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FINITE GROUPS HAVING AT MOST 27 NON-NORMAL PROPER SUBGROUPS OF NON-PRIME-POWER ORDER

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Abstract. We prove that any finite group having at most 27 non-normal proper subgroups of non-prime-power order is solvable except for $G \cong A_5$, the alternating group of degree 5.

1. Introduction

All groups are considered to be finite. Note that a group of non-prime-power order in which every non-trivial subgroup has prime-power order is a minimal group of non-prime-power order. In [3], Gallian and Moulton obtained a complete classification of non-abelian minimal groups of non-prime-power order. Obviously any non-abelian minimal group of non-prime-power-order is solvable. In [6] and [7], we showed that if a group G has either less than three conjugacy classes of proper subgroups of non-prime-power order or less than three classes of proper subgroups of the same non-prime-power order then G is solvable, and G is a non-solvable group having exactly either three conjugacy classes of proper subgroups of non-prime-power order or three classes of proper subgroups of the same non-prime-power order if and only if $G \cong A_5$. Moreover, we proved that a non-solvable group G has exactly four conjugacy classes of proper subgroups of non-prime-power order if and only if $G \cong PSL(2,8)$, and a non-solvable group G has exactly four classes of proper subgroups of the same non-prime-power order if and only if $G \cong PSL(2,7)$ or $PSL(2,8)$, see [8].

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Note that there always exists at least one solvable group G such that G has exactly n proper subgroups of non-prime-power order for any positive integer $n \geq 1$. For example, let $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{p^{n+1}}$, where $p \geq 3$ is a prime. Then G is a solvable group having exactly n proper subgroups of non-prime-power order.

From the above fact, a natural question arises:

QUESTION 1.1. *Does there always exist a non-solvable group G such that G has exactly n proper subgroups of non-prime-power order for any positive integer $n \geq 1$?*

As an answer to the above question, we have the following result, the proof of which is given in Section 3.

THEOREM 1.1. *Any group G having at most 27 proper subgroups of non-prime-power order is solvable except for $G \cong A_5$, the alternating group of degree 5.*

As an extension of Theorem 1.1, we further obtain the following result, the proof of which is given in Section 4.

THEOREM 1.2. *Any group G having at most 27 non-normal proper subgroups of non-prime-power order is solvable except for $G \cong A_5$.*

2. Preliminaries

In this section, we prove two essential lemmas needed in the sequel.

LEMMA 2.1. *Suppose that G is a minimal non-abelian simple group.*

(1) *If $G \cong A_5$, then G has exactly 21 proper subgroups of non-prime-power order.*
(2) *If $G \ncong A_5$, then G has at least 50 proper subgroups of non-prime-power order.*

PROOF. It is obvious that $G \cong A_5$ has exactly 21 proper subgroups of non-prime-power order by [1]. Next, suppose that G is a minimal non-abelian simple group that is not isomorphic to A_5. By [10], G might be isomorphic to one of the following groups: $PSL(2,p)$, $p > 5$ is a prime such that $5 \nmid p^2 - 1$; $PSL(2,2^q)$, q is an odd prime; $PSL(2,3^q)$, q is an odd prime; $PSL(3,3)$; $Sz(2^q)$, q is an odd prime.

(1) Suppose that $G \cong PSL(2,p)$, $p > 5$ is a prime such that $5 \nmid p^2 - 1$. Let $p = 7$. By [1], $PSL(2,7)$ has exactly 22 maximal subgroups that have non-prime-power order. Note that S_3 is also a proper subgroup of $PSL(2,7)$ of non-prime-power order. The number of conjugates of S_3 in $PSL(2,7)$ is equal to $|PSL(2,7) : N_{PSL(2,7)}(S_3)| = |PSL(2,7) : S_3| = 28$. Therefore, $PSL(2,7)$ has at least 50 proper subgroups of non-prime-power order. If $p > 7$, then by the hypothesis, $p \geq 13$. By [2], $PSL(2,p)$ has a maximal subgroup A that

