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APOSYNDETIC PROPERTIES OF THE n−FOLD SYMMETRIC PRODUCT SUSPENSION OF A CONTINUUM

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Abstract. In this paper the n−fold symmetric product suspension of a continuum is investigated with respect to the properties of aposyndesis such as: aposyndesis, finite aposyndesis, mutual aposyndesis and strictly nonmutual aposyndesis.

1. Introduction

In 1941 F. B. Jones introduced the notion of aposyndesis ([8]). In [6, Theorem 1], [11, Theorem 4], [12, Corollary 5.2], [13, Corollary 4.3], [15, Corollary 2.36] and [7, Theorem 2.4] we can find some results of aposyndesis in hyperspaces of continua. Given a continuum X and an integer $n \geq 1$, in 1931 K. Borsuk and S. Ulam ([4]) defined the n−fold symmetric product of X, $F_n(X) = \{A \subset X : A \text{ has at most } n \text{ points}\}$. For each integer $n \geq 2$, in the year 2009 we defined the n−fold symmetric product suspension of X ([1]), denoted by $SF_n(X)$, as the quotient space $F_n(X)/F_1(X)$. In [2] we investigated the induced maps between these spaces.

In 1997 S. Macías showed in [11, Theorem 4], that $F_n(X)$ is colocally connected (in particular aposyndetic), for every $n \geq 2$. In [11, Theorem 8] he also proved that $F_n(X)$ is countable closed aposyndetic, for every $n \geq 2$. In the present paper, we prove that $SF_n(X)$ is colocally connected (in particular aposyndetic), for every $n \geq 3$. Also we prove that $F_n(X)$ is finitely aposyndetic, for every $n \geq 3$. Furthermore, we show a class of continua for which their second symmetric product suspensions are not

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aposyndetic. Moreover, we prove that if a continuum is aposyndetic, then its second symmetric product suspension is aposyndetic.

On the other hand, in 1999 J. M. Martinez-Motejano showed in [16, Theorem 1] that $F_n(X)$ is mutually aposyndetic, for every $n \geq 3$. In [11, Theorem 15] S. Macías proved that if X is a chainable continuum such that its second symmetric product is mutually aposyndetic, then X is the arc. In this paper we prove that $SF_n(X)$ is mutually aposyndetic, for every $n \geq 3$. Also we show that if a continuum is aposyndetic, then its second symmetric product is mutually aposyndetic. Furthermore, we proved that if a continuum is 2-aposyndetic and mutually aposyndetic, then its second symmetric product suspension is mutually aposyndetic. Moreover, we verify that if X is a chainable continuum such that its second symmetric product suspension is mutually aposyndetic, then X is the arc.

In [11, Theorem 16] S. Macías proved that a chainable continuum is indecomposable if and only if its second symmetric product is strictly nonmutually aposyndetic. We show that a continuum with span zero is indecomposable if and only if its second symmetric product suspension is strictly nonmutually aposyndetic.

2. Definitions

The symbol $\mathbb{N}$ will denote the set of positive integers. A *continuum* is a nonempty compact, connected metric space. A *subcontinuum* is a continuum contained in a space X. If X is a continuum, then given $A \subset X$ and $\epsilon > 0$, the open ball around A of radius ϵ is denoted by $N(\epsilon, A)$, the closure of A in X by $Cl_X(A)$, the interior of A in X is denoted by $int_X(A)$ and the boundary of A in X is denoted by $Bd(A)$. If $A = \{a\}$, we let $N(a, \epsilon)$ denote the open ball around a of radius ϵ. An *arc* is any space which is homeomorphic to the closed interval $[0, 1]$. A *ray* is a space homeomorphic to $[0, \infty)$. An *Elsa continuum*, denoted E-continuum, is a compactification of the ray with an arc as the remainder. A *map* means a continuous function. An onto map $f : X \to Y$ between continua is said to be:

- *monotone* provided that $f^{-1}(y)$ is connected for each $y \in Y$;
- *open* provided that if U is any open subset of X, then $f(U)$ is an open subset of Y.

A continuum X is said to be *colocally connected*, provided that each point of it has a local base of open sets whose complements are connected. A continuum X is said to be *aposyndetic at x with respect to y*, provided that there exists a subcontinuum W of X such that $x \in int_X(W) \subset W \subset X \setminus \{y\}$, it is said to be *aposyndetic at x*, if it is aposyndetic at x with respect to any point of $X \setminus \{x\}$, and it is said to be *aposyndetic*, if it is aposyndetic at each of its points. A continuum X is *2-aposyndetic at x* if for every pair of points p and q lying in $X \setminus \{x\}$, there exists a subcontinuum W of X such that $x \in int_X(W) \subset W \subset X \setminus \{p,q\}$. A continuum is *2-aposyndetic* if it is 2-aposyndetic at every one of its points. Let $\mathcal{F}$ be a collection of finite subsets of a continuum X. Then X is said to be *finitely aposyndetic* if for each $x \in X$ and each $F \in \mathcal{F}$ such that $x \notin F$, there exists a subcontinuum W of X such that $x \in int_X(W)$ and $W \cap F = \emptyset$. A continuum X is *mutually aposyndetic*, provided that if x and y are two distinct points of X then there exist two disjoint subcontinua W_x and W_y of X such that $x \in int_X(W_x)$ and $y \in int_X(W_y)$. A continuum X is said to be *strictly nonmutually aposyndetic* if each pair of subcontinua of X which have nonempty interior intersect. A point x of a continuum X is called a *cut point* of X, provided that $X \setminus \{x\}$ is not connected. A point x of a continuum X is called a *not-cut point* of X, provided that $X \setminus \{x\}$ is connected.

A *chain* in a continuum X is a finite collection $\{U_1, U_2, \ldots, U_m\}$ of open subsets of X such that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leq 1$ for each $i, j \in \{1, 2, \ldots, m\}$. The elements of a chain are called *links*. For $\epsilon > 0$ an *ϵ-chain* is a chain in which each link has diameter less than ϵ. A continuum is *chainable* if for each $\epsilon > 0$, it can be covered by an ϵ-chain. A continuum is *decomposable* provided that it can be written as the union of two of its proper subcontinua. A continuum is *indecomposable* if it is not decomposable.

Given a continuum X and $n \in \mathbb{N}$, the product of X with itself n times will be denoted by X^n. The symbol Δ_{X^2} will denote the diagonal of X^2, that is, $\Delta_{X^2} = \{(x,x) \in X^2 : x \in X\}$. If W is a subset of X^2, we let W^* denote the subset $\{(y,x) \in X^2 : (x,y) \in W\}$ of X^2. Denote the first and second projections of X^2 onto X by π_1 and π_2, respectively. The *span* of a continuum (X,d) (respectively, the *semispan* of X), denote by $\sigma(X)$ (respectively, $\sigma_0(X)$), is the least upper bound of the set of all numbers $\epsilon \geq 0$ for which there exists a subcontinuum Z of X^2 such that $\pi_1(Z) = \pi_2(Z)$ (respectively, $\pi_2(Z) \subset \pi_1(Z)$) and $d(x,y) \geq \epsilon$ for each $(x,y) \in Z$.

