GLASNIK MATEMATIČKI
Vol. 49(69)(2014), 163 – 178

POLYHARMONIC MAPPINGS AND J. C. C. NITSCHE TYPE INEQUALITIES

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Abstract. In this paper a J. C. C. Nitsche type inequality for polyharmonic mappings between rounded annuli on the Euclidean space $\mathbf{R}^d$ is considered. The case of radial biharmonic mappings between annuli on the complex plane and the corresponding inequality is studied in detail.

1. Introduction

By d we denote a positive integer and by $\mathbf{R}^d$ we denote the Euclidean space. The unit sphere is $S^{d-1} = \{x \in \mathbf{R}^d : |x| = 1\}$, where the norm of a vector $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$ is defined by

$$|x| := \sqrt{\sum_{k=1}^{d} x_k^2}.$$

By $A(r, R) = \{x \in \mathbf{R}^d : r < |x| < R\}$ we denote an annulus with inner and outer radii r and R, respectively. A mapping u is called *radial* if $u(x) = |x|u(x/|x|)$. The bi-harmonic equation is

$$\Delta^2 u = \Delta\Delta u = 0.$$

The polyharmonic equation is defined by the induction:

$$\Delta^m u := \Delta\left(\Delta^{m-1} u\right) = 0, \tag{1.1}$$

where m is a positive integer. By the Almansi representation theorem ([1]), the class of $m-$harmonic functions coincides with the class of functions of the

2010 *Mathematics Subject Classification.* 30C55, 31C05.
Key words and phrases. Harmonic mappings, poly-harmonic mappings, annuli.

form

$$f(x) = \sum_{k=0}^{m-1} |x|^{2k} f_k(x)$$

where f_k are harmonic functions. Thus, a planar mapping u is bi-harmonic if and only if $u(z) = |z|^2 g(z) + h(z)$, where g and h are harmonic mappings, i.e. the mappings w satisfying the Laplace equation $\Delta w = 0$ in some simply connected subdomain Ω of the complex plane $\mathbf{C}$. Every analytic function is a harmonic mapping and every bi-holomomorphic function is a harmonic diffeomorphism. A well known Schottky theorem asserts that two annuli on the complex plane can be mapped by means of a bi-holomorphic mapping if and only if they have the same modulus. The aim of this paper to study rigidity of polyharmonic mappings between annuli in the Euclidean space.

J. C. C. Nitsche ([9]) by considering the complex-valued univalent harmonic functions

$$f(z) = \frac{ts - t^2}{(1 - t^2)} \frac{1}{\bar{z}} + \frac{1 - ts}{1 - t^2} z, \tag{1.2}$$

showed that an annulus $1 < |z| < t$ can then be mapped onto any annulus $1 < |w| < s$ with

$$s \geq n(t) := \frac{1 + t^2}{2t}. \tag{1.3}$$

J. C. C. Nitsche conjectured that, condition (1.3) is necessary as well. The critical Nitsche map with zero initial speed is

$$f(z) = \frac{1 + |z|^2}{2\bar{z}}.$$

This means that this function makes the maximal distortion of rounded annuli $A(1, t)$.

Nitsche also showed that $s \geq s_0$ for some constant $s_0 = s_0(t) > 1$. Thus, although the annulus $1 < |z| < t$ can be mapped harmonically onto a punctured disk, it cannot be mapped onto any annulus that is "too thin". For the generalization of this conjecture to $\mathbb{R}^d$ and some related results we refer to [6]. For the case of hyperbolic harmonic mappings we refer to [2]. Some other generalizations have been done in [7]. The Nitsche conjecture for Euclidean harmonic mappings is settled recently in [3] by Iwaniec, Kovalev and Onninen, showing that, only radial harmonic mappings $g(z) = e^{i\alpha} f(z)$, where f is defined in (1.2), which inspired the Nitsche conjecture, making the extremal distortion of rounded annuli. For some partial results toward the Nitsche conjecture and some other generalizations we refer to the papers [4], [8] and [11].

In this paper, we consider a similar problem for polyharmonic mappings. We begin by the case when d is not an odd integer smaller or equal to $2m - 1$ and show that no Nitsche phenomenon can occur in this case. This is shown in Theorem 2.3. It is interesting that such kind of Nitsche phenomenon does not occur for the biharmonic mappings in Euclidean space $\mathbf{R}^3$ (contrary to the case of harmonic mappings, [7]). The main result (see Theorem 2.4(a)) implies in particular that: if f is a radial $m-$harmonic mapping between annuli $A(1,t)$ and $A(1,s)$ of the Euclidean space $\mathbf{R}^d$ such that d is not an odd integer smaller than $2m$, then $s > s(t) > 1$. In the second part of Theorem 2.4 we present a Nitsche type inequality for harmonic mappings in the space. It can be considered as a counterpart of all related results for harmonic mappings, because we do not assume either injectivity or surjectivity of a harmonic mapping. In Section 3 we find examples of radial bi-harmonic maps between annuli in the complex plane and then we establish some quantitative estimates of J. C. C. Nitsche type rigidity for radial biharmonic mappings. In addition, Section 3 contains some hard but elementary computer aided computations.

