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FINITE p-GROUPS WITH $A \cap B$ BEING MAXIMAL IN A OR B FOR ANY TWO NON-INCIDENT SUBGROUPS A AND B

Zvonimir Janko
University of Heidelberg, Germany

Abstract. We determine here the structure of the title groups up to isomorphism.

The purpose of this paper is to classify the title groups. Two subgroups A and B of a p-group G are incident if $A \leq B$ or $B \leq A$. If we know something about the intersection $A \cap B$ of any two non-incident subgroups A and B of a finite p-group G, then this has a very strong influence on the structure of G and in some cases we can even determine the structure of G up to isomorphism. All groups considered here will be finite p-groups and our notation is standard (as introduced in [1]).

We prove here the following result.

Theorem 1. *Let G be a nonabelian finite p-group such that $A \cap B$ is maximal in A or B for any two non-incident subgroups A and B in G. Then G is one of the following groups:*

(a) *minimal nonabelian groups:* D_8, Q_8, $\mathrm{S}(p^3)$ *(nonabelian group of order p^3 and exponent p), $p > 2$, M_{p^n}, $n \geq 3$ with $n \geq 4$ in case $p = 2$,*
(b) *2-groups of maximal class and order $\geq 2^4$,*
(c) $G \cong \mathrm{Q}_8 \times \mathrm{C}_2$,
(d) *$G = Q * S$ with $Q \cong \mathrm{Q}_8$, $S \cong \mathrm{C}_4$ and $Q \cap S = \mathrm{Z}(Q)$,*
(e) *a regular p-group G of order p^4 and exponent p^2, $p > 2$, with $\mho_1(G) \cong \mathrm{C}_p$ and $\Omega_1(G) \cong \mathrm{S}(p^3)$,*
(f) *an irregular 3-group G of maximal class, $|G| = 3^4$ and $\exp(G) = 9$, where G has no elementary abelian subgroup of order 27.*

Conversely, all the above p-groups satisfy the assumption of the theorem.

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Proof. Let G be a p-group such that $A \cap B$ is maximal in A or B for any two non-incident subgroups A and B in G. We note that this hypothesis is hereditary for subgroups and factor-groups.

(i) Let A be an abelian subgroup of G. Then A is either cyclic or elementary abelian with $p^2 \leq |A| \leq p^3$ or $A \cong \mathrm{C}_p^e \times \mathrm{C}_p$, $e \geq 2$. From now on we assume that G is nonabelian.

Indeed, first suppose that A is elementary abelian. If $|A| \geq p^4$, then A has a subgroup $A_1 \times A_2$ with $A_1 \cong A_2 \cong \mathrm{E}_{p^2}$, contrary to our hypothesis. Hence $|A| \leq p^3$. Now suppose that A is not elementary abelian so that $\exp(A) = p^e$, $e \geq 2$. Let A_1 be a cyclic subgroup of order p^e in A and let A_2 be a complement of A_1 in A. By our hypothesis, $|A_2| \leq p$ and we are done.

(ii) Let M be a minimal nonabelian subgroup of G. Then

$$M \cong \mathrm{D}_8,\ \mathrm{Q}_8,\ \mathrm{S}(p^3),\ p > 2, \text{ or } M \cong \mathrm{M}_{p^n},\ n \geq 3 \text{ with } n \geq 4 \text{ in the case } p = 2.$$

All these minimal nonabelian subgroups satisfy the hypothesis of our theorem. From now on we assume that G is not minimal nonabelian.

Indeed, D_8, Q_8, and $\mathrm{S}(p^3)$ satisfy the hypothesis of our theorem. Let

$$M = \langle a, b \mid a^{p^u} = b^{p^v} = c^p = 1,\ [a,b] = c,\ [a,c] = [b,c] = 1\rangle,$$

where $u \geq 2$, $v \geq 1$. Then we get

$$\langle a\rangle \cap \langle b, c\rangle = \{1\} \text{ with } |\langle a\rangle| \geq p^2 \text{ and } |\langle b, c\rangle| \geq p^2,$$

a contradiction. Let

$$M = \langle a, b \mid a^{p^u} = b^{p^v} = 1,\ [a,b] = a^{p^{u-1}}\rangle, \text{ where } u \geq 2,\ v \geq 2.$$

But then

$$\langle a\rangle \cap \langle b\rangle = \{1\} \text{ with } |\langle a\rangle| \geq p^2 \text{ and } |\langle b\rangle| \geq p^2,$$

which is a contradiction. Hence we have $u \geq 2$, $v = 1$ and so $M \cong \mathrm{M}_{p^n}$, $n \geq 3$ with $n \geq 4$ in the case $p = 2$.

