}, A_4, D_{14}, $(\mathrm{C}_4, \mathrm{C}_5)$ or $(\mathrm{C}_3, \mathrm{C}_7)$.

From the proof of paragraph above, we have that both $G \backslash F_2(G)$ and $F_2(G) \backslash F(G)$ are contained in $\mathrm{Van}(G)$. Since $F_2(G) \backslash F(G)$ contains at least one conjugacy class of G, it follows by the hypothesis that $k_G(G \backslash F_2(G)) = 2$ or 3 (if $k_G(G \backslash F_2(G)) = 1$, then G is a Frobenius group with kernel G' and complement of order 2, and thus G is a $V3C$-group.).

CASE 2.1. $k_G(G \backslash F_2(G)) = 2$.

In this case, by Proposition 4.6, we have to discuss the following two cases:

CASE 2.1.1. $|F_2(G) : G'| = 2$, $|G : F_2(G)| = 2$, $G = G'P$, where G' is a normal 2-complement of G, and $P \in \mathrm{Syl}_2(G)$, $|P| = 4$.

Note that $[G : F_2(G)] = 2$ and $k_G(F_2(G) \backslash F(G)) \leq 2$, we have that $G/F(G) \cong \mathrm{S}_3$ or D_{10}, and thus $|G/F(G) : (G/F(G))'| = 2$. If $O_2(G) = 1$, then $F(G) \leq G'$ and so $|G/F(G) : (G/F(G))'| = |G : G'| = 4$, a contradiction. Hence $|O_2(G)| = 2$ and so $O_2(G) = Z(G)$. Clearly,

$$k_{G/Z(G)}(G/Z(G) \backslash F_2(G)/Z(G)) = 1,$$

so $G/Z(G)$ is a Frobenius group with kernel $(G/Z(G))' \cong G'$ and complement $P/Z(G)$ of order 2. Then G' is abelian and thus $G' \leq F(G)$, a contradiction (note that $\mathrm{nl}(G) = 3$).

CASE 2.1.2. $G' = F_2(G)$, $|G : G'| = 2$ and $G/O_{2'}(G) \cong \mathrm{S}_4$.

Write $\overline{G} = G/O_{2'}(G)$. Then $F_2(G)/O_{2'}(G) = F_2(\overline{G})$. Observe that $G/F(G) \cong \mathrm{S}_3$ or D_{10}; thus $|O_2(G)| = 4$ and so $\overline{F(G)} = F(\overline{G})$. Hence we easily see that $G/F(G) \cong \mathrm{S}_3$ and that $F(G) = O_2(G) \times O_{2'}(G)$.

Assume that $F_2(G) \backslash F(G) = x^G + y^G$, where x is a 3-element of G. Observe that $|C_G(x)| = |C_G(y)| = 6$ and so x is an element of order 3. we easily see that x commutes with an involution t in $F(G)$, we obtain a contradiction (note that $\overline{F_2(G)}$ is a Frobenius group with kernel $\overline{F(G)}$). The contradiction shows that $k_G(F_2(G) \backslash F(G)) = 1$ and so $k_G(G \backslash F(G)) = 3$. Recall that $G/F(G) \cong \mathrm{S}_3$; then by Proposition 4.7, $G \cong \mathrm{S}_4$ and thus G is of type (6) of Theorem 1.4, so a $V3C$-groups.

CASE 2.2. $k_G(G \backslash F_2(G)) = 3$.

Recall that $\mathrm{nl}(G) = 3$. We then in a position to apply Proposition 4.7 to G, with $F_2(G)$ playing the role of N, obtaining that G is of types (6), (7) or (8) of Proposition 4.7. In particular, $[G : F_2(G)] = 2$. On the other hand, it easily see that $k_G(F_2(G) \backslash F(G)) = 1$, and so $F_2(G)/F(G)$ contains exactly two conjugacy classes of $G/F(G)$. Hence $G/F(G) \cong \mathrm{S}_3$. Set $F_2(G) \backslash F(G) = g^G$ such that g is a 3-element. Observe that $|C_G(g)| = 3$ and thus $|G|_3 = 3$. Recall that $k_G(G \backslash F_2(G)) = 3$ and $G/F(G) \cong \mathrm{S}_3$; thus $|G : F_2(G)| = 2$ and G satisfies (6) or (7) of Proposition 4.7. Then G contains an element x with $|C_G(x)| = 6$, which shows that there exists an element y in $F_2(G) \backslash F(G)$ such that $C_G(y)$ contains an involution, a contradiction.