2. General polyharmonic mappings

In this section we consider Nitsche phenomenon of $m-$harmonic mappings between annuli on $\mathbf{R}^d$. We will treat two possible cases.

- Rigidity case: d is not an odd integer smaller or equal to $2m-1$.
- Non-rigidity case: d is an odd integer smaller or equal to $2m-1$.

We begin by the following lemma

Lemma 2.1. *For the mapping $f(x) = x/|x|$, $x \in \mathbf{R}^d \setminus \{0\}$, there holds the following formula*

$$\Delta^m f = \prod_{k=1}^{m} (2k-1)(2k-d-1)\frac{x}{|x|^{2m+1}}.$$

Thus the mapping $f(x) = x/|x|$ is m-harmonic on the space $\mathbf{R}^d \setminus \{0\}$ if and only if d is an odd integer smaller or equal to $2m-1$.

Proof. We use the following formula which can be proved by direct calculation

$$\Delta\left(\frac{x}{|x|^\alpha}\right) = \frac{\alpha(\alpha-d)x}{|x|^{\alpha+2}}.$$

By using the mathematical induction we obtain

$$\Delta^m\left(\frac{x}{|x|}\right) = \prod_{k=1}^{m} (2k-1)(2k-d-1)\frac{x}{|x|^{2m+1}}.$$

□

It follows from Lemma 2.1 that the mapping $f(z) = z/|z|$ is not bi-harmonic in the complex plane, and the question arises: is there any nonconstant biharmonic mapping with constant modulus? The following example gives a positive answer to the previous question.

EXAMPLE 2.2. The function

$$f(z) = \frac{z}{\bar{z}} = \frac{z^2}{|z^2|}$$

is a biharmonic mapping of $\mathbf{C} \setminus \{0\}$ onto the unit circle $\mathbf{T}$.

THEOREM 2.3. *Let d be an odd integer smaller or equal to $2m-1$. Then for every $t > 0$ and $s > 0$ there exists an $m-$harmonic diffeomorphism between the annuli $A(1,t)$ and $A(1,s)$.*

PROOF. Define

$$f(x) = \alpha \frac{x}{|x|} + (1-\alpha)x.$$

Then for $0 < \alpha < 1$ the mapping f is an $m-$harmonic diffeomorphism between the annuli $A(1,t)$ and $A(1,s)$, where $s = \alpha+(1-\alpha)t$. Namely $r \to \alpha+(1-\alpha)r$ is increasing in $[1,t]$, $\alpha + (1-\alpha)t \leq t$,

$$f\left(\frac{x}{|x|}\right) = \frac{x}{|x|}$$

and

$$f\left(t\frac{x}{|x|}\right) = (\alpha + (1-\alpha)t)\frac{x}{|x|}.$$

The existence of $m-$harmonic mappings between the annuli $A(1,t)$ and $A(1,s)$ for $s \geq t$ is established in [6]. Namely for

$$\rho \geq \frac{dr}{d-1+r^d} \tag{2.1}$$

the mapping defined by

$$f(x) = \left(\frac{1-r^{d-1}\rho}{1-r^d} + \frac{r^{d-1}\rho - r^d}{(1-r^d)|x|^d}\right)x \tag{2.2}$$

is a harmonic mapping between the annuli $\mathbf{A}(r,1)$ and $\mathbf{A}(\rho,1)$. Then the mapping

$$g(x) = \frac{1}{\rho}f(rx)$$

defines the harmonic mapping between $A(1,t)$ and $A(1,s)$ for $s \geq t$. ◻

THEOREM 2.4. *Assume d is a positive integer and $\mathbf{R}^d$ is the Euclidean space.*

(a) *Assume that d is not an odd integer smaller or equal to $2m-1$ and let $\mathfrak{P}_t$ be a family of $m-$harmonic orientation preserving mappings of $A(1,t)$ into itself, such that its closure w.r.t. sum norm contains no $m-$harmonic mapping u between $A(1,t)$ and S^{d-1}. Then there exists a positive constant $\delta(t) > 0$ such that $s \geq 1+\delta(t)$ and $\|u\| \geq 1+\delta(t)$ for $u \in \mathfrak{P}_t$.*

(b) *Let u be a harmonic mapping of the annulus $A(1,t)$ into $A(1,s)$ of the Euclidean space $\mathbf{R}^d$ such that $u(\frac{1+t}{2}S^{d-1})$ separates the boundary components of $A(1,s)$. Then there exists a positive constant $\delta(t)>0$ such that $s \geq 1+\delta(t)$.*

REMARK 2.5. Notice that in the statement (b) Theorem 2.4, we do not assume either injectivity or surjectivity. The theorem gives essentially new proof of Nitsche type rigidity for harmonic mappings. From Lemma 3.1 we obtain that the family $\mathfrak{P}_t$ of radial $m-$harmonic mappings satisfies the condition of Theorem 2.4(a), so the rigidity occur for this special class of mappings.