Conversely, let

$$\mathrm{M}_{p^{u+1}} \cong M = \langle a, b \mid a^{p^u} = b^p = 1,\ [a,b] = a^{p^{u-1}}\rangle,$$

where $u \geq 2$ and $u \geq 3$ if $p = 2$. Let X_1, X_2 be any two non-incident subgroups of M such that setting $X_1 \cap X_2 = X$, we have $|X_1 : X| \geq p^2$ and $|X_2 : X| \geq p^2$. If both X_1 and X_2 are noncyclic, then $X_1 \geq S$, $X_2 \geq S$, where $S = \Omega_1(M) \cong \mathrm{E}_{p^2}$. In this case X_1 and X_2 are incident since M/S is cyclic, a contradiction. We may assume that X_1 is cyclic so that $|X_1 : \mho_1(X_1)| = p$ and $\mho_1(X_1) > X$. Since $X_1 \geq M' \cong \mathrm{C}_p$, we have $X_1 \trianglelefteq M$. Suppose that $\mho_1(X_2) \not\leq X$. Then $\mho_1(X_1X_2)$ is noncyclic, contrary to the fact that $\mho_1(M)$ is cyclic. Hence $\mho_1(X_2) \leq X$. But $M' \leq \mho_1(X_1)$ and so $\Phi(X_1X_2) \leq \mho_1(X_1)$ giving $\mathrm{d}(X_1X_2) \geq 3$, contrary to the fact that each subgroup Y of M possesses a cyclic subgroup of index p and so $\mathrm{d}(Y) \leq 2$. We have proved that our group M satisfies the hypothesis of our theorem.

(iii) Suppose that G has no abelian normal subgroup of type (p, p). Then G is a 2-group of maximal class and order $\geq 2^4$. All these 2-groups of maximal class satisfy the hypothesis of our theorem. In the sequel we assume that G has a normal abelian subgroup U of type (p, p).

Let G be a 2-group of maximal class and order $\geq 2^4$. Then G possesses a unique cyclic subgroup $\langle a \rangle \cong \mathrm{C}_{2^n}$, $n \geq 3$, of index 2 which is a unique abelian maximal subgroup of G. Set $\langle z \rangle = \langle a^{2^{n-1}} \rangle$ so that for each $x \in G - \langle a \rangle$,

$$x^2 \in \langle z \rangle \text{ and } \mathrm{C}_{\langle a \rangle}(x) = \langle z \rangle.$$

Let X_1 and X_2 be two non-incident subgroups of G and set $X = X_1 \cap X_2$. Assume, by way of contradiction, $|X_1 : X| \geq 4$ and $|X_2 : X| \geq 4$. Since $X_i \cap \langle a \rangle \neq \{1\}$, $i = 1, 2$, we get $X_i \geq \langle z \rangle$ and so $X \geq \langle z \rangle$ implying $|X_i| \geq 8$. Assume that one of X_i, $i = 1, 2$, is contained in $\langle a \rangle$, say $X_1 \leq \langle a \rangle$. Since X_1 and X_2 are non-incident, we have $X_2 \not\leq \langle a \rangle$ and $X_2 \cap \langle a \rangle \not\geq X_1$ so that $X_2 \cap \langle a \rangle = X_2 \cap X_1 = X$. But then $|X_2 : X| = 2$, a contradiction. We have proved that both X_1 and X_2 are not contained in $\langle a \rangle$ and since $|X_i| \geq 8$, $i = 1, 2$, we see that both X_1 and X_2 are of maximal class. On the other hand, $X_1 \cap \langle a \rangle$ and $X_2 \cap \langle a \rangle$ are incident of orders ≥ 4 and so we may assume without loss of generality that $X_1 \cap \langle a \rangle \leq X_2 \cap \langle a \rangle$. We may set

$$X_1 = (X_1 \cap \langle a \rangle)\langle x_1 \rangle, \text{ where } x_1 \in G - \langle a \rangle \text{ and } x_1^2 \in \langle z \rangle.$$