Given a continuum X and $n \in \mathbb{N}$, the *n-fold symmetric product of* X, denoted by $F_n(X)$, is the hyperspace:

$$F_n(X) = \{A \subset X : A \text{ has at most } n \text{ points}\}$$

topologized with the Hausdorff metric, which is defined as follows:

$$\mathcal{H}(A,B) = \inf\{\epsilon > 0 : A \subset N(\epsilon, B) \text{ and } B \subset N(\epsilon, A)\}.$$

On the other hand, $F_n(X)$ can be topologized with the topology defined as follows: given a finite collection, $U_1, \ldots, U_m$, of open sets of X, $\langle U_1, \ldots, U_m \rangle_n$, denotes the following subset of $F_n(X)$:

$$\left\{ A \in F_n(X) : A \subset \bigcup_{k=1}^{m} U_k \text{ and } A \cap U_k \neq \emptyset \text{ for each } k \in \{1, \ldots, m\} \right\}.$$

It is known that the family of all subset of $F_n(X)$ of the form $\langle U_1, \ldots, U_m \rangle_n$, as defined above, form a basis for a topology for $F_n(X)$ (see [18, 0.11]) called the *Vietoris topology*, and that the Vietoris topology and the topology induced by the Hausdorff metric coincide [18, 0.13].

Given a continuum X and $n \in \mathbb{N}$ with $n \geq 2$, we defined (in [1]) the *n-fold symmetric product suspension of the continuum* X, denoted by $SF_n(X)$, as the quotient space:

$$SF_n(X) = F_n(X)/F_1(X),$$

with the quotient topology.

Give a continuum X and an integer $n \geq 2$, $q_X^n : F_n(X) \to SF_n(X)$ denotes the quotient map. Also, let F_X^n denotes the point $q_X^n(F_1(X))$. We denote by $f_2 : X^2 \to F_2(X)$ the map given by $f_2(x, y) = \{x, y\}$, for each $(x, y) \in X^2$.

Remark 2.1. Let X be a continuum and let $n \geq 2$ be an integer. Then $SF_n(X) \setminus \{F_X^n\}$ is homeomorphic to $F_n(X) \setminus F_1(X)$.

3. Aposyndesis

Theorem 3.1. *Let X be a continuum and let $n \geq 2$ be an integer. Then (1) and (2) below are true.*

1. *$SF_n(X)$ is aposyndetic at F_X^n.*
2. *If $\mathcal{A} \in SF_n(X) \setminus \{F_X^n\}$, then $SF_n(X)$ is aposyndetic at $\mathcal{A}$ respect to any point of $SF_n(X) \setminus \{F_X^n, \mathcal{A}\}$.*

Proof. To prove (1), let $\mathcal{B} \in SF_n(X) \setminus \{F_X^n\}$. Let Γ and Λ be open subsets of $SF_n(X)$ such that $F_X^n \in \Gamma$, $\mathcal{B} \in \Lambda$ and $\Gamma \cap \Lambda = \emptyset$. Let $B \in F_n(X) \setminus F_1(X)$ be such that $q_X^n(B) = \mathcal{B}$. Hence, $(q_X^n)^{-1}(\Lambda)$ is an open subset of $F_n(X)$ such that $B \in (q_X^n)^{-1}(\Lambda) \subset F_n(X) \setminus F_1(X)$. By [11, Theorem 4], there exists an open subset $\mathcal{U}$ of $F_n(X)$ such that $B \in \mathcal{U} \subset (q_X^n)^{-1}(\Lambda)$ and $F_n(X) \setminus \mathcal{U}$ is connected. It follows that $\mathcal{B} \in q_X^n(\mathcal{U}) \subset \Lambda$ and $q_X^n(F_n(X) \setminus \mathcal{U})$ is connected. Since $\mathcal{U} \cap F_1(X) = \emptyset$, by Remark 2.1, we have that $q_X^n(\mathcal{U})$ is an open subset of $SF_n(X)$ and $q_X^n(F_n(X) \setminus \mathcal{U}) = SF_n(X) \setminus q_X^n(\mathcal{U})$. Then $SF_n(X) \setminus q_X^n(\mathcal{U})$ is a subcontinuum of $SF_n(X)$. We note that $F_X^n \in \Gamma \subset SF_n(X) \setminus q_X^n(\mathcal{U})$, thus $F_X^n \in int_{SF_n(X)}(SF_n(X) \setminus q_X^n(\mathcal{U}))$. Therefore, $SF_n(X) \setminus q_X^n(\mathcal{U})$ is a subcontinuum of $SF_n(X)$ such that $F_X^n \in int_{SF_n(X)}(SF_n(X) \setminus q_X^n(\mathcal{U})) \subset SF_n(X) \setminus q_X^n(\mathcal{U}) \subset SF_n(X) \setminus \{\mathcal{B}\}$. Hence, $SF_n(X)$ is aposyndetic at F_X^n.

A similar argument can be used to verify (2). ☐

Theorem 3.2. *Let X be a continuum and let $n \geq 3$ be an integer. Then $SF_n(X)$ is aposyndetic.*

Proof. By Theorem 3.1, it is sufficient to prove that, for each element $\mathcal{A} \in SF_n(X) \setminus \{F_X^n\}$, $SF_n(X)$ is aposyndetic at $\mathcal{A}$ with respect to F_X^n. Let $\mathcal{A} \in SF_n(X) \setminus \{F_X^n\}$. We take $A \in F_n(X) \setminus F_1(X)$ such that $q_X^n(A) = \mathcal{A}$. Fix points $p, q \in A$ with $p \neq q$. Let U and V be open subsets of X such that $p \in U$, $q \in V$, $Cl_X(U) \cap Cl_X(V) = \emptyset$ and $A \subset U \cup V$. We define $\mathcal{C} = \langle Cl_X(U), Cl_X(V), X\rangle_n \cup \langle Bd(U), Bd(V), X\rangle_n \cup \langle Bd(U), \{p\}, X\rangle_n \cup$

$\langle \{p\}, \{q\}, X\rangle_n$. By [16, Lemma 2], $\mathcal{C}$ is a subcontinuum of $F_n(X)$. Since $A \in \langle U, V\rangle_n \subset \mathcal{C}$, we obtain that $A \in int_{F_n(X)}(\mathcal{C}) \subset \mathcal{C}$. Moreover, we note that $\mathcal{C} \cap F_1(X) = \emptyset$. Hence, by Remark 2.1, $q_X^n(\mathcal{C})$ is a subcotinuum of $SF_n(X)$ such that $\mathcal{A} \in int_{SF_n(X)}(q_X^n(\mathcal{C})) \subset q_X^n(\mathcal{C}) \subset SF_n(X) \setminus \{F_X^n\}$. Consequently, $SF_n(X)$ is aposyndetic at $\mathcal{A}$ with respect to F_X^n. ☐

A continuum X is said to be *unicoherent* provided that whenever A and B are proper subcontinua of X with $X = A \cup B$, then $A \cap B$ is connected. We proved in [1, Theorem 4.1] that for each integer $n \geq 3$, $SF_n(X)$ is unicoherent. Moreover, it is known that a unicoherent aposyndetic continuum is finitely aposyndetic ([3]). Therefore, by Theorem 3.2 and [1, Theorem 4.1], we have the following corollary.