PROOF. Let $C[A(1,t)]$ be the Banach space of continuous mappings of the annulus $A(1,t)$ into itself with the norm

$$\|f\| = \sup_{x\in A(1,t)} |f(x)|.$$

Let $\mathcal{B}_t$ be the closure of $\mathfrak{P}_t \subset C[A(t,1)]$. We prove that $\mathcal{B}_t$ contains only polyharmonic function. We will prove first this fact for biharmonic mappings. The general situation follows by mathematical induction. To prove that $\mathcal{B}_t$ contains only $m-$harmonic function, assume that for some sequence of functions $u_n \in \mathcal{B}_t$ and $u \in C[A(1,t)]$ we have $\lim_{n\to\infty}\|u_n - u\| = 0$. Then for all $\varphi \in C_0^4[A(1,t)]$ we have

$$0 = \int_{A(1,t)} \varphi(x)\Delta^2 u_n(x)dA(x) = \int_{A(1,t)} u_n(x)\Delta^2\varphi(x)dA(x).$$

Letting $n\to\infty$ we have

$$\int_{A(1,t)} u(x)\Delta^2\varphi(x)dA(x) = 0.$$

By Sobolev embedding theorem we have

$$W^{4,1} \subset W^{2,2}.$$

It follows that

$$v(x) = \Delta u(x)\Delta\varphi(x) \in L^2(A(1,t))$$

and therefore,

$$\int_{A(1,t)} u(x)\Delta^2\varphi(x)dA(x) = \int_{A(1,t)} \Delta u\Delta\varphi(x)dA(x) = 0.$$

This implies that $w = \Delta u$ is a weak solution of the Laplace equation $\Delta w = 0$. By a well-known result, it follows that w is a smooth harmonic function. Finally we obtain that u is a bi-harmonic function.

Let

$$\tilde{f} := \frac{f}{|f|}$$

be the projection of the mapping f on the unit sphere. Define

$$\mathcal{P}_t = \{\tilde{f} : f \in \mathcal{B}_t\}.$$

Then $\mathcal{P}_t$ is a bounded closed subset of $C[A(1,t)]$.

Next let us show that the class of bounded $m-$harmonic mappings is a normal (compact) family of functions. From [10, Lemma 5] for a $m-$harmonic function f we have the inequality

$$|\nabla f(x)| \leq \frac{C'(m,d)}{r^{n+1}} \int_{|y-x|<r} |f(y)|dy. \tag{2.3}$$

If $M = \sup |f(x)|$, then it follows from (2.3), by taking $r = \operatorname{dist}(x, \partial A(1,t))$ that

$$|\nabla f(x)| \leq \frac{C(m,d)}{\operatorname{dist}(x, \partial A(1,t))} M. \tag{2.4}$$

Now the claim follows from Ascoli-Arzela theorem.

It follows that the class of bounded $m-$harmonic mappings is a normal (compact) family of functions. Take $\epsilon < (t-1)/2$ and define

$$A_{t,\epsilon} = \overline{A(1+\epsilon, t-\epsilon)}.$$

In the following let $\mathcal{P}_{t,\epsilon}$ ($\mathcal{B}_{t,\epsilon}$) denote the restriction of the class of mappings $\mathcal{P}_t$ (i.e. of $\mathcal{B}_t$) to the closed annulus $A_{t,\epsilon}$. Since the mapping

$$F_\epsilon : \mathcal{B}_{t,\epsilon} \to \mathbf{R}$$

defined by $F_\epsilon(f) = \|f - \tilde{f}\|$ is continuous and $\mathcal{B}_{t,\epsilon}$ is compact, by the Weierstrass theorem, we have

$$\begin{aligned}
\delta_{t,\epsilon} =: \operatorname{dist}(\mathcal{P}_{t,\epsilon}, \mathcal{B}_{t,\epsilon}) &:= \inf_{f \in \mathcal{B}_{t,\epsilon}, g \in \mathcal{P}_{t,\epsilon}} \|f - g\| \\
&= \inf_{f_\epsilon \in \mathcal{B}_{t,\epsilon}} \|f_\epsilon - \tilde{f}_\epsilon\| = \|g_\epsilon - \tilde{g}_\epsilon\| \leq \inf_{f \in \mathcal{B}_t} \|f - \tilde{f}\|,
\end{aligned}$$

where g_ϵ is the restriction of a mapping $g \in \mathfrak{P}_t$. Since $\tilde{g} \notin \mathfrak{P}_t$, it follows that for some $\epsilon > 0$ we have $\delta_{t,\epsilon} > 0$. We define

$$\delta_t = \sup_{0 \leq \epsilon \leq (t-1)/2} \delta_{t,\epsilon}.$$

Now if f is a mapping between $A(1,t)$ and $A(1,s)$, with $s < 1 + \delta_t$, then

$$\|f - \tilde{f}\| = \sup_{z \in A(1,t)} (|f(z)| - 1) < \delta_t,$$

which implies that $f \notin \mathcal{B}_t$.

We now prove (b). In order to use the proof of (a), we show that if $\{u_n\}$ is a sequence of harmonic mappings satisfying the condition of the theorem, and $u = \lim_{n\to\infty} u_n$ then u is not a constant modulus. Since $u_n(\frac{1+t}{2} S^{d-1})$

separates the boundary components of $A(1,s)$, there exist two convergent sequences $\{x_n\}$ and $\{y_n\}$ from $\frac{1+t}{2}S^{d-1}$ such that

$$|u_n(x_n) - u_n(y_n)| > 2.$$

If $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$, then $|u(x) - u(y)| \geq 2$, implying that u is not a constant function.