Since $X_1 \not\leq X_2$, we have $x_1 \notin X_2$ and $X = X_1 \cap X_2 = X_1 \cap \langle a \rangle$ so that $|X_1 : X| = 2$, a contradiction.

(iv) If G possesses an elementary abelian subgroup E of order p^3, then $\exp(G) = p$.

Suppose that this is false. Then there is a cyclic subgroup $C \cong \mathrm{C}_{p^2}$. By hypothesis, $|C \cap E| = p$. Then $E = (C \cap E) \times R$, where $C \cap R = \{1\}$ and $|C| = |R| = p^2$, contrary to our hypothesis.

(v) Let A be a maximal normal abelian subgroup of G containing U. Then we have

$$A = \langle a \rangle \times \langle u \rangle,$$

where

$$\langle a \rangle \cong \mathrm{C}_{p^e},\ e \geq 2,\ \langle u \rangle \cong \mathrm{C}_p,\ a^{p^{e-1}} = z, \langle z \rangle \leq \mathrm{Z}(G),\ U = \langle z, u \rangle$$

and G has no elementary abelian subgroups of order p^3.

Let A be a maximal normal abelian subgroup of G containing U so that $A < G$. Assume that G has an elementary abelian subgroup of order p^3. By (iv), $\exp(G) = p$ and so $p > 2$. By (i), we get either $A = U \cong \mathrm{E}_{p^2}$ or $A \cong \mathrm{E}_{p^3}$. However, if $A = U$, then $|G| = p^3$ and $G \cong \mathrm{M}_{p^3}$ or $G \cong \mathrm{S}(p^3)$, contrary to our assumption that G is not minimal abelian. Hence we have $A \cong \mathrm{E}_{p^3}$. Let U_0 be a subgroup of order p in U such that $U_0 \leq \mathrm{Z}(G)$ and let $g \in G - A$ so that

$o(g) = p$ and $S = \langle U_0, g\rangle \cong \mathrm{E}_{p^2}$ with $S \cap A = U_0$. Let T be a complement of U_0 in A so that $A = T \times U_0$. But then $T \cap S = \{1\}$ and $|T| = |S| = p^2$, contrary to our hypothesis.

We have proved that G has no elementary abelian subgroups of order p^3. If $A = U$, then G is minimal nonabelian of order p^3, contrary to our assumption in (ii). By (i), A is abelian of type (p^e, p) with $e \geq 2$ and we are done.

(vi) If $\mathrm{C}_G(U) > A$, then $G \cong \mathrm{Q}_8 \times \mathrm{C}_2$ and so G is a group appearing in part (c) of our theorem. From now on we may assume that $\mathrm{C}_G(U) = A$ so that $|G : A| = p$ and $\exp(G) = p^e$.

Indeed, assume that $\mathrm{C}_G(U) > A$. Since G has no elementary abelian subgroups of order p^3, we have

$$\Omega_1(\mathrm{C}_G(U)) = U \leq \mathrm{Z}(\mathrm{C}_G(U)).$$

By [2, Lemma 57.1], for each $x \in \mathrm{C}_G(U) - A$, there is $y \in A - U$ such that $M = \langle x, y\rangle$ is minimal nonabelian, where $\Omega_1(M) \leq \mathrm{Z}(M)$. By (ii), this forces $p = 2$ and $M \cong \mathrm{Q}_8$. Hence $o(x) = o(y) = 4$ and $x^2 = y^2 = z$ (see (v)). All elements $x \in \mathrm{C}_G(U) - A$ are of order 4 with $x^2 = z$ and since for an element y of order 4 in $A - U$,