Corollary 3.3. *Let X be a continuum and let $n \geq 3$ be an integer. Then $SF_n(X)$ is finitely aposyndetic.*

Lemma 3.4. *Let X be a continuum and let $n \geq 2$ be an integer. Then $F_n(X) \setminus F_1(X)$ is connected.*

Proof. We consider two cases.
(1) First we consider $n = 2$. Respect to X, we have two possibilities.
(1.1) The continuum X is the arc. Then $F_2(X) \setminus F_1(X)$ is connected.
(1.2) The continuum X is not the arc. Then, by [9, Theorem 2], it follows that $F_2(X) \setminus F_1(X)$ is connected.
(2) Now we consider $n \geq 3$. Let $A, B \in F_n(X) \setminus F_1(X)$ be such that $A \neq B$. Without loss of generality, suppose that there exists a point $b \in B \setminus A$. Take $c \in B \setminus \{b\}$. Fix points $x, y \in A$ with $x \neq y$. Let U and V be open subsets of X such that $x \in U$, $y \in V$, $b \notin Cl_X(U)$, $Cl_X(U) \cap Cl_X(V) = \emptyset$ and $A \subset U \cup V$. Let $\mathcal{C} = \langle Cl_X(U), Cl_X(V), X\rangle_n \cup \langle Bd(U), Bd(V), X\rangle_n \cup \langle Bd(U), \{b\}, X\rangle_n \cup \langle \{b\}, \{c\}, X\rangle_n$. By [16, Lemma 2], $\mathcal{C}$ is connected. Furthermore, $\mathcal{C} \subset F_n(X) \setminus F_1(X)$, $A \in \langle Cl_X(U), Cl_X(V), X\rangle_n$ and $B \in \langle \{b\}, \{c\}, X\rangle_n$. Therefore, $\mathcal{C}$ is a connected subset of $F_n(X) \setminus F_1(X)$ such that $A, B \in \mathcal{C}$. ☐

Theorem 3.5. *Let X be a continuum and let $n \geq 2$ be an integer. Then each point of $SF_n(X)$ is a not-cut point of $SF_n(X)$.*

Proof. Let $\mathcal{A} \in SF_n(X)$. We have the following cases.
(1) Suppose that $\mathcal{A} = F_X^n$. By Lemma 3.4, we have that $F_n(X) \setminus F_1(X)$ is a connected subset of $F_n(X)$. Then, by Remark 2.1, $q_X^n(F_n(X) \setminus F_1(X)) = SF_n(X) \setminus \{F_X^n\}$ is a connected subset of $SF_n(X)$. This implies that $\mathcal{A}$ is a not-cut point of $SF_n(X)$.
(2) Suppose that $\mathcal{A} \in SF_n(X) \setminus \{F_X^n\}$. We take $A \in F_n(X) \setminus F_1(X)$ such that $q_X^n(A) = \mathcal{A}$. By [11, Corollary 5], it follows that $F_n(X) \setminus \{A\}$ is a connected subset of $F_n(X)$. Hence, $q_X^n(F_n(X) \setminus \{A\}) = SF_n(X) \setminus \{\mathcal{A}\}$ is a connected subset of $SF_n(X)$. Consequently, we conclude that $\mathcal{A}$ is a not-cut point of $SF_n(X)$. ☐

It is known that if each point of X is a not-cut point of X, then the continuum X is colocally connected if and only if X is aposyndetic ([21, 4.14, p. 50]). Hence, by Theorems 3.2 and 3.5, we have the following:

COROLLARY 3.6. *Let X be a continuum and let $n \geq 3$ be an integer. Then $SF_n(X)$ is colocally connected.*

LEMMA 3.7. *Let X be a continuum. Then the following are equivalent:*

(1) *for each point $(x_1, x_2) \in X^2 \setminus \triangle_{X^2}$, there exists a subcontinuum C of X^2 such that $(x_1, x_2) \in int_{X^2}(C)$ and $C \cap \triangle_{X^2} = \emptyset$,*
(2) *for each element $\mathcal{A} \in SF_2(X) \setminus \{F_X^2\}$, $SF_2(X)$ is aposyndetic at $\mathcal{A}$ with respect to F_X^2.*

PROOF. We prove that (1) implies (2). Let $\mathcal{A} \in SF_2(X) \setminus \{F_X^2\}$. We take $A \in F_2(X) \setminus F_1(X)$ such that $q_X^2(A) = \mathcal{A}$. Suppose that $A = \{a_1, a_2\}$. Since $(a_1, a_2) \in X^2 \setminus \triangle_{X^2}$, by (1), there exists a subcontinuum C of X^2 such that $(a_1, a_2) \in int_{X^2}(C)$ and $C \cap \triangle_{X^2} = \emptyset$. Thus, since the map $f_2 : X^2 \to F_2(X)$ given by $f_2((x_1, x_2)) = \{x_1, x_2\}$ is an open map ([11, Lemma 9]), we obtain that $f_2((a_1, a_2)) = A \in int_{F_2(X)}(f_2(C))$. Furthermore, since $C \cap \triangle_{X^2} = \emptyset$, it follows that $f_2(C) \cap F_1(X) = \emptyset$. Hence, by Remark 2.1, $q_X^2(f_2(C))$ is a subcontinuum of $SF_2(X)$ such that $\mathcal{A} \in int_{SF_2(X)}(q_X^2(f_2(C))) \subset q_X^2(f_2(C)) \subset SF_2(X) \setminus \{F_X^2\}$. This proves that $SF_2(X)$ is aposyndetic at $\mathcal{A}$ with respect to F_X^2.

Next we prove that (2) implies (1). Let $(a_1, a_2) \in X^2 \setminus \triangle_{X^2}$. We define $A = \{a_1, a_2\}$. Then $f_2((a_1, a_2)) = A \in F_2(X) \setminus F_1(X)$. Let $\mathcal{A} = q_X^2(A)$. Thus, $\mathcal{A} \in SF_2(X) \setminus \{F_X^2\}$. By (2), there exists a subcontinuum Θ of $SF_2(X)$ such that $\mathcal{A} \in int_{SF_2(X)}(\Theta) \subset \Theta \subset SF_2(X) \setminus \{F_X^2\}$. Define $\mathcal{C} = (q_X^2)^{-1}(\Theta)$. Since q_X^2 is a monotone map, and by [21, 2.2, p. 138], $\mathcal{C}$ is a subcontinuum of $F_2(X)$, also $A \in int_{F_2(X)}(\mathcal{C})$. Since $F_X^2 \notin \Theta$, we obtain that $\mathcal{C} \cap F_1(X) = \emptyset$. This implies that $f_2^{-1}(\mathcal{C}) \cap \triangle_{X^2} = \emptyset$.