Finally, we prove that there exists no nonconstant harmonic mapping on the space with a constant modulus. To show the last fact we only need to recall the formula

$$\Delta|u| = \rho\|DS\|^2$$

for a harmonic mapping $u(x) = \rho(x)S(x)$, $|S(x)| = 1$, and $\|DS\|^2$ is the Hilbert-Schmidt norm of differential matrix DS (see eg. [6]). Thus if $|u|$ is a constant function then $DS \equiv 0$. Therefore S is a constant mapping, i.e. u is a constant mapping, provided that $|u|$ is a constant function. □

3. Radial biharmonic mappings

A mapping f is called *radial* if there exists a constant φ and a real function g such that

$$f(re^{i\theta}) = g(r)e^{i(\theta+\varphi)}.$$

It is well known that, a radial solution u of the harmonic equation is given by $u(z) = Az + B/\bar{z}$, where a and b are two complex constants. To prove this we start by the Laplacian in polar coordinates. Let $U(r,\theta) = u(re^{i\theta})$. Then $\Delta u = 0$ if and only if

$$\Delta U := \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0.$$

Assuming that $U(r,\theta) = p(r)e^{i\theta}$ we obtain the equation

$$\frac{1}{r^2}(r^2 p''(r) + rp'(r) - p(r)) = 0.$$

By taking the change of variables $t = \log r$, $P(t) = p(r)$ we obtain $P(t) = Ae^t + Be^{-t}$ and therefore $p(r) = Ar + B/r$. Thus

$$u(z) = Az + \frac{B}{\bar{z}}.$$

If v is bi-harmonic, then the mapping $u = \Delta v$ is harmonic. If v is radial, then u is radial as well. It follows that

$$\Delta v = Az + \frac{B}{\bar{z}}$$

for some real constants A and B. Take $V(r,\theta) = v(re^{i\theta})$. Then we have

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = \left(Ar + \frac{B}{r}\right)e^{i\theta}. \tag{3.1}$$

Put

$$V(r,\theta) = g(r)e^{i\theta}.$$

Then (3.1) is equivalent with

$$\frac{1}{r^2}(r^2 g''(r) + r g'(r) - g(r)) = Ar + \frac{B}{r}.$$

By taking again the change of variables $t = \log r$, $G(t) = g(r)$ we arrive at the equation

$$G''(t) - G(t) = Ae^{3t} + Be^{t}.$$

Thus

$$G(t) = de^{-t} + ae^{t} + bte^{t} + ce^{3t}, \quad a,b,c,d \in \mathbf{R}$$

and therefore

$$g(r) = \frac{d}{r} + ar + br\log r + cr^3. \tag{3.2}$$

It follows that, every radial solution of bi-harmonic equation has the form:

$$f(z) = \frac{d}{\bar{z}} + az + bz\log|z| + c|z|^2 z.$$

3.1. *The technical lemmas.* First of all we would like to notice that some of calculations presented in this subsection are aided by using Mathematica 8 software. Throughout the section we will assume that the bi-harmonic mapping $f(re^{it}) = g(r)e^{it} : A(1,t) \to A(1,s)$ maps the inner boundary onto the inner boundary of corresponding annulus, i.e. g defined in (3.2) is increasing. A similar analysis works for the case when g is a decreasing function. We call a radial harmonic mapping f *homogeneous* if the initial and the final speeds are equal to zero, i.e. if $g'(1) = g'(t) = 0$.

Lemma 3.1. *Assume that $t > 1$ and $s > 1$. If the function g defined by (3.2) satisfies $g(1) = 1$, $g(t) = s$, $g'(1) = x > 0$, $g'(t) = y > 0$, then there exists a function h, such that $h(1) = 1$, $h(t) = s$, $h'(1) = 0$, $h'(t) = 0$, and*

$$g(r) = A(r) + B(r)s + U(r)x + V(r)y,$$

and

$$h(r) = A(r) + B(r)s$$

where

$$\begin{aligned}
A &= [(3+r^2)(r^2-t^2)(t^2-1) + 2r^2(3-2t^2-t^4)\log r \\
&\quad + 2(r^4+t^4+r^2(-3+t^4))\log t]/D, \\
B &= [2r^2(-1-2t^2+3t^4)\log r \\
&\quad - (-1+r^2)((-1+t^2)(r^2+3t^2) + 2(-1+r^2)t^2\log t)]/(tD),
\end{aligned}$$

$$
\begin{aligned}
U = {} & [-2r^2(-1+t^2)^2\log r \\
& + (-1+r^2)((r^2-t^2)(-1+t^2) - 2(r^2-t^4)\log t)]/D, \\
V = {} & [-2r^2(-1+t^2)^2\log r \\
& + (-1+r^2)(-(r^2-t^2)(-1+t^2) + 2(-1+r^2)t^2\log t)]/D,
\end{aligned}
$$

where $D = 4r(-1+t^2)(1-t^2+(1+t^2)\log t)$.