is isomorphic to a dihedral group of order $p+1$ and a maximal subgroup B that is isomorphic to a dihedral group of order $p-1$. Obviously $p+1$ and $p-1$ cannot be a 2-power at the same time. If $p+1$ is not a 2-power, then $PSL(2,p)$ has at least $|PSL(2,p):N_{PSL(2,p)}(A)| = |PSL(2,p):A| = \frac{p(p-1)}{2} \geq \frac{13(13-1)}{2} = 78$ proper subgroups of non-prime-power order. If $p-1$ is not a 2-power, then $PSL(2,p)$ has at least $|PSL(2,p):N_{PSL(2,p)}(B)| = |PSL(2,p):B| = \frac{p(p+1)}{2} \geq \frac{13(13+1)}{2} = 91$ proper subgroups of non-prime-power order. Therefore, whenever $p > 5$ is a prime such that $5 \nmid p^2-1$, $PSL(2,p)$ has at least 50 proper subgroups of non-prime-power order.

(2) Suppose that $G \cong PSL(2,2^q)$, q is an odd prime. By [2], G has a maximal subgroup C that is isomorphic to a dihedral group of order $2\cdot(2^q+1)$ and a maximal subgroup D that is isomorphic to a dihedral group of order $2\cdot(2^q-1)$. Since $|G:N_G(C)| = |G:C| = 2^{q-1}\cdot(2^q-1) \geq 2^2\cdot(2^3-1) = 28$ and $|G:N_G(D)| = |G:D| = 2^{q-1}\cdot(2^q+1) \geq 2^2\cdot(2^3+1) = 36$, G has at least 64 proper subgroups of non-prime-power order.

(3) Suppose that $G \cong PSL(2,3^q)$, q is an odd prime. By [2], G has a maximal subgroup E that is isomorphic to a dihedral group of order 3^q-1. Since $|G:N_G(E)| = |G:E| = \frac{3^q(3^q+1)}{2} \geq \frac{3^3(3^3+1)}{2} = 378$, G has at least 378 proper subgroups of non-prime-power order.

(4) Suppose that $G \cong PSL(3,3)$. By [1], G has a maximal subgroup F of order 39 that is isomorphic to the normalizer of a Sylow 13-subgroup of G. Since $|G:N_G(F)| = |G:F| = 144$, G has at least 144 proper subgroups of non-prime-power order.

(5) Suppose that $G \cong Sz(2^q)$, q is an odd prime. By [9], G has a maximal subgroup S of order $2^{2q}(2^q-1)$ that is isomorphic to a Frobenius group. Since $|G:N_G(S)| = |G:S| = 2^{2q}+1 \geq 2^6+1 = 65$, G has at least 65 proper subgroups of non-prime-power order.

Therefore, G has at least 50 proper subgroups of non-prime-power order whenever G is a minimal non-abelian simple group that is not isomorphic to A_5. ∎

LEMMA 2.2. *Suppose that G is a group such that $G/\Phi(G) \cong A_5$. If $\Phi(G) \neq 1$, then G has at least* 37 *proper subgroups of non-prime-power order.*

PROOF. (1) Suppose that $|\Phi(G)|$ is not a prime-power. Since A_5 has exactly 58 proper subgroups, G has exactly 58 proper subgroups H such that $\Phi(G) \leq H$. Obviously H has non-prime-power order. It follows that G has at least 58 proper subgroups of non-prime-power order.

(2) Suppose that $|\Phi(G)|$ is a prime-power. Since $G/\Phi(G) \cong A_5$, $|\Phi(G)|$ might only be a 2-power or a 3-power or a 5-power. Let $|\Phi(G)|$ be a 2-power. Since A_5 has exactly 37 non-trivial subgroups of non-2-power order, G has at least 37 proper subgroups of non-prime-power order. Let $|\Phi(G)|$ be a 3-power. Since A_5 has exactly 47 non-trivial subgroups of non-3-power order, G has at

least 47 proper subgroups of non-prime-power order. If $|\Phi(G)|$ is a 5-power. Since A_5 has exactly 51 non-trivial subgroups of non-5-power order, G has at least 51 proper subgroups of non-prime-power order.