On the other hand, by [11, Lemma 12], $f_2^{-1}(\mathcal{C})$ has at most two components. Let $f_2^{-1}(\mathcal{C}) = C_1 \cup C_2$, where C_1 and C_2 are the components of $f_2^{-1}(\mathcal{C})$. Hence, $C_1 \cap \triangle_{X^2} = \emptyset$ and $C_2 \cap \triangle_{X^2} = \emptyset$. Since $(a_1, a_2) \in f_2^{-1}(\mathcal{C})$, we suppose without loss of generality, that $(a_1, a_2) \in C_1$. Since $A \in int_{F_2(X)}(\mathcal{C})$, we obtain that $(a_1, a_2) \in int_{X^2}(f_2^{-1}(\mathcal{C}))$. Consequently, by [14, Lemma 1.6.2], we have that $(a_1, a_2) \in int_{X^2}(C_1)$. Therefore, C_1 is a subcontinuum of X^2 such that $(a_1, a_2) \in int_{X^2}(C_1)$ and $C_1 \cap \triangle_{X^2} = \emptyset$. ∎

THEOREM 3.8. *If X is an E-continuum, then $SF_2(X)$ is not aposyndetic.*

PROOF. Let $X = J \cup S$, where J is the remainder and S is homeomorphic to $[0, \infty)$. Let $(a_1, a_2) \in J \times J$ be such that $(a_1, a_2) \in X^2 \setminus \triangle_{X^2}$. Suppose that there exists a subcontinuum C of X^2 such that $(a_1, a_2) \in int_{X^2}(C)$. Since the map $\pi_1 : X^2 \to X$, given by $\pi_1(x_1, x_2) = x_1$, is an open map, it follows that $a_1 \in int_X(\pi_1(C))$. Hence, since $\pi_1(C)$ is a subcontinuum of X, we have that

$J \subset \pi_1(C)$. With similar arguments, we obtain that $J \subset \pi_2(C)$. Therefore, we have that either $\pi_1(C) \subset \pi_2(C)$ or $\pi_2(C) \subset \pi_1(C)$.

On the other hand, by [17, Lemma 6, p. 126], X is a chainable continuum. Thus, by [10, p. 210], it follows that X has zero span, and by [5, Theorem 6], X has zero semispan. This implies that $C \cap \triangle_{X^2} \neq \emptyset$. Then, by Lemma 3.7, there exists $\mathcal{A} \in SF_2(X) \setminus \{F_X^2\}$ such that $SF_2(X)$ is not aposyndetic at $\mathcal{A}$ with respect to F_X^2. Consequently, $SF_2(X)$ is not aposyndetic. □

LEMMA 3.9. *Let X be a continuum, let $x_1, x_2 \in X$ with $x_1 \neq x_2$ and let A_1 and A_2 be subcontinua of X such that $x_1 \in int_X(A_1)$, $x_2 \in int_X(A_2)$, $x_2 \notin A_1$ and $x_1 \notin A_2$. Then, for each $\epsilon > 0$ with $\epsilon < \frac{1}{2}\min\{d(x_1, A_2), d(x_2, A_1)\}$,*

$$C = (A_1 \times A_2) \setminus (N(\epsilon, A_2) \times N(\epsilon, A_1))$$

is a subcontinuum of X^2 such that $(x_1, x_2) \in int_{X^2}(C)$ and $C \cap \triangle_{X^2} = \emptyset$.

PROOF. Let $\epsilon > 0$ be such that $\epsilon < \frac{1}{2}\min\{d(x_1, A_2), d(x_2, A_1)\}$. Note that C is a closed subset of X^2, and thus, C is compact. To prove that C is connected, let (p, q) and (r, s) be two distinct points in C, and we will check that there exists a connected subset of C containing the points (p, q) and (r, s). Since $(p, q), (r, s) \in C$, it follows that $p, r \in A_1$ and $q, s \in A_2$. In order to prove that C is connected, we consider the following cases:

CASE (1). $p \notin N(\epsilon, A_2)$ and $r \notin N(\epsilon, A_2)$. For this case, it follows that $\{p\} \times A_2$ and $\{r\} \times A_2$ are connected subsets of C such that $(p, q), (p, x_2) \in \{p\} \times A_2$ and $(r, s), (r, x_2) \in \{r\} \times A_2$. Furthermore, since $x_2 \notin N(\epsilon, A_1)$, we obtain that $A_1 \times \{x_2\}$ is a connected subset of C such that $(p, x_2), (r, x_2) \in A_1 \times \{x_2\}$. We define $K = [\{p\} \times A_2] \cup [\{r\} \times A_2] \cup [A_1 \times \{x_2\}]$. Then, K is a connected subset of C containing the points (p, q) and (r, s).

CASE (2). $p \notin N(\epsilon, A_2)$ and $s \notin N(\epsilon, A_1)$. We note that $\{p\} \times A_2$ and $A_1 \times \{s\}$ are connected subsets of C such that $(p, q), (p, s) \in \{p\} \times A_2$ and $(p, s), (r, s) \in A_1 \times \{s\}$. Let $K = [\{p\} \times A_2] \cup [A_1 \times \{s\}]$. It follows that K is a connected subset of C such that $(p, q), (r, s) \in K$.

CASE (3). $q \notin N(\epsilon, A_1)$ and $r \notin N(\epsilon, A_2)$. Observe that $A_1 \times \{q\}$ and $\{r\} \times A_2$ are connected subsets of C such that $(p, q), (r, q) \in A_1 \times \{q\}$ and $(r, s), (r, q) \in \{r\} \times A_2$. We define $K = [A_1 \times \{q\}] \cup [\{r\} \times A_2]$. Hence, K is a connected subset of C containing the points (p, q) and (r, s).

CASE (4). $q \notin N(\epsilon, A_1)$ and $s \notin N(\epsilon, A_1)$. We obtain that $A_1 \times \{q\}$ and $A_1 \times \{s\}$ are connected subsets of C such that $(p, q), (x_1, q) \in A_1 \times \{q\}$ and $(r, s), (x_1, s) \in A_1 \times \{s\}$. Since $x_1 \notin N(\epsilon, A_2)$, we have that $\{x_1\} \times A_2$ is a connected subset of C such that $(x_1, q), (x_1, s) \in \{x_1\} \times A_2$. We define $K = [A_1 \times \{q\}] \cup [A_1 \times \{s\}] \cup [\{x_1\} \times A_2]$. It follows that K is a connected subset of C containing the points (p, q) and (r, s).

By the Cases (1)–(4), we conclude that C is connected, and thus, C is a subcontinuum of X^2.

On the other hand, we note that $x_1 \in int_X(A_1) \setminus Cl_X(N(\epsilon, A_2))$ and $x_2 \in int_X(A_2) \setminus Cl_X(N(\epsilon, A_1))$. Hence, $(x_1, x_2) \in int_{X^2}(C)$. Furthermore, it easy to see that $C \cap \triangle_{X^2} = \emptyset$. □

THEOREM 3.10. *If X is an aposyndetic continuum, then $SF_2(X)$ is aposyndetic.*

PROOF. By Theorem 3.1 and Lemma 3.7, it is sufficient to show that for each point $(x_1, x_2) \in X^2 \setminus \triangle_{X^2}$, there exists a subcontinuum C of X^2 such that $(x_1, x_2) \in int_{X^2}(C)$ and $C \cap \triangle_{X^2} = \emptyset$. Let $(x_1, x_2) \in X^2 \setminus \triangle_{X^2}$. Since X is an aposyndetic continuum, there exist two subcontinua A_1 and A_2 of X such that $x_1 \in int_X(A_1)$, $x_2 \in int_X(A_2)$, $x_2 \notin A_1$ and $x_1 \notin A_2$. We take $\epsilon > 0$ such that $\epsilon < \frac{1}{2}\min\{d(x_1, A_2), d(x_2, A_1)\}$. Then, by Lemma 3.9, the subset

$$C = (A_1 \times A_2) \setminus (N(\epsilon, A_2) \times N(\epsilon, A_1))$$

is a subcontinuum of X^2 such that $(x_1, x_2) \in int_{X^2}(C)$ and $C \cap \triangle_{X^2} = \emptyset$. □

4. Mutual Aposyndesis

LEMMA 4.1. *Let X be a continuum, let $n \geq 2$ be an integer, and let U and V be nonempty, proper, open subsets of X such that $U \subset Cl_X(U) \subset V$. If $\mathcal{K} = \langle Cl_X(V)\rangle_n \cup \langle X \setminus U\rangle_n$, then $\mathcal{K}$ is a subcontinuum of $F_n(X)$ such that $F_1(X) \subset int_{F_n(X)}(\mathcal{K})$.*

PROOF. It is clear that $\mathcal{K}$ is a closed subset of $F_n(X)$, and thus $\mathcal{K}$ is compact. Furthermore, since $X = V \cup (X \setminus Cl_X(U))$, it follows that $F_1(X) \subset int_{F_n(X)}(\mathcal{K})$.