PROOF. The proof is lengthy but straightforward and is therefore omitted. $\square$

LEMMA 3.2. *Under the conditions and notation of Lemma 3.1 there hold the following relations*

$$B'(r) > 0, \quad \text{for} \quad 1 < r < t < \infty, \tag{3.3}$$

$$-A'(r) > 0, \quad \text{for} \quad 1 < r < t < \infty, \tag{3.4}$$

$$-\frac{d}{dr}\frac{A'(r)}{B'(r)} > 0, \quad \text{for} \quad 1 < r < t < \infty, \tag{3.5}$$

$$-\frac{d}{dr}\frac{U'(r)}{B'(r)} > 0, \quad \text{for} \quad 1 < r < t < \infty, \tag{3.6}$$

$$\frac{d}{dr}\frac{V'(r)}{B'(r)} > 0, \quad \text{for} \quad 1 < r < t < \infty, \tag{3.7}$$

$$\lim_{r\to t-0}\frac{-A'(r)}{B'(r)} = \frac{t(3-4t^2+t^4+4t^2\log t)}{2-2t^2+\log t+3t^4\log t}, \tag{3.8}$$

$$\lim_{r\to t-0}\frac{-U'(r)}{B'(r)} = \frac{t(-1+t^4-4t^2\log t)}{2-2t^2+\log t+3t^4\log t}, \tag{3.9}$$

$$\lim_{r\to t-0}\frac{-V'(r)}{B'(r)} = -\infty, \tag{3.10}$$

$$\lim_{r\to 1+0}\frac{-A'(r)}{B'(r)} = -\frac{t(-2t^2(-1+t^2)+(3+t^4)\log t)}{1-4t^2+3t^4-4t^2\log t}, \tag{3.11}$$

$$\lim_{r\to 1+0}\frac{-U'(r)}{B'(r)} = -\infty, \tag{3.12}$$

$$\lim_{r\to 1+0}\frac{-V'(r)}{B'(r)} = \frac{t(-1+t^4-4t^2\log t)}{1-4t^2+3t^4-4t^2\log t}. \tag{3.13}$$

The proof of Lemma 3.1 lies on the following lemma.

LEMMA 3.3. *For all* $1 < t$, $1 < r < t$,

a)
$$\begin{aligned} &2r^2(-1-2t^2+3t^4)\log r \\ &\quad + (1-r^2)(3(r^2-t^2)(-1+t^2)+2(1+3r^2)t^2\log t) > 0, \end{aligned}$$

b)
$$\begin{aligned} &-2r^2(-3+2t^2+t^4)\log r \\ &\quad + (r^2-1)(3(r^2-t^2)(-1+t^2)+2(3r^2+t^4)\log t) < 0, \end{aligned}$$

c)
$$\begin{aligned} &(-1+t^2)(3(-1+r^2)(-r^2+t^2)+2r^2(1+3t^2)\log r) \\ &\quad + 2(1+2r^2-3r^4)t^2\log t > 0, \end{aligned}$$

d)
$$\begin{aligned} &2(r^4-t^2)(t^2-1)\log r \\ &\quad + (1-r^2)((r^2-t^2)(-1+t^2)+2(r^2-1)t^2\log t) > 0 \end{aligned}$$

e)
$$1-t^2+(1+t^2)\log t > 0.$$

PROOF OF LEMMA 3.3. By taking the substitution $\rho = r^2$, $\tau = t^2$, the inequality d) of the lemma is equivalent with the inequality

$$h(\tau) := (\rho^2-\tau)(-1+\tau)\log\rho+(\rho-1)((\tau-\rho)(\tau-1)+(1-\rho)\tau\log\tau) \geq 0. \tag{3.14}$$

Then

$$h'(\tau) = (1+\rho^2-2\tau)\log\rho - (-1+\rho)(2(\rho-\tau)+(-1+\rho)\log\tau),$$

$$h''(\tau) = -\frac{(-1+\rho)(-1+\rho-2\tau)}{\tau} - 2\log\rho$$

and

$$h'''(\tau) = \frac{(-1+\rho)^2}{\tau^2}.$$

It follows that h'' is increasing, and therefore

$$h''(\tau) \geq h''(\rho) = -1/\rho + \rho - 2\log\rho.$$

But

$$(-1/\rho+\rho-2\log\rho)' = \frac{(\rho-1)^2}{\rho^2},$$

and therefore $-1/\rho+\rho-2\log\rho \geq -1+1-2\log 1 = 0$. It follows that

$$h''(\tau) \geq 0.$$

Thus

$$h'(\tau) \geq h'(\rho) = 0.$$

It follows finally that

$$h(\rho) \geq h(\tau) = 0.$$

The proof of a), b), c) and e) are similar to the proof of d) and are therefore omitted. $\square$