$$y^x = y^{-1} = yz \quad \text{and} \quad (yu)^x = (yu)z = (yu)^{-1},$$

it follows that x acts invertingly on $\Omega_2(A) \cong \mathrm{C}_4 \times \mathrm{C}_2$ which gives $|\mathrm{C}_G(U) : A| = 2$. If $e > 2$, then

$$|\langle a\rangle| \geq 8,\ \langle x, u\rangle \cong \mathrm{C}_4 \times \mathrm{C}_2 \quad \text{with} \quad \langle a\rangle \cap \langle x, u\rangle = \langle z\rangle,$$

contrary to our hypothesis. Hence $e = 2$ and so $\mathrm{C}_G(U) = Q \times \langle u\rangle \cong \mathrm{Q}_8 \times \mathrm{C}_2$, where $\mathrm{Z}(Q) = \langle z\rangle$ and $A \cong \mathrm{C}_4 \times \mathrm{C}_2$. If $G = \mathrm{C}_G(U)$, then we are done. Therefore we may assume that $G > \mathrm{C}_G(U)$ so that $|G : \mathrm{C}_G(U)| = 2$ and $|G| = 2^5$. Suppose that there is $g \in G - \mathrm{C}_G(U)$ such that $g^2 \in \langle z, u\rangle$. Since $D = \langle z, u\rangle\langle g\rangle \cong \mathrm{D}_8$, we have $g^2 \in \langle z\rangle$. But then $Q \cap D = \langle z\rangle$, contrary to our hypothesis. We have proved that for each $g \in G - \mathrm{C}_G(U)$, $g^2 \in \mathrm{C}_G(U) - \langle z, u\rangle$ and so $\Omega_2(G) \cong \mathrm{Q}_8 \times \mathrm{C}_2$. By [2, Theorem 52.1], G is isomorphic to a uniquely determined group of order 2^5 given in part (A2)(a) of [2,Theorem 49.1]. In particular, there is an element $y \in G - \mathrm{C}_G(U)$ of order 8 such that $\langle y\rangle \cap Q = \mathrm{Z}(Q)$, contrary to our hypothesis.

From now on we may assume that $\mathrm{C}_G(U) = A$ so that $|G : A| = p$ and $\exp(G) = p^e$. Indeed, if $\exp(G) = p^{e+1}$, then G has a cyclic subgroup of index p and so G is either a 2-group of maximal class or $G \cong \mathrm{M}_{p^n}$ which is minimal nonabelian. But all these groups are excluded by our assumptions in (ii) and (iii).

(vii) If there are elements of order $\leq p^2$ in $G - A$, then $e = 2$, $A \cong \mathrm{C}_{p^2} \times \mathrm{C}_p$, $|G| = p^4$ and G is isomorphic to a group given in parts (d), (e) or (f) of our theorem. Conversely, all groups from parts (c) to (f) of our theorem satisfy the hypothesis of that theorem.

Assume that there is $g \in G - A$ such that $g^p \in \langle z, u\rangle = \Omega_1(A)$ and so $g^p \in \langle z\rangle$, where

$$\langle g, u\rangle \cong \mathrm{M}_{p^3} \text{ or } \mathrm{S}(p^3) \text{ with } p > 2 \text{ or } \langle g, u\rangle \cong \mathrm{D}_8.$$

If $e > 2$, then

$$|\langle a\rangle : \langle z\rangle| \geq p^2 \text{ and } |\langle g, u\rangle : \langle z\rangle| \geq p^2, \text{ where } \langle a\rangle \cap \langle g, u\rangle = \langle z\rangle,$$

contrary to our hypothesis. Thus, $e = 2$, $A \cong \mathrm{C}_{p^2} \times \mathrm{C}_p$ and $|G| = p^4$.

First consider the case $p = 2$. Since G is not of maximal class, a result of O. Taussky implies $G' = \langle z\rangle$. Hence $D = \langle g, u\rangle \cong \mathrm{D}_8$ with $D' = \langle z\rangle$ so that $G = D * S$, where $|S| = 4$ and $D \cap S = \langle z\rangle$. But G has no elementary abelian subgroups of order 8 and so $S \cong \mathrm{C}_4$. Since $\mathrm{D}_8 * \mathrm{C}_4 \cong \mathrm{Q}_8 * \mathrm{C}_4$, we have obtained the group from part (d) of our theorem.