Next we prove that $\mathcal{K}$ is connected. Since $F_1(X)$ is a connected subset of $\mathcal{K}$, it is sufficient to show that for each point $A \in \mathcal{K}$, there exists a connected subset $\mathcal{E}$ of $\mathcal{K}$ such that $A \in \mathcal{E}$ and $\mathcal{E} \cap F_1(X) \neq \emptyset$.

Let $A \in \mathcal{K}$. Suppose that $A \in \langle Cl_X(V)\rangle_n$. We assume that $A = \{a_1, \ldots, a_k\}$ with $k \leq n$. For each $i \in \{1, \ldots, k\}$, let C_i be the component of $Cl_X(V)$ such that $a_i \in C_i$. By [19, Theorem 5.4], for each $i \in \{1, \ldots, k\}$, it follows that $C_i \cap Bd(V) \neq \emptyset$. Consequently, for each $i \in \{1, \ldots, k\}$, let $b_i \in C_i \cap Bd(V)$. We define $\mathcal{C} = \langle C_1, \ldots, C_k\rangle_n$ and $B = \{b_1, \ldots, b_k\}$. Then, by [16, Lemma 1], $\mathcal{C}$ is a connected set. Moreover, it is clear that $\mathcal{C} \subset \mathcal{K}$ and $A, B \in \mathcal{C}$.

For each $i \in \{1, \ldots, k-1\}$, we define $\mathcal{D}_i = \{\{b_1, \ldots, b_i\} \cup K : K \in F_1(X)\}$. Note that, for each $i \in \{1, \ldots, k-1\}$, $\mathcal{D}_i$ is a continuum. Furthermore, since for each $i \in \{1, \ldots, k\}$, $b_i \in Cl_X(V) \cap (X \setminus U)$ and $X = Cl_X(V) \cup (X \setminus U)$, we obtain that $\mathcal{D}_1, \ldots, \mathcal{D}_{k-1}$ are subcontinua of $\mathcal{K}$. Moreover, we have that $\mathcal{D}_i \cap \mathcal{D}_{i+1} \neq \emptyset$, $\{b_1\} \in \mathcal{D}_1$ and $B \in \mathcal{D}_{k-1}$. Hence, $\mathcal{D} = \cup_{i=1}^{k-1}\mathcal{D}_i$ is a subcontinuum of $\mathcal{K}$ such that $B, \{b_1\} \in \mathcal{D}$. We define $\mathcal{E} = \mathcal{C} \cup \mathcal{D}$. It follows that $\mathcal{E}$ is a subcontinuum of $\mathcal{K}$ such that $A \in \mathcal{E}$ and $\mathcal{E} \cap F_1(X) \neq \emptyset$.

Similiarly, we can show the case $A \in \langle X \setminus U\rangle_n$. □

THEOREM 4.2. *Let X be a continuum and let $n \geq 3$ be an integer. Then $SF_n(X)$ is mutually aposyndetic.*

PROOF. Let $\mathcal{A}, \mathcal{B} \in SF_n(X)$ be such that $\mathcal{A} \neq \mathcal{B}$. We assume, without loss of generality, that $\mathcal{A} \neq F_X^n$. We take $A \in F_n(X) \setminus F_1(X)$ such that $q_X^n(A) = \mathcal{A}$, and let $B \in F_n(X)$ such that $q_X^n(B) = \mathcal{B}$. Hence, we have the following cases:

CASE (1). $A \not\subset B$. For this case, let $a \in A \setminus B$ and let $b \in A$ with $a \neq b$. We define $B_1 = B \cup \{b\}$. Let U, V, U_1 and V_1 be proper open subsets of X such that $B_1 \subset U \subset Cl_X(U) \subset U_1 \subset Cl_X(U_1) \subset V_1 \subset X$, $a \in V$ and $Cl_X(V_1) \cap Cl_X(V) = \emptyset$. We define the subsets, $\mathcal{H}$ and $\mathcal{K}$, of $F_n(X)$ as follows: $\mathcal{H} = \langle Cl_X(V), Cl_X(U), X\rangle_n \cup \langle Bd(V), Bd(U), X\rangle_n \cup \langle Bd(V), \{b\}, X\rangle_n \cup \langle \{a\}, \{b\}, X\rangle_n$ and $\mathcal{K} = \langle Cl_X(V_1)\rangle_n \cup \langle X \setminus U_1\rangle_n$. By [16, Lemma 2], $\mathcal{H}$ is a subcontinuum of $F_n(X)$. Furthermore, $A \in int_{F_n(X)}(\mathcal{H})$ and $\mathcal{H} \cap F_1(X) = \emptyset$. Applying Remark 2.1, we obtain that $q_X^n(\mathcal{H})$ is a subcontinuum of $SF_n(X)$ such that $\mathcal{A} \in int_{SF_n(X)}(q_X^n(\mathcal{H}))$. On the other hand, by Lemma 4.1, we have that $\mathcal{K}$ is a subcontinuum of $F_n(X)$ such that $F_1(X) \subset int_{F_n(X)}(\mathcal{K})$. Also, $B \in int_{F_n(X)}(\mathcal{K})$. By Remark 2.1, it follows that $q_X^n(\mathcal{K})$ is a subcontinuum of $SF_n(X)$ such that $\mathcal{B}, F_X^n \in int_{SF_n(X)}(q_X^n(\mathcal{K}))$. Since $\mathcal{H} \cap \mathcal{K} = \emptyset$ and $F_1(X) \subset \mathcal{K}$, we obtain that $q_X^n(\mathcal{H}) \cap q_X^n(\mathcal{K}) = \emptyset$. This proves Case (1).

CASE (2). $A \subset B$. Since $A \in F_n(X) \setminus F_1(X)$, it follows that $B \in F_n(X) \setminus F_1(X)$. Since $A \neq B$, we obtain that $B \not\subset A$. Let $b \in B \setminus A$. Since $B \in F_n(X) \setminus F_1(X)$, there exists a point $a \in B \setminus \{b\}$. Now, we proceed as in Case (1).

Finally, by Cases (1) and (2), we conclude that $SF_n(X)$ is mutually aposyndetic. □

THEOREM 4.3. *Let X a chainable continuum. Then $SF_2(X)$ is mutually aposyndetic if and only if X is the arc.*

PROOF. If X is the arc then, by [1, Example 3.1], $SF_2(X)$ is homeomorphic to $[0,1]^2$, and hence, $SF_2(X)$ is mutually aposyndetic .