Proof of Lemma 3.1. First of all

$$\begin{aligned}A'(r) &= [2r^2(3-2t^2-t^4)\log r\\ &\quad + (r^2-1)(3(r^2-t^2)(-1+t^2)+2(3r^2+t^4)\log t)]/(rD),\\ B'(r) &= [2r^2(3t^4-2t^2-1)\log r\\ &\quad + (1-r^2)(3(r^2-t^2)(-1+t^2)+2(1+3r^2)t^2\log t)]/(rtD),\\ U'(r) &= [(1+3r^2)(r^2-t^2)(t^2-1)+2r^2(1-t^2)^2\log r\\ &\quad - 2(3r^4-t^4-r^2(1+t^4))\log t]/(rD),\\ V'(r) &= [2r^2(1-t^2)^2\log r\\ &\quad + (1-r^2)((-1+t^2)(3r^2+t^2)+2(1+3r^2)t^2\log t)]/(rD),\end{aligned}$$

where $D = 4r(-1+t^2)(1-t^2+(1+t^2)\log t)$. Lemma 3.3, a) and b) implies that $A'(r) < 0$ and $B'(r) > 0$.

The derivative of $-A'(r)/B'(r)$ is

$$12rt(-1+t^2)(1-t^2+(1+t^2)\log t)\times \frac{2(r^4-t^2)(t^2-1)\log r-(r^2-1)((r^2-t^2)(t^2-1)+2(r^2-1)t^2\log t)}{(2r^2(1+2t^2-3t^4)\log r+(r^2-1)(3(r^2-t^2)(t^2-1)+2(1+3r^2)t^2\log t))^2}.$$

Lemma 3.3, d), e) implies that the last expression is positive. Thus (3.5) is proved. The proof of (3.4), (3.3), (3.6) and (3.7) are similar. The proof of relations (3.8)-(3.13) are similar and follows by l'Hôspital rule. See Figure 1 for the geometric interpretation of (3.12) and (3.13). ∎

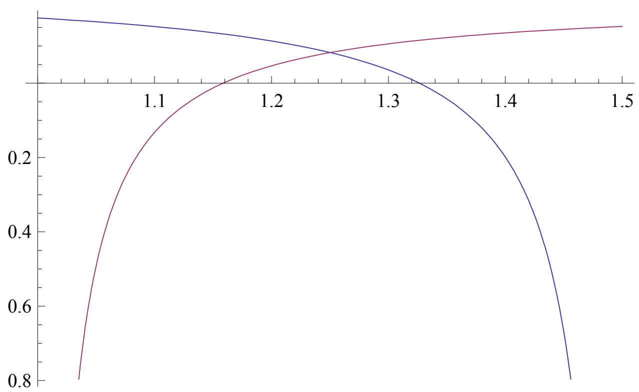

Figure 1. These two curves are graphs of the functions $-U'(r)/B'(r)$ and $-V'(r)/B'(r)$ for $t = 3/2$, and $1 < r < t$.

LEMMA 3.4. *For every $t > 1$, and $\tau = \frac{1+t}{2}$,*

$$\frac{-A'(\tau)}{B'(\tau)} > 1.$$

PROOF. Namely

$$\frac{-A'(\tau)}{B'(\tau)} > 1$$

if and only if $\varphi(t) > 0$, where

$$\varphi(t) := (1-t^2)(9+30t+9t^2+8(1+t)^2\log\tau) - 2t(9+18t+17t^2+4t^3)\log t.$$

On the other hand

$$\varphi^{(5)}(t) = \frac{12(9+6t+2t^2+6t^3+9t^4)}{t^4(1+t)^2} > 0$$

and $\varphi^{(k)}(1) \geq 0$, for $k = 1, 2, 3, 4$. It follows that $\varphi^{(4)}(t) > 0$, $\varphi^{(3)}(t) > 0$, $\varphi''(t) > 0$, $\varphi'(t) > 0$ and $\varphi(t) > 0$ for $t > 1$. ∎

LEMMA 3.5. *Under the conditions and notation of Lemma 3.1 we have $U'(r) = V'(r)$ if and only if*

$$r = \rho := \sqrt{\frac{1}{6} + \frac{t^2}{6} + \frac{1}{6}\sqrt{1+14t^2+t^4}}. \tag{3.15}$$

Moreover

$$U'(\rho) = V'(\rho) > 0, \quad \frac{-A'(\rho)}{B'(\rho)} > 1.$$

PROOF. As

$$U'(r) - V'(r) = \frac{-3r^4+t^2+r^2(1+t^2)}{2r^2(-1+t^2)}$$

it follows that

$$U'(r) = V'(r) \text{ if and only if } r = \rho := \sqrt{\frac{1}{6} + \frac{t^2}{6} + \frac{1}{6}\sqrt{1+14t^2+t^4}}$$

or what is the same

$$t = \frac{\rho\sqrt{-1+3\rho^2}}{\sqrt{1+\rho^2}}.$$

By taking the substitution $\kappa = \rho^2$, $\eta = t^2$ we obtain

$$\begin{aligned}-2U'(\rho) &= \frac{-2\rho^2(t^2-1) - 2\rho^2(t^2-1)\log\rho + (3\rho^4+t^2+\rho^2(t^2-1))\log t}{2\rho^2(1-t^2+(1+t^2)\log t)} \\ &= \frac{-2\kappa(\eta-1) - \kappa(t^2-1)\log\kappa + (3\kappa^2+\eta+\kappa/2(\eta-1))\log\eta}{\kappa(2-2\eta+(1+\eta)\log\eta)}.\end{aligned}$$