CASE 3. $\mathrm{nl}(G) = 4$.

Then $\mathrm{nl}(G/F(G)) = 3$ and thus $G/F(G) \cong \mathrm{S}_4$. Consequently,

$$G/F(G) \backslash F_2(G)/F(G) \subseteq Van(G/F(G)).$$

It follows that $G \backslash F_2(G) \subseteq \mathrm{Van}(G)$. Note that $F_2(G)/F(G)$ is an elementary abelian group of order 4, then Lemma 4.3 together with Lemma 5.2 yield that $F_2(G) \backslash F(G) \subseteq \mathrm{Van}(G)$ and that $k_G(F_2(G) \backslash F(G)) \geq 2$, thus we reach a contradiction (recall that $k_G(G \backslash F_2(G)) \geq 3$). The contradiction completes the proof. ∎

LEMMA 5.3 ([1, Theorem 2.3]). *There is some vanishing sum* $v_1 + v_2 + ... + v_n = 0$ *of* n *mth roots of unity if and only if* n *is a linear combination, with non-negative integer coefficients, of the prime divisors of* m.

COROLLARY 5.4. *Suppose that* $\chi(1) = 2$ *or* 4 *for all* $\chi \in \mathrm{Irr}_1(G)$. *Then* $\mathrm{Van}(G)$ *cannot contain elements of odd orders.*

Let χ be an irreducible character of G. Note that if $x \in v(\chi)$, $z \in Z(G)$, then $xz \in v(\chi)$. Indeed, if D is a representation of G with character χ, then

$D(xz) = D(x)D(z) = (\lambda(z)I)D(x)$, where λ is a linear character of $Z(G)$ and I is the identity matrix with degree $\chi(1)$. So $\chi(xz) = \mathrm{tr}(D(xz)) = \lambda(z)\chi(x) = 0$.

Remark 5.5. We show that G is one of types except for (1) and (5) in Theorem 1.2, then G is a $V4C$-group (but not a $V3C$-group). Suppose that G is of type (2) of Theorem 1.2. Then by Corollary 5.4, $F(G)$ does not contain vanishing elements of G (note that $F(G) = G' \times Z(G)$). Note that $[G : F(G)] = 2$, thus by Lemma 4.3, $G \backslash F(G) = \mathrm{Van}(G)$. Set $G \backslash F(G) = n_1^G + \cdots + n_s^G$. Then, we get

$$|G \backslash F(G)| = \frac{|G|}{|C_G(n_1)|} + \cdots + \frac{|G|}{|C_G(n_s)|},$$

It follows that

$$\frac{1}{2} = \frac{1}{|C_G(n_1)|} + \cdots + \frac{1}{|C_G(n_s)|}.$$

Since $G/Z(G)$ is a Frobenius group with kernel $(G/Z(G))' \cong G'$ and complement $P/Z(G)$ of order 2, n_i acts fixed point freely on G', and so $|C_G(n_i)| = 8$ (note that P is abelian). Hence $s = 4$. If G satisfies (4) of Theorem 1.2, then by Corollary 5.4, G is a $V4C$-group (but not a $V3C$-group). Clearly, the same is true if G is of type (3) in Theorem 1.2.

It is worth mentioning that there exists Frobenius groups G with complement of order 5 and abelian kernel N, such that $N \cap \mathrm{Van}(G)$ is non-empty (see [1, Example 2]), so G is not a $V4C$-group.