From above arguments, G has at least 37 proper subgroups of non-prime-power order. □

3. Proof of Theorem 1.1

The proof of Theorem 1.1 follows from the following three lemmas.

Lemma 3.1. *Suppose that G is a group having at most 20 proper subgroups of non-prime-power order. Then G is solvable.*

Proof. Let G be a counterexample of minimal order. It follows that G is a minimal non-solvable group. Therefore, $G/\Phi(G)$ is a minimal non-abelian simple group. However, $G/\Phi(G)$ has at least 21 proper subgroups of non-prime-power order by Lemma 2.1, which implies that G has at least 21 proper subgroups of non-prime-power order, a contradiction. Therefore, G is solvable. □

Lemma 3.2. *Suppose that G is a non-solvable group having exactly 21 proper subgroups of non-prime-power order. Then $G \cong A_5$.*

Proof. By the hypothesis, every maximal subgroup of G has at most 20 proper subgroups of non-prime-power order. Then every maximal subgroup of G is solvable by Lemma 3.1. It follows that G is a minimal non-solvable group and then $G/\Phi(G)$ is a minimal non-abelian simple group. Since G has exactly 21 proper subgroups of non-prime-power order, by Lemma 2.1, $G/\Phi(G)$ might only be isomorphic to A_5. Then by Lemma 2.2, one has $\Phi(G) = 1$. Therefore, $G \cong A_5$. □

Lemma 3.3. *Suppose that G has exactly n proper subgroups of non-prime-power order, where $22 \leq n \leq 27$. Then G is solvable.*

Proof. Let G be a counterexample of minimal order. That is, for any proper subgroup M of G if M has exactly $k\,(22 \leq k \leq 27)$ proper subgroups of non-prime-power order then M is solvable.

(1) Suppose that G has at least one non-solvable maximal subgroup. Let N be a non-solvable maximal subgroup of G, by Lemmas 3.1 and 3.2, $N \cong A_5$. We claim that

$$N \trianglelefteq G.$$

Otherwise, assume that $N \ntrianglelefteq G$. By the hypothesis, the number of conjugates of N in G is not greater than 6. That is, $|G : N_G(N)| = |G : N| = k \leq 6$. Then $G/N_G \lesssim S_k$, where $k \leq 6$ and N_G is the largest normal subgroup of G that is contained in N. Since N is simple and $N \ntrianglelefteq G$, one has $N_G = 1$. It follows that $G \lesssim S_k$, where $k \leq 6$. Since G is non-solvable and $A_5 \cong N$ is maximal in G but $N \ntrianglelefteq G$, G might only be isomorphic to A_6. Obviously A_6 has more than 27 proper subgroups of non-prime-power order, a contradiction. Therefore, $N \trianglelefteq G$.

If G has a normal maximal subgroup $T \neq N$, then $T \cap N = 1$, as N is simple. It follows that $G = T \times N$ and then $N \cong G/T$ is a cyclic group of prime order, a contradiction. Therefore, N is the unique normal maximal subgroup of G.

Since G is non-solvable, by [4], G has at least three conjugacy classes of maximal subgroups. If G has a maximal subgroup of prime-power order H, by [5, Theorem 10.4.2], H must have 2-power order. That is, H is a Sylow 2-subgroup of G. Since all Sylow 2-subgroups of G are conjugate, G has at least one maximal subgroup K of non-prime-power order such that $K \neq N$. By the hypothesis, $|G : N_G(K)| \leq 5$. Since N is the unique normal maximal subgroup of G, we have $K \ntrianglelefteq G$. Then $|G : N_G(K)| = |G : K| = t$, where $3 \leq t \leq 5$. If $3 \leq t \leq 4$, one has that $G/K_G \lesssim S_t$ is solvable. It follows that G/K_G has at least one normal maximal subgroup, say A/K_G. Obviously, $K_G \nleq N$. Then $A \neq N$, a contradiction. If $|G : K| = 5$, one has $G/K_G \lesssim S_5$. Note that G/K_G must be non-solvable. Then $G/K_G \cong S_5$ or A_5. If $K_G \leq N$, one has $K_G = 1$. Obviously $G \ncong A_5$, and $G \cong S_5$ has more than 27 proper subgroups of non-prime-power order, a contradiction. If $K_G \nleq N$, one has $K_G \cap N = 1$ and then $G = K_G \times N$ also has more than 27 proper subgroups of non-prime-power order, a contradiction.