To prove the converse, by [20, Theorem 1] and [11, Theorem 15], it is sufficient to show that given two different points of the form (x, q) and (y, q) (or (q, x) and (q, y)) in X^2, there exist subcontinua $\mathcal{A}$ and $\mathcal{B}$ of $F_2(X)$ such that $f_2((x, q)) \in int_{F_2(X)}(\mathcal{A})$, $f_2((y, q)) \in int_{F_2(X)}(\mathcal{B})$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. To this end, let (x, q) and (y, q) be two different points in X^2. We denote $A = f_2((x, q))$ and $B = f_2((y, q))$. Then $q_X^2(A)$ and $q_X^2(B)$ are two different points in $SF_2(X)$. We take two subcontinua Γ and Λ of $SF_2(X)$ such that $q_X^2(A) \in int_{SF_2(X)}(\Gamma)$, $q_X^2(B) \in int_{SF_2(X)}(\Lambda)$ and $\Gamma \cap \Lambda = \emptyset$. Let $\mathcal{A} = (q_X^2)^{-1}(\Gamma)$ and let $\mathcal{B} = (q_X^2)^{-1}(\Lambda)$. Since q_X^2 is a monotone map, by [21, 2.2, p.138], we obtain that $\mathcal{A}$ and $\mathcal{B}$ are subcontinua of $F_2(X)$. Furthermore, $A \in int_{F_2(X)}(\mathcal{A})$,

$B \in int_{F_2(X)}(\mathcal{B})$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Then, proceeding as in [11, Theorem 15] and [20, Theorem 1], we can conclude that X is the arc. □

As an easy consequence of [11, Lemma 9], we have the following:

LEMMA 4.4. *Let X be a continuum and let (a_1, a_2) and (b_1, b_2) be points in X^2. If A and B are subcontinua of X^2 such that $(a_1, a_2) \in int_{X^2}(A)$, $(b_1, b_2) \in int_{X^2}(B)$ and $(A \cup A^*) \cap (B \cup B^*) = \emptyset$, then $f_2(A)$ and $f_2(B)$ are subcontinua of $F_2(X)$ such that $\{a_1, a_2\} \in int_{F_2(X)}(f_2(A))$, $\{b_1, b_2\} \in int_{F_2(X)}(f_2(B))$ and $f_2(A) \cap f_2(B) = \emptyset$.*

THEOREM 4.5. *If X is an aposyndetic continuum, then $F_2(X)$ is mutaully aposyndetic.*

PROOF. Let $A, B \in F_2(X)$ such that $A \neq B$. Without loss of generality, suppose that there exists a point $b_1 \in B \setminus A$. Take $a_1 \in A$. Then we have the following cases:

CASE (1). $A \in F_1(X)$. Let A_1 be a subcontinuum of X such that $a_1 \in int_X(A_1)$ and $b_1 \notin A_1$. Thus, there exists an open subset U of X such that $b_1 \in U \subset Cl_X(U) \subset X \setminus A_1$. We define $C = A_1 \times A_1$ and $K = (Cl_X(U) \times X) \cup (X \times Cl_X(U))$. It follows that C and K are subcontinua of X^2 such that $(a_1, a_1) \in int_{X^2}(C)$, $(b_1, b_2) \in int_{X^2}(K)$ (with $b_2 \in X$ such that $B = \{b_1, b_2\}$) and $(C \cup C^*) \cap (K \cup K^*) = \emptyset$. Therefore, by Lemma 4.4, $f_2(C)$ and $f_2(K)$ are two subcontinua of $F_2(X)$ such that $A \in int_{F_2(X)}(f_2(C))$, $B \in int_{F_2(X)}(f_2(K))$ and $f_2(C) \cap f_2(C) = \emptyset$.

CASE (2). $A \in F_2(X) \setminus F_1(X)$. Let $A = \{a_1, a_2\}$. Since $b_1 \notin A$, there exist subcontinua A_1 and A_2 of X such that $a_1 \in int_X(A_1)$, $a_2 \in int_X(A_2)$, $b_1 \notin A_1$ and $b_1 \notin A_2$. Let U be an open subset of X such that $b_1 \in U \subset Cl_X(U) \subset X \setminus (A_1 \cup A_2)$. We define $C = A_1 \times A_2$ and $K = (Cl_X(U) \times X) \cup (X \times Cl_X(U))$. Then C and K are two subcontinua of X^2 such that $(a_1, a_2) \in int_{X^2}(C)$, $(b_1, b_2) \in int_{X^2}(K)$ (with $b_2 \in X$ such that $B = \{b_1, b_2\}$) and $(C \cup C^*) \cap (K \cup K^*) = \emptyset$. Hence, by Lemma 4.4, we have that $f_2(C)$ and $f_2(K)$ are subcontinua of $F_2(X)$ such that $A \in int_{F_2(X)}(f_2(C))$, $B \in int_{F_2(X)}(f_2(K))$ and $f_2(C) \cap f_2(K) = \emptyset$. □

THEOREM 4.6. *If a continuum X is 2-aposyndetic and mutually aposyndetic, then $SF_2(X)$ is mutually aposyndetic.*

PROOF. Let $\mathcal{A}, \mathcal{B} \in SF_2(X)$ be such that $\mathcal{A} \neq \mathcal{B}$. Without loss of generality, we suppose that $\mathcal{A} \neq F_X^2$. Hence, there exists $A \in F_2(X) \setminus F_1(X)$ such that $q_X^2(A) = \mathcal{A}$. We put $A = \{a_1, a_2\}$. We have the following cases:

CASE (1). $\mathcal{B} \neq F_X^2$. Let $B \in F_2(X) \setminus F_1(X)$ be such that $q_X^2(B) = \mathcal{B}$. We assume that $B = \{b_1, b_2\}$. Suppose that $a_1 \in A \setminus B$. Then, since X is 2-aposyndetic, there exist two subcontinua W_1 and W_2 of X such that $b_1 \in int_X(W_1)$, $W_1 \cap \{b_2, a_1\} = \emptyset$, $b_2 \in int_X(W_2)$ and $W_2 \cap \{b_1, a_1\} = \emptyset$. We fix a number $\epsilon > 0$ such that $\epsilon < \frac{1}{2}\min\{d(b_2, W_1), d(a_1, W_1), d(b_1, W_2), d(a_1, W_2)\}$. Applying Lemma 3.9, we obtain that

$$C = (W_1 \times W_2) \setminus (N(\epsilon, W_2) \times N(\epsilon, W_1))$$

is a subcontinuum of X^2 such that $(b_1, b_2) \in int_{X^2}(C)$ and $C \cap \triangle_{X^2} = \emptyset$.

We define $U = N(a_1, \epsilon)$. We note that $Cl_X(U) \cap (N(\epsilon, W_1) \cup N(\epsilon, W_2)) = \emptyset$. Hence, $Cl_X(U) \cap (W_1 \cup W_2) = \emptyset$. We define

$$Z = (Cl_X(U) \times X) \cup (X \times Cl_X(U)).$$

It follows that Z is a subcontinuum of X^2 such that $(a_1, a_2) \in int_{X^2}(Z)$ and $Z = Z^*$. Since $(a_1, a_2) \in int_{X^2}(Z) \cap (X^2 \setminus \triangle_{X^2})$, there exists an open subset V of X^2 with $(a_1, a_2) \in V \subset Z$ and $V \subset X^2 \setminus \triangle_{X^2}$. Moreover, it is easy to see that $(C \cup C^*) \cap (Z \cup Z^*) = \emptyset$. By Lemma 4.4, we obtain that $f_2(C)$ and $f_2(Z)$ are subcontinua of $F_2(X)$ such that $A \in int_{F_2(X)}(f_2(Z))$, $B \in int_{F_2(X)}(f_2(C))$ and $f_2(C) \cap f_2(Z) = \emptyset$. Furthermore, by [11, Lemma 9], $f_2(V)$ is an open subset of $F_2(X)$ with $A \in f_2(V) \subset f_2(Z)$ and $f_2(V) \cap F_1(X) = \emptyset$. Applying Remark 2.1, we conclude that $q_X^2(f_2(Z))$ and $q_X^2(f_2(C))$ are two subcontinua of $SF_2(X)$ such that $\mathcal{A} \in int_{SF_2(X)}(q_X^2(f_2(Z)))$, $\mathcal{B} \in int_{SF_2(X)}(q_X^n(f_2(C)))$ and $SF_2(X)(q_X^2(f_2(C)) \cap SF_2(X)(q_X^n(f_2(Z))) = \emptyset$.