In the following, we give three groups G_1, G_2 and G_3 satisfying (2), (3) and (4) in Theorem 1.2, respectively (see [18]).

$$G_1 = [C(3)], \quad C(8) = \langle a, b \,|\, a^8 = b^3 = 1, a^{-1}ba = b^{-1} \rangle,$$

$$G_2 = A_4 \times C(2), \quad G_3 = [C(15)],$$

$$C(4) = \langle a, b, c \,|\, a^5 = b^3 = c^4 = 1, ab = ba, c^{-1}ac = a^2, c^{-1}bc = b^{-1} \rangle.$$

However, if G satisfies (1) of Theorem 1.2, that is, G is a Frobenius group with kernel M and complement Q_8, then we do not know whether $M \cap \mathrm{Van}(G)$ is empty or not.

6. Non-solvable $V4C$-groups

In this section, we study the non-solvable $V4C$-groups.

Let p be a prime number. Recall that a character $\chi \in \mathrm{Irr}(G)$ is said to be of p-defect zero if p does not divide $|G|/\chi(1)$. By a fundamental result of Brauer (see [8, Theorem 8.17]), if $\chi \in \mathrm{Irr}(G)$ is of p-defect zero then, for every element $g \in G$ such that p divides $o(g)$, we have $\chi(g) = 0$.

The following Lemma comes from [3, Proposition 2.1].

Lemma 6.1. *Let G be a non-abelian simple group and p a prime number. If G is of Lie type, or if $p \geq 5$, then there exists $\chi \in \mathrm{Irr}(G)$ of p-defect zero.*

LEMMA 6.2. *Let G be a non-abelian simple group. If G is a $V4C$-group, then G is isomorphic to A_5.*

PROOF. Let G be a simple $V4C$-group. Let $G \cong \mathrm{A}_n$ for some $n \geq 14$. Then $\{5, 10, 7, 14, 11\} \subseteq \pi_e(G)$ and thus by Lemma 6.1, we obtain a contradiction. Hence the hypothesis implies that $G \cong \mathrm{A}_n$ for some $n \leq 13$, and so $G \cong \mathrm{A}_5$ from [2].

By [2], G cannot be a sporadic simple groups. By the classification theorem of the finite simple groups we can now suppose that G is a simple group of Lie type. Then, by Lemma 6.1, for each prime factor p of $|G|$ there exists some $\chi \in \mathrm{Irr}_1(G)$ such that χ is of p-defect zero. Hence any non-identity element of G is contained in $\mathrm{Van}(G)$. It follows by the hypothesis that G consists of five conjugacy classes, and then, by [18], G is isomorphic to A_5. ∎

LEMMA 6.3. *Let G be non-solvable group. If G is a $V4C$-group, then G has the unique non-abelian composite factor.*

PROOF. By induction, we may assume that $\mathrm{Sol}(G)$, the maximal solvable normal subgroup of G, is trivial. Let N be a (non-solvable) minimal normal subgroup of G. If N is not a non-abelian simple group, then $N = N_1 \times ... \times N_s$ is a direct product of isomorphic simple groups N_i, where $s \geq 2$. If $p \geq 5$, then there exists $\theta_i \in \mathrm{Irr}(N_i)$ of p-defect zero (see Lemma 6.1), and set

$$\theta = \theta_1 \times \ldots \times \theta_s.$$

Let χ_0 be an irreducible constituent of θ^G, let $x_1 \in N_1$ be of a prime order p, $x_2 \in N_2$ be of a prime order q. Notice that

$$x_1^G \subseteq N_1 \cup \ldots \cup N_s.$$

So $x_1 x_2$ is not conjugate to x_1. Clearly, θ^g is of p-defect zero for any $g \in G$, then we have

$$\theta^g(x_1) = \theta^g(x_1 x_2) = 0.$$

This implies that

$$\chi_0(x_1) = \chi_0(x_1 x_2) = 0.$$

Then the hypothesis yields that N_1 has only one prime divisor p greater than 3, that N_1 is a simple K_3-group (a simple group G is called a simple K_3-group if the number of prime factors of $|G|$ is 3), and also that N_1 has no irreducible character of 2-defect zero or 3-defect zero. By [2] and [5], we obtain a contradiction. Hence N is a simple group.