(2) Suppose that every maximal subgroup of G is solvable. It follows that G is a minimal non-solvable group and then $G/\Phi(G)$ is a minimal non-abelian simple group. By Lemma 2.1, $G/\Phi(G)$ might only be isomorphic to A_5. If $\Phi(G) \neq 1$, by Lemma 2.2, G has at least 37 proper subgroups of non-prime-power order, a contradiction. If $\Phi(G) = 1$, then $G \cong A_5$ has exactly 21 proper subgroups of non-prime-power order, also a contradiction.

From arguments (1) and (2), the counterexample does not exist and so G is solvable. ∎

Lemmas 3.1, 3.2 and 3.3 combined together give Theorem 1.1.

4. Proof of Theorem 1.2

Lemma 4.1. *Suppose that G is group having at most 20 non-normal proper subgroups of non-prime-power order. Then G is solvable.*

Proof. Let G be a counterexample of minimal order. If G has no normal proper subgroups of non-prime-power order, by Lemma 3.1, G is solvable, a contradiction. Suppose G has a normal proper subgroup of non-prime-power order, say R. For the group G/R, by the minimality of G, both R and G/R are solvable. It follows that G is solvable, also a contradiction. ∎

Lemma 4.2. *Suppose that G is a non-solvable group having exactly 21 non-normal proper subgroups of non-prime-power order. Then $G \cong A_5$.*

Proof. By the hypothesis and Lemma 4.1, every maximal subgroup of G is solvable. Then G is a minimal non-solvable group and so $G/\Phi(G)$ is a minimal non-abelian simple group. By Lemma 2.1, $G/\Phi(G)$ might only be isomorphic to A_5. If $\Phi(G) \neq 1$, by Lemma 2.2, G has at least 37 proper subgroups of non-prime-power order, which are non-normal in G, a contradiction. Therefore, $\Phi(G) = 1$, and then $G \cong A_5$. ∎

Lemma 4.3. *Suppose that G is a group having exactly n non-normal proper subgroups of non-prime-power order, where $22 \leq n \leq 27$. Then G is solvable.*

Proof. Let G be a counterexample of minimal order. That is, for any proper subgroup T of G, if T has exactly m $(22 \leq m \leq 27)$ non-normal proper subgroups of non-prime-power order, then T is solvable.

(1) Suppose that every maximal subgroup of G is solvable. It follows that G is a minimal non-solvable group and then $G/\Phi(G)$ is a minimal non-abelian simple group. Arguing as in Lemma 4.2, we can get a contradiction.

(2) Suppose that G has a non-solvable maximal subgroup, say M. By the hypothesis, and Lemmas 4.1 and 4.2, one has $M \cong A_5$. Moreover, arguing as in Lemma 3.3, we can conclude that $M \trianglelefteq G$. If G has no other normal proper subgroups of non-prime-power order except M, by the hypothesis, G has at most 6 proper subgroups of non-prime-power order not contained in M. Arguing as in Lemma 3.3, we can get a contradiction. So suppose now that G has a normal proper subgroup N of non-prime-power order such that $N \neq M$. Since M is simple, $N \cap M = 1$. It follows that $G = N \times M \cong N \times A_5$. Obviously, $N \times A_5$ has more than 27 non-normal proper subgroups of non-prime-power order, a contradiction.

Therefore, the counterexample does not exist and so G is solvable. ∎

Lemmas 4.1, 4.2 and 4.3 combined together give Theorem 1.2.

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