Case (2). $\mathcal{B} = F_X^2$. Since X is mutually aposyndetic, there exist two subcontinua W_1 and W_2 of X such that $a_1 \in int_X(W_1)$, $a_2 \in int_X(W_2)$ and $W_1 \cap W_2 = \emptyset$. Since X is 2-aposyndetic, for each $x \in X \setminus A$, let W_x be a subcontinuum of X such that $x \in int_X(W_x)$ and $W_x \cap A = \emptyset$. Then $\{int_X(W_x) : x \in X \setminus A\} \cup \{int_X(W_1), int_X(W_2)\}$ is an open cover of X. Hence, there exist $x_1, \ldots, x_m \in X \setminus A$ such that $X = \cup_{i=1}^m int_X(W_{x_i}) \cup (int_X(W_1) \cup int_X(W_2))$.

We define a subset, W, of X^2 as follows:

$$W = \cup_{i=1}^m W_{x_i}^2 \cup (W_1^2 \cup W_2^2).$$

Then W is a subcontinuum of X^2 with $\triangle_{X^2} \subset int_{X^2}(W)$.

Let $J = \{i \in \{1, \ldots, m\} : W_{x_i} \cap W_1 \neq \emptyset\}$ and let $K = \{i \in \{1, \ldots, m\} : W_{x_i} \cap W_2 \neq \emptyset\}$. We note $J \neq \emptyset$ and $K \neq \emptyset$.

Let $\epsilon_1 > 0$ and $\epsilon_2 > 0$ be such that $\epsilon_1 < \frac{1}{2}\min\{d(W_{x_i}, a_1) : i \in \{1, \ldots, m\}\}$ and $\epsilon_2 < \frac{1}{2}\min\{d(W_{x_i}, a_2) : i \in \{1, \ldots, m\}\}$. Let $\epsilon > 0$ be such that $\epsilon < \min\{\epsilon_1, \epsilon_2\}$. We define a subset, Z, of X as $Z = \cup_{i=1}^m W_{x_i}$. We note that $a_1 \notin N(\epsilon, Z)$. To see that $a_1 \notin N(\epsilon, Z)$, suppose that $a_1 \in N(\epsilon, Z)$. Then there exists $z \in Z$ such that $d(a_1, z) < \epsilon$. Thus, there exists $j \in \{1, \ldots, m\}$ such that $z \in W_{x_j}$, so that $\epsilon < d(a_1, W_{x_j}) \leq d(a_1, z) < \epsilon$, which is a contradiction. Therefore, $a_1 \notin N(\epsilon, Z)$. Similarly, $a_2 \notin N(\epsilon, Z)$.

Let $U_1 = int_X(W_1) \setminus N(\epsilon, Z)$, and let $U_2 = int_X(W_2) \setminus N(\epsilon, Z)$. It follows that $U_1 \times U_2$ is an open subset of X^2 such that $(a_1, a_2) \in U_1 \times U_2$. We define

$$C = (W_1 \times W_2) \setminus (N(\epsilon, Z) \times N(\epsilon, Z)).$$

Since $U_1 \times U_2 \subset C$, we obtain that $(a_1, a_2) \in int_{X^2}(C)$. Moreover, we have that C is a closed subset of X, and thus, C is compact.

Next, we prove that C is connected. Let (p, q) and (r, s) be two distinct points in C. We are going to show that there is a connected subset in C containing (p, q) and (r, s). Since $(p, q), (r, s) \in C$, it follows that $(p, q), (r, s) \in W_1 \times W_2$ and $(p, q), (r, s) \notin N(\epsilon, Z) \times N(\epsilon, Z)$. Hence, $p, r \in W_1$, $q, s \in W_2$, $[p \notin N(\epsilon, Z)$ or $q \notin N(\epsilon, Z)]$ and $[r \notin N(\epsilon, Z)$ or $s \notin N(\epsilon, Z)]$. Consequently, we have the following possibilities:

(a) $p \notin N(\epsilon, Z)$ and $r \notin N(\epsilon, Z)$. In this case, it follows that $\{p\} \times W_2$ and $\{r\} \times W_2$ are two connected subsets of C such that $(p, q), (p, a_2) \in \{p\} \times W_2$ and $(r, s), (r, a_2) \in \{r\} \times W_2$. Moreover, since $a_2 \notin N(\epsilon, Z)$, we have that $W_1 \times \{a_2\}$ is a connected subset of C such that $(p, a_2), (r, a_2) \in W_1 \times \{a_2\}$. We define $D = [\{p\} \times W_2] \cup [\{r\} \times W_2] \cup [W_1 \times \{a_2\}]$. Then D is a connected subset of C containing (p, q) and (r, s).

(b) $p \notin N(\epsilon, Z)$ and $s \notin N(\epsilon, Z)$. It follows that $\{p\} \times W_2$ and $W_1 \times \{s\}$ are two connected subsets of C such that $(p, q), (p, s) \in \{p\} \times W_2$ and $(p, s), (r, s) \in W_1 \times \{s\}$. We define $D = [\{p\} \times W_2] \cup [W_1 \times \{s\}]$. Then D is a connected subset of C containing (p, q) and (r, s).

(c) $q \notin N(\epsilon, A_1)$ and $r \notin N(\epsilon, A_2)$. This case is similar to (b), with $D = [W_1 \times \{q\}] \cup [\{r\} \times W_2]$.

(d) $q \notin N(\epsilon, A_1)$ and $s \notin N(\epsilon, A_1)$.

This case is similar to (a), with $D = [W_1 \times \{q\}] \cup [W_1 \times \{s\}] \cup [\{a_1\} \times W_2]$.

By (a)–(d), we conclude that C is a connected subset of X^2. Consequently, C is a subcontinuum of X^2 such that $(a_1, a_2) \in int_{X^2}(C)$.

Next we prove that $C \cap W = \emptyset$. Suppose that $(c_1, c_2) \in C \cap W$. This implies that $(c_1, c_2) \in W_1 \times W_2$, $(c_1, c_2) \notin N(\epsilon, Z) \times N(\epsilon, Z)$ and $(c_1, c_2) \in W$. Since $W = \cup_{i=1}^{m} W_{x_i}^2 \cup (W_1^2 \cup W_2^2)$, we have that either $(c_1, c_2) \in W_1^2$ or $(c_1, c_2) \in W_2^2$ or there exists $i \in \{1, \ldots, r\}$ such that $(c_1, c_2) \in W_{x_i}^2$. If $(c_1, c_2) \in W_1^2$, then $c_2 \in W_1 \cap W_2$, which is a contradiction. If $(c_1, c_2) \in W_2^2$, then $c_1 \in W_1 \cap W_2$, which is a contradiction. If there exists $i \in \{1, \ldots, r\}$ such that $(c_1, c_2) \in W_{x_i}^2$, then $c_1, c_2 \in W_{x_i}$, thus, $c_1, c_2 \in Z$, this implies that $(c_1, c_2) \in N(\epsilon, Z) \times N(\epsilon, Z)$, which is a contradiction. Therefore, $C \cap W = \emptyset$.