Suppose that G/N is non-solvable. Note that $\mathrm{out}(N)$ is solvable by the the classification of the finite simple groups, it follows that $C_G(N)$ is non-solvable and hence contains a non-solvable minimal normal subgroup M of G as $\mathrm{Sol}(C_G(N)) = 1$. Set $T = M \times N$. Let $\psi \in \mathrm{Irr}(M)$ be q-defect zero, and let $\theta \in \mathrm{Irr}(N)$ be a p-defect zero, where q, p are prime divisors of $|M|$ and $|N|$, respectively. Let x_1, $x_2 \in M$ be of order q, s, respectively, where $s \neq p$ and

$s \neq q$. Let y_1, $y_2 \in N$ be of order p, r, respectively, where $r \neq p$ and $r \neq q$. Then for any irreducible constituent χ of $(\psi \times \theta)^G$, we see that

$$\chi(x_1) = \chi(x_1 y_1) = \chi(x_1 y_2) = \chi(y_1) = \chi(y_1 x_2) = 0.$$

The contradiction completes the proof. □

LEMMA 6.4. *Let G be non-solvable group. If G is a $V4C$-group, then G is perfect (i.e. $G = G'$).*

PROOF. Otherwise, we may assume that $G' < G$. By Lemma 6.3, there exist two normal subgroups N and M of G such that $N < M \leq G'$ and M/N is a non-abelian simple group. From the argument of the proof of Lemma 6.3, we get that $(M/N) \cap \mathrm{Van}(G/N)$ is non-empty, and thus $G' \cap \mathrm{Van}(G)$ is also non-empty.

Suppose that there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi_{G'}$ is not irreducible. It follows by [8, Theorem 6.22] that G has a proper subgroup H such that $G' \leq H < G$ and $G \backslash H \subseteq v(\chi)$. Since $G' \cap$ $\mathrm{Van}(G)$ is also non-empty, it follows by the hypothesis that $k_G(G\backslash H) \leq 3$. If $k_G(G\backslash H) \leq 2$, then G is solvable (see [15, Theorem 2.2]), a contradiction. Hence we may assume that $k_G(G\backslash H) = 3$. Note that G is non-solvable; it follows by [15, Theorem 3.5] that G has a normal subgroup E such that $G/E \cong \mathrm{S}_5$ or M_{10}, then we obtain a contradiction from [2] and the proof is complete. □

LEMMA 6.5 ([12, Theorem]). *Suppose that a group G contains a subgroup X of order 3 such that $C_G(X) = X$. If,for every $g \in G$, the subgroup $\langle X, X^g \rangle$ is finite, then one of the following holds:*

(1) *$G = NN_G(X)$ for a periodic nilpotent subgroup N of nilpotent class 2, and NX is a Frobenius group with kernel N and complement X.*
(2) *$G = NA$, where A is isomorphic to $A_5 \cong SL_2(4)$ and N is a normal elementary Abelian 2-subgroup, here, N is a direct product of order 16 subgroups normal in G and isomorphic to the natural $SL_2(4)$-module of dimension 2 over a field of order 4.*
(3) *G is isomorphic to $L_2(7)$.*

Next we are ready to prove Theorem 1.3.

PROOF OF THEOREM 1.3. Clearly our hypothesis is inherited by any factor group. By Lemma 6.2, it suffices to show that G is a non-abelian simple group. Assume that G is a minimal counter-example.

First, we show that G has a minimal normal subgroup N such that G/N is non-solvable. Otherwise, we may assume that G has the unique minimal normal subgroup N such that G/N is solvable. Recall that $(G/N)' = G'N/N$, thus it follows by Lemma 6.4 that $(G/N)' = G/N$, which yields that $N = G$, a contradiction (note that G is a minimal counter-example). Hence G/N is non-solvable. Now it follows by induction that G/N is a non-abelian simple group. Applying Lemma 6.2, we conclude that $G/N \cong \mathrm{A}_5$. Then G/N has exactly one conjugacy class of elements of order 3. Choose a 3-element a of G such that $(aN)^{G/N}$ is the class of elements of order 3 in G/N. Set $A = (aN)^{G/N}$.