Since

$$\eta = \frac{\kappa(3\kappa-1)}{1+\kappa}$$

and

$$\kappa(2-2\eta+(1+\eta)\log\eta)>0$$

we have to prove that

$$L(\kappa):=-2\kappa(\eta-1)-\kappa(t^2-1)\log\kappa+(3\kappa^2+\eta+\kappa/2(\eta-1))\log\eta>0.$$

Then

$$\begin{aligned}L(\kappa)&=\kappa\frac{(-1-2\kappa+3\kappa^2)(2+\log\kappa)}{1+\kappa}-\kappa(-1+3\kappa)\log\left(\frac{\kappa(-1+3\kappa)}{1+\kappa}\right)\\&=\frac{\kappa}{1+\kappa}K(\kappa),\end{aligned}$$

where

$$K(\kappa)=(-1-2\kappa+3\kappa^2)(2+\log\kappa)-\kappa(-1+3\kappa)(1+\kappa)\log\left(\frac{\kappa(-1+3\kappa)}{1+\kappa}\right).$$

Further,

$$K'''(\kappa)=\frac{4(1-4\kappa+14\kappa^2+12\kappa^3+9\kappa^4)}{\kappa^2(-1+2\kappa+3\kappa^2)^2}>0.$$

Moreover,

$$K''(1)\geq 0,\quad K'(1)\geq 0,\quad K(1)\geq 0$$

and therefore

$$K(\kappa)>0.$$

Since $r\to -A'(r)/B'(r)$ is increasing, and

$$\rho=\sqrt{\frac{1}{6}+\frac{t^2}{6}+\frac{1}{6}\sqrt{1+14t^2+t^4}}>\tau=\frac{1+t}{2},$$

by Lemma 3.4 and (3.5) we obtain

$$\frac{-A'(\rho)}{B'(\rho)}>\frac{-A'(\tau)}{B'(\tau)}>1. \tag{3.16}$$

□

3.2. *Rigidity of radial biharmonic mappings.* As a direct corollary of Lemma 3.1 we obtain

Theorem 3.6. *If $f(re^{i\theta})=h(r)e^{i\theta}$, $h(1)=1$, $h(t)=s$, $h'(1)=0$, $h'(t)=0$, is a radial homogeneous bi-Harmonic mapping of the annulus $A(1,t)$ onto the annulus $A(1,s)$, then*

$$s\geq\sigma_0(t):=\frac{t(3-4t^2+t^4+4t^2\log t)}{2-2t^2+\log t+3t^4\log t}.$$

The critical bi-Nitsche homogeneous bi-harmonic mapping is

$$f(z)=h_0(r)e^{i\theta},\quad z=re^{i\theta},$$

where

$$h_0(r) = \frac{(1-t^2)(3t^2+3(3-t^2)r^2-r^4)}{4r(2-2t^2+\log t+3t^4\log t)} + \frac{(6t^4+6(1+t^4)r^2-2r^4)\log t+6(1-t^2)^2r^2\log r}{4r(2-2t^2+\log t+3t^4\log t)}.$$

The condition is sufficient as well. The function $\sigma_0(t)$ is smaller than the corresponding function $n(t)$ for harmonic mappings. See Figure 2.

Theorem 3.7. *If $f(re^{i\theta}) = g(r)e^{i\theta}$ is a radial bi-harmonic diffeomorphism of the annulus $A(1,t)$ onto the annulus $A(1,s)$, mapping the inner boundary onto the inner boundary, then $s \ge \sigma(t)$ where the constant*

$$\sigma(t) = \inf_{x\ge 0, y\ge 0} \sup_{1\le r\le t} \left\{ \frac{-A'(r)}{B'(r)} + x\frac{-U'(r)}{B'(r)} + y\frac{-V'(r)}{B'(r)} \right\}$$

is bigger than $\frac{-A'(\rho)}{B'(\rho)} (> 1)$ (cf. (3.15), (3.16)) and smaller than $\sigma_0(t)$. The condition is sufficient as well. Moreover there exists a critical bi-Nitsche mapping $f(re^{it}) = g_0(r)e^{it}$, between annuli $A(1,t)$ and $A(1,\sigma(t))$ and it satisfies the conditions $g_0'(1) > 0$ and $g_0'(t) > 0$.