Now we consider the case $p > 2$, where

$$D = \langle g, u\rangle \cong \mathrm{M}_{p^3} \text{ or } \mathrm{S}(p^3).$$

By (vi), $\exp(G) = p^2$ and so for each $x \in G - A$, $x^p \in \langle z\rangle$ and this implies $\mho_1(G) = \langle z\rangle$. We may use [2, Theorem 74.1]. If G is a group of part (a) of that theorem, then [3, Proposition 149.1] implies that G is minimal nonabelian, a contradiction. Hence G is a group of order p^4 from parts (c) or (d) of [2, Theorem 74.1]. We have obtained the groups stated in parts (e) and (f) of our theorem.

Conversely, let G be any group of order p^4 stated in parts (c), (d), (e), (f) of our theorem. Note that in all these cases, $\mho_1(G) \cong \mathrm{C}_p$ and G has no elementary abelian subgroups of order p^3. We claim that G satisfies the hypothesis of our theorem. Indeed, let X_1, X_2 be two non-incident subgroups of G. If X_1 or X_2 is a maximal subgroup of G, say X_1, then $|X_2 : (X_1 \cap X_2)| = p$. We may assume, by way of contradiction, that $|X_1| = |X_2| = p^2$ and $X_1 \cap X_2 = \{1\}$. Set $\mho_1(G) = \langle z\rangle \cong \mathrm{C}_p$, where $\langle z\rangle \leq \mathrm{Z}(G)$. If X_1 or X_2, say X_1, is elementary abelian of order p^2, then $z \in X_1$ because G has no elementary abelian subgroups of order p^3. But then $\mho_1(X_2) = \{1\}$ and so $\langle z\rangle \times X_2 \cong \mathrm{E}_{p^3}$, a contradiction. It follows that both X_1 and X_2 are cyclic of order p^2, which contradicts the fact that $\mho_1(G) \cong \mathrm{C}_p$.

(viii) Finally, we consider the remaining case that $\Omega_2(G) = \Omega_2(A) \cong \mathrm{C}_{p^2} \times \mathrm{C}_p$ and we shall obtain in this case a contradiction.

Indeed, in this case we must have $e \geq 3$ and so $|G| = p^{e+2} \geq p^5$ and by (vi), G is of exponent p^e.

First we consider the case $p = 2$. Since G is not minimal nonabelian, it follows that G is not an L_2-group and so [1, Lemma 42.1] implies that

$$G = \langle a, b \mid a^{2^e} = b^8 = 1,\ a^b = a^{-1},\ a^{2^{e-1}} = b^4 = z\rangle, \quad e \geq 3.$$

But then

$$o(a) \geq 8, \; o(b) = 8 \text{ and } \langle a \rangle \cap \langle b \rangle = \langle z \rangle \cong \mathrm{C}_2,$$

contrary to our hypothesis.

Now suppose $p > 2$. Since $|G| = p^{e+2} \geq p^5$ and G is of exponent p^e, $e \geq 3$, we may use [2, Theorem 74.1]. Hence G is one of the groups of parts (a) or (b) of that theorem. However, if G is metacyclic, then the fact that A is an abelian maximal subgroup of G implies together with [3, Proposition 149.1] that G is minimal nonabelian, a contradiction. Hence G is an L_3-group. But this contradicts our assumption that $\Omega_2(G) = \Omega_2(A) \cong \mathrm{C}_{p^2} \times \mathrm{C}_p$. Our theorem is proved. □

References

[1] Y. Berkovich, Groups of prime power order, Vol. 1, Walter de Gruyter, Berlin-New York, 2008.

[2] Y. Berkovich and Z. Janko, Groups of prime power order, Vol. 2, Walter de Gruyter, Berlin-New York, 2008.

[3] Y. Berkovich and Z. Janko, Groups of prime power order, Vol. 4, Walter de Gruyter, Berlin-New York, 2014.

Z. Janko
Mathematical Institute
University of Heidelberg
69120 Heidelberg
Germany
E-mail: janko@mathi.uni-heidelberg.de

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