Recall that $k_{G/N}(\mathrm{Van}(G/N)) = 4$; thus $k_G(A) = 1$. Then each $\chi \in \mathrm{Irr}(G|N)$ vanishes on A. By the second orthogonality relation we have

$$|C_G(a)| = |C_{G/N}(aN)| = 3.$$

Hence G has an element a with $C_G(a)$ of order 3. Hence G satisfies the hypothesis of Lemma 6.5.

If G is the group in Lemma 6.5(1), then G is solvable, which is a contradiction.

Suppose that G has the structure described in Lemma 6.5(3). Then we obtain a contradiction from [2]. Hence G is the group in Lemma 6.5(2). But G is not a $V4C$-group (see [18, p. 310]), which is the final contradiction. ◻

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References

[1] D. Bubboloni, S. Dolfi and P. Spiga, *Finite groups whose irreducible characters vanish only on p-elements,* J. Pure Appl. Algebra **213** (2009), 370–376.

[2] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, Atlas of finite groups, Oxford University Press, Eynsham, 1985.

[3] S. Dolfi, E. Pacifici, L. Sanus and P. Spiga, *On the orders of zeros of irreducible characters,* J. Algebra **321** (2009), 345–352.

[4] S. Dolfi, E. Pacifici, L. Sanus and P. Spiga, *On the vanishing prime graph of solvable groups,* J. Group Theory **13** (2010), 189–206.

[5] M. Herzog, *On finite simple groups of order divisible by three primes only,* J. Algebra **10** (1968), 383–388.

[6] B. Huppert, Character theory of finite groups, de Gruyter, Berlin, 1998.

[7] B. Huppert, Endliche gruppen, Vol. 1, Springer-Verlag, Berlin-New York, 1967.

[8] I. M. Isaacs, Character theory of finite groups, Academic Prees, New York-London 1976.

[9] I. M. Isaacs, G. Navarro and T. R. Wolf, *Finite group elements where no irreducible character vanishes,* J. Algebra **222** (1999), 413–423.

[10] G. James and M. Liebeck, Representations and character of groups, Cambridge University Press, Cambridge, 1993.

[11] O. Manz and T. R. Wolf, Representations of solvable groups, Cambridge University Press, Cambridge, 1993.

[12] V. D. Mazurov, *On groups that contain a self-centralizing subgroup of order* 3*,* Algebra and Logic **42** (2003), 29–36.

[13] A. Moretó and J. Sangroniz, *On the number of conjugacy classes of zeros of characters,* Israel J. Math. **142** (2004), 163–187.

[14] G. Qian, *Bounding the fitting height of a solvable group by the number of zeros in a character table,* Proc. Amer. Math. Soc., **142** (2002), 3171–3176.

[15] G. Qian, W. Shi and X. You, *Conjugacy classes outside a normal subgroup,* Comm. Algebra **32** (2004), 4809–4820.

[16] Y. C. Ren, X. H. Liu and J. S. Zhang, *Notes on the restriction and the zeros of an irreducible character of a finite group,* Sichuan Daxue Xuebao **44** (2007), 1183–1188.

[17] C. Shao, S. Humphries, X. You and J. Zhang, *A note on conjugacy classes outside a normal subgroup,* Comm. Algebra **37** (2009), 3306–3308.

[18] A. Vera Lopez and J. Vera Lopez, *Classification of finite groups according to the number of conjugacy classes,* Israel J. Math. **51** (1985), 305–338.

[19] W. J. Wong, *Finite groups with a self-centralizing subgroup of order* 4*,* J. Austral. Math. Soc. **7** (1967), 570–576.

[20] J. S. Zhang, W. J. Shi and Z. C. Shen, *Finite groups in which every irreducible character vanishes on at most three conjugacy classes,* J. Group Theory **13** (2010), 799–819.

J. Zhang
School of Science
Sichuan University of Science and Engineering
643000 Zigong
P. R. China
E-mail: zjscdut@163.com

Z. Shen
College of Science
China Agricultural University
100083 Beijing
P. R. China
E-mail: zhencai688@sina.com

J. Shi
School of Mathematics and Information Science
Yantai University
264005 Yantai
P. R. China
E-mail: shijt@math.pku.edu.cn

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