Similarly, we obtain that $C^* \cap W = \emptyset$. Moreover, since $W = W^*$, it follows that $(C \cup C^*) \cap (W \cup W^*) = \emptyset$.

By Lemma 4.4, we have that $f_2(C)$ and $f_2(W)$ are two subcontinua of $F_2(X)$ such that $A \in int_{F_2(X)}(f_2(C)) \subset F_2(X) \setminus F_1(X)$, $F_1(X) \subset int_{F_2(X)}(f_2(W))$ and $f_2(C) \cap f_2(W) = \emptyset$. Hence, applying Remark 2.1, we conclude that $q_X^2(f_2(C))$ and $q_X^2(f_2(W))$ are two subcontinua of $SF_2(X)$ such that $\mathcal{A} \in int_{SF_2(X)}(q_X^2(f_2(C)))$, $\mathcal{B} \in int_{SF_2(X)}(q_X^2(f_2(W)))$ and $q_X^2(f_2(C)) \cap q_X^2(f_2(W)) = \emptyset$.

Finally, by Cases (1) and (2), we conclude that $SF_2(X)$ is mutually aposyndetic. $\square$

THEOREM 4.7. *If X is a decomposable continuum, then $SF_2(X)$ is not strictly nonmutually aposyndetic.*

PROOF. Let X be a decomposable continuum. Let A_1 and A_2 be proper subcontinua of X such that $X = A_1 \cup A_2$. Hence, there exist two points $x_1 \in X \setminus A_2$ and $x_2 \in X \setminus A_1$. We note that $x_1 \in int_X(A_1)$ and $x_2 \in int_X(A_2)$. Fix a number $\epsilon > 0$ such that $\epsilon < \frac{1}{2} \min\{d(x_1, A_2), d(x_2, A_1)\}$. Consequently, by Lemma 3.9, the set

$$C = (A_1 \times A_2) \setminus (N(\epsilon, A_2) \times N(\epsilon, A_1))$$

is a subcontinuum of X^2 such that $(x_1, x_2) \in int_{X^2}(C)$ and $C \cap \triangle_{X^2} = \emptyset$.

We take $y \in N(A_1, \epsilon) \cap (X \setminus A_1)$. Since $N(A_1, \epsilon) \cap (X \setminus A_1)$ is an open subset of X, there exists an open subset U of X such that $y \in U \subset Cl_X(U) \subset N(A_1, \epsilon) \cap (X \setminus A_1)$. We define

$$K = (Cl_X(U) \times A_2) \cup (A_2 \times Cl_X(U)).$$

It follows that K is a subcontinuum of X^2 such that $int_{X^2}(K) \neq \emptyset$.

Note that $(C \cup C^*) \cap (K \cup K^*) = \emptyset$. By Lemma 4.4, we have that $f_2(C)$ and $f_2(K)$ are subcontinua of $F_2(X)$ such that $int_{F_2(X)}(f_2(C)) \neq \emptyset$, $int_{F_2(X)}(f_2(K)) \neq \emptyset$ and $f_2(C) \cap f_2(K) = \emptyset$.

Since $C \cap \triangle_{X^2} = \emptyset$, it follows that $f_2(C) \subset F_2(X) \setminus F_1(X)$. Applying Remark 2.1 we obtain that $q_X^2(f_2(C))$ is a subcontinuum of $SF_2(X)$ such that $int_{SF_2(X)}(q_X^2(f_2(C))) \neq \emptyset$.

On the other hand, let $z \in U \setminus \{y\}$. Since $(y, z) \in U \times U \subset K$, $\{y, z\} \in f_2(U \times U) \subset f_2(K)$. By [11, Lemma 9], we have that $f_2(U \times U)$ is an open subset of $F_2(X)$. Let $\mathcal{U}$ be an open subset of $F_2(X)$ such that $\{y, z\} \in \mathcal{U} \subset f_2(U \times U) \cap (F_2(X) \setminus F_1(X))$. Using Remark 2.1 we obtain that $q_X^2(f_2(\mathcal{U}))$ is an open subset of $SF_2(X)$ with $q_X^2(\{y, z\}) \in q_X^2(f_2(\mathcal{U}) \subset q_X^2(f_2(K))$. Thus, $q_X^2(f_2(K))$ is a subcontinuum of $SF_2(X)$ such that $int_{SF_2(X)}(q_X^2(f_2(K))) \neq \emptyset$.

Finally, since $f_2(C) \cap f_2(K) = \emptyset$ and $f_2(C) \subset F_2(X) \setminus F_1(X)$, by Remark 2.1, we conclude that $q_X^2(f_2(C)) \cap q_X^2(f_2(K)) = \emptyset$. This prove that $SF_2(X)$ is not strictly nonmutually aposyndetic. ☐

THEOREM 4.8. *Let X a continuum with zero span. Then X is indecomposable if and only if $SF_2(X)$ is strictly nonmutually aposyndetic.*

PROOF. Suppose that X is indecomposable. Let $\mathfrak{A}$ be a subcontinuum of $SF_2(X)$ such that $int_{SF_2(X)}(\mathfrak{A}) \neq \emptyset$. We put $\mathcal{B} = (q_X^2)^{-1}(\mathfrak{A})$. Since q_X^2 is a monotone map, by [21, 2.2, p.138], we obtain that $\mathcal{B}$ is a subcontinuum of $F_2(X)$, furthermore, we have that $int_{F_2(X)}(\mathcal{B}) \neq \emptyset$. We take a component W of $(f_2)^{-1}(\mathcal{B})$. Applying [11, Lemma 14] it follows that $int_{X^2}(W) \neq \emptyset$. Since π_1 and π_2 are open maps, we have that $\pi_1(W)$ and $\pi_2(W)$ are subcontinua of X such that $int_X(\pi_1(W)) \neq \emptyset$ and $int_X(\pi_2(W)) \neq \emptyset$. Since X is indecomposable, by [14, Corollary 1.7.21], it follows that $\pi_1(W) = X$ and $\pi_2(W) = X$. Since X is a continuum with zero span, we conclude that

$W \cap \triangle_{X^2} \neq \emptyset$. This implies that $\mathcal{B} \cap F_1(X) \neq \emptyset$. Consequently, $F_X^2 \in \mathfrak{A}$. This prove that, $SF_2(X)$ is strictly nonmutually aposyndetic.

Applying Theorem 4.7 we obtain that if $SF_2(X)$ is strictly nonmutually aposyndetic, then X is indecomposable. □

It is known, for instance [10, p. 210], that if X is a chainable continuum then X is a continuum with zero span. Hence, by Theorem 4.8, we have the following corollary.

Corollary 4.9. *Let X a chainable continuum. Then X is indecomposable if and only if $SF_2(X)$ is strictly nonmutually aposyndetic.*

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