Proof. Under the conditions of the theorem g is non-decreasing. Then

$$g'(r) \ge 0, \quad \text{for} \quad 1 \le r \le t, \quad \text{and } t > 1$$

if and only if

$$A'(r) + B'(r)s + U'(r)x + V'(r)y \ge 0 \ \text{ for } \quad 1 \le r \le t, \quad \text{and } t > 1.$$

Here $x = g'(0)$ and $y = g'(t)$. If

$$X_n(t) := \frac{-A'(r_n)}{B'(r_n)} + x_n\frac{-U'(r_n)}{B'(r_n)} + y_n\frac{-V'(r_n)}{B'(r_n)} \to \sigma(t)$$

then because of (3.16)

$$X_n \ge \frac{-A'(\rho)}{B'(\rho)} + x_n\frac{-U'(\rho)}{B'(\rho)} + y_n\frac{-V'(\rho)}{B'(\rho)} > \frac{-A'(\rho)}{B'(\rho)} > 1.$$

It follows that the sequences x_n and y_n stay bounded when $n \to \infty$. On the other hand, it follows from (3.8), (3.9), (3.10), (3.11), (3.12) that there exist $1 < \tau_1(t) < \tau_2(t) < t$ such that the function

$$p(r) = -\frac{U'(r) + V'(r)}{B'(r)}$$

is negative in intervals $[1, \tau_1]$ and $[\tau_2, t]$. From (3.5), the maximum of $\frac{-A'(r)}{B'(r)}$ is $\frac{-A'(t)}{B'(t)} := \frac{-A'(t-0)}{B'(t-0)}$ defined in (3.8). Thus there exists a small enough $x > 0$ such that

$$\frac{-A'(t)}{B'(t)} > \frac{-A'(r)}{B'(r)} + xp(r)$$

for all $r : 1 < r < t$ and fixed t. This means that

$$\sigma(t) < \sigma_0(t).$$

Assume without loss of generality that $x_n \to x_0$ and $y_n \to y_0$ and $r_n \to r_0$. The sequence g_n is monotonic and converges to a strictly monotonic function g_0. The resulting bi-harmonic mapping is critical. Since $\sigma < \sigma_0$, because $A'(t-0) = B'(t-0) = 0$, $U'(t-0) = 0$, $V'(t-0) = 1$ and $(-U'/B')(t-0) > 0$ it follows that $x_0 > 0$, $y_0 > 0$ and $r_0 < t$. □

EXAMPLE 3.8. By using the previous theorem we obtain that there does not exist a radial biharmonic mapping between annuli $A(1, 2)$ and $A(1, 1.00098)$.

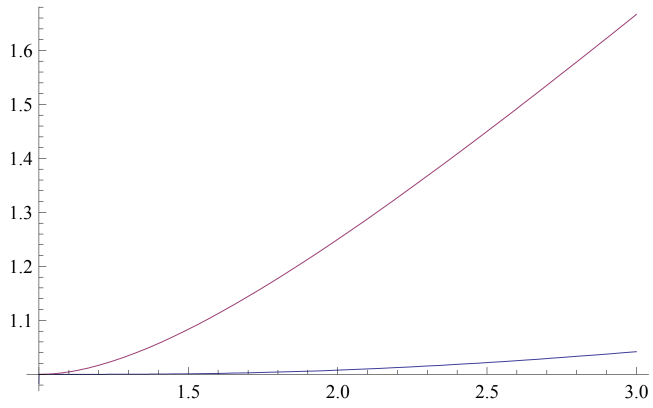

FIGURE 2. The curve above (below) corresponds to the critical harmonic (bi-harmonic) mappings between annuli $A(1, t)$, and $A(1, \omega(t))$, where $\omega(t) = \sigma_0(t)$ and $\omega(t) = n(t)$ respectively $(1 < t \leq 3)$.

REFERENCES

[1] N. Aronszajn, T. M. Creese and L. J. Lipkin, Polyharmonic functions, Clarendon Press, Oxford, 1983.
[2] Z.-C. Han, *Remarks on the geometric behavior of harmonic maps between surfaces*, in Elliptic and parabolic methods in geometry, Minneapolis, 1994, Chow, Ben (ed.) et al., A K Peters, Wellesley, 1996, 57–66.
[3] T. Iwaniec, L. K. Kovalev and J. Onninen, *The Nitsche conjecture*, J. Amer. Math. Soc. **24** (2011), 345–373.
[4] T. Iwaniec, L. K. Kovalev and J. Onninen, *Doubly connected minimal surfaces and extremal harmonic mappings,* J. Geom. Anal. **22** (2012), 726–762.
[5] D. Kalaj, *Harmonic maps between annuli on Riemann surfaces*, Isr. J. Math. **182** (2011), 123–147.

[6] D. Kalaj, *On the univalent solution of PDE $\Delta u = f$ between spherical annuli*, J. Math. Anal. Appl. **327** (2007), 1–11.
[7] D. Kalaj, *On the Nitsche conjecture for harmonic mappings in $\mathbb{R}^2$ and $\mathbb{R}^3$*, Israel J. Math. **150** (2005), 241–251.
[8] A. Lyzzaik, *The modulus of the image annuli under univalent harmonic mappings and a conjecture of Nitsche*, J. London Math. Soc. (2) **64** (2001), 369–384.
[9] J. C. C. Nitsche, *On the modulus of doubly connected regions under harmonic mappings*, Amer. Math. Monthly **69** (1962), 781–782.
[10] M. Pavlović, *Decompositions of L^p and Hardy spaces of polyharmonic functions*, J. Math. Anal. Appl. **216** (1997), 499–509.
[11] A. Weitsman, *Univalent harmonic mappings of annuli and a conjecture of J. C. C. Nitsche*, Israel J. Math. **124** (2001), 327–331.

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Received: 14.4.2013.