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A SHAPE THEORETIC APPROACH TO GENERALIZED COHOMOLOGICAL DIMENSION WITH RESPECT TO TOPOLOGICAL SPACES

Takahisa Miyata
Kobe University, Japan

Abstract. A. N. Dranishnikov introduced the notion of generalized cohomological dimension of compact metric spaces with respect to CW spectra. In this paper, taking an inverse system approach, we generalize this definition and obtain two types of generalized comological dimension with respect to general topological spaces, which are objects in the stable shape category. We characterize those two types of generalized cohomological dimension in terms of maps and obtain their fundamental properties. In particular, we obtain their relations to the integral cohomological dimension and the covering dimension. Moreover, we study the generalized cohomological dimensions of compact Hausdorff spaces with respect to the Kahn continuum and the Hawaiian earing.

1. Introduction

The theory of cohomological dimension originates in the work of P. S. Alexandroff in the late 1920's ([3]). J. Dydak ([8]) outlines the development of the cohomological dimension theory from the beginning to mid 1990's. For every finite-dimensional compact metric space X, the integral cohomological dimension $\text{c-dim}_{\mathbb{Z}} X$ equals the covering dimension $\dim X$ ([4]), but there exists an infinite dimensional compact metric space with finite integral cohomological dimension ([5]). To distinguish the class of infinite dimensional spaces, Dranishnikov ([6]) introduced the notion of generalized cohomological dimension c-dim_E with respect to CW spectra E. If $K(G)$ is the Eilenberg-MacLane spectrum $\{K(G,n)\,(n \geq 0), \text{a singleton}\,(n < 0)\}$, where G is an

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abelian group, then the generalized cohomological dimension c-$\dim_{K(G)} X$ coincides with the ordinary cohomological dimension c-$\dim_G X$. If S is the sphere spectrum $\{S^n\,(n \geq 0)$, a singleton$\,(n < 0)\}$, then c-$\dim_S X$ coincides with the cohomotopical dimension.

If E is a CW spectrum with some additional condition, then c-$\dim_{\mathbb{Z}} X \leq$ c-$\dim_E X \leq \dim X$ for every compact metric space X ([5, Theorems 1, 2]). There exists a compact metric space X such that c-$\dim_{\mathbb{Z}} X = 3$ and c-$\dim_S X = \infty$ ([5, Theorem 4]). In 1986, S. Nowak raised the following question: does the cohomotopical dimension coincide with its covering dimension for compact metric spaces? It is still an open problem ([20]).

For given spaces X and Y, for a closed subset A of X, and a map $f : A \to Y$, the extension problem is to determine whether the map f extends to a map $\overline{f} : X \to Y$. Here the codomain Y is usually assumed to be a nice space such as a CW complex, a polyhedron, or an ANR, so that the homotopy extension property holds for maps into Y. Recently, generalizing the extension property, the author introduced the notion of approximate extension property to allow more general spaces for codomains ([15]). Roughly, the idea is based on the extendability of maps from closed subsets of spaces to inverse systems, which are parts of expansions in the sense of [13].

This paper is to unify the idea of approximate extension property and the idea of generalized cohomological dimension with respect to CW spectra and obtain generalized cohomological dimension with respect to topological spaces, which represents the right bottom box in the following diagram.

Extension property with respect to CW complexes	$\longrightarrow$	Generalized cohomological dimension with respect to CW-spectra
$\downarrow$		$\downarrow$
Approximate extension property with respect to topological spaces	$\longrightarrow$	Generalized cohomological dimension with respect to topological spaces

From another point of view, homotopy equivalent CW complexes and shape equivalent shapes have the same extension type and approximate extension type, respectively; homotopy equivalent CW spectra have the same generalized cohomology type. In our new theory, stable shape equivalent spaces have the same generalized cohomology type. For convenience, the generalized stable shape category in the sense of [17], which is an extended version of the stable shape category and whose objects include all topological spaces together with CW spectra, is considered.

Stable shape originates in the work of E. Lima [12] in the study of the Spanier-Whitehead duality for compact subsets of the n-th sphere, and its theory has been developed by various mathematicians. More recently, stable cohomotopy groups are studied in this framework ([20–22]). Another aspect is that duality between compact and CW spectra holds in the (generalized) stable shape category ([14, 18]). We are interested in seeing how our generalized cohomological dimension theory will fit this framework.

In this paper we define generalized cohomological dimension theory with respect to stable shape object and study its fundamental properties. In order to do so, we define generalized comological dimension with respect to inverse systems of CW spectra. Allowing more objects for dimension coefficients, we have more dimension types and structures for the theory.

For every inverse system $\boldsymbol{E}$ of CW spectra we introduce two definitions for generalized cohomological dimension with respect to $\boldsymbol{E}$: small dimension $\text{g-dim}_{\boldsymbol{E}}$ (see Section 3) and large dimension $\text{G-dim}_{\boldsymbol{E}}$ (see Section 4). The two types of generalized cohomological dimension coincide if the inverse system consists of a single CW spectrum and $\text{g-dim}_{\boldsymbol{E}} \leq \text{G-dim}_{\boldsymbol{E}}$ holds in general. Using the notion of shape theoretical expansion we define the corresponding types of generalized cohomological dimension g-dim_Z and G-dim_Z with respect to topological spaces Z. We show that thus defined types of generalized cohomological dimension are invariant in the generalized stable shape category. We obtain their characterizations in terms of maps between CW spectra or between CW complexes. We will see that the large generalized cohomological dimension has a better characterization if the coefficient inverse system of CW spectra satisfies some additional condition.

We consider variable coefficient spaces for the generalized cohomological dimensions (Section 5). In particular, we obtain relations between large generalized cohomological dimensions G-dim_{Z_1}, G-dim_{Z_2} and G-dim_{Z_3} with respect to spaces in a cofibre sequence $Z_1 \to Z_2 \to Z_3$. We apply the notion of duality between compact metric spaces and CW spectra ([14]) to our generalized cohomological dimension.

Generalizing the case for generalized cohomological dimension with respect to CW spectra ([5, Theorems 1, 2]), we introduce the notion of approximately perfectly connected space (Section 6). We show that if Z is an approximately perfectly connected space, then $\text{g-dim}_Z X = \dim X$ for every finite dimensional compact Hausdorff space X and if Z satisfies some additional condition, then $\text{c-dim}_{\mathbb{Z}} X \leq \text{g-dim}_Z X$ for every compact metric space X.

As an example, we discuss generalized cohomological dimension with respect to Kahn continuum and Hawaiian earing (Sections 5, 6). In particular, we show that if Z is the Kahn continuum, then $\text{g-dim}_Z X = 0$ for every finite dimensional compact Hausdorff space X.

Throughout the paper, unless otherwise stated, space means topological space and map means continuous map. All spaces are assumed to have base points, and maps and homotopies preserve base points. For spaces X and Y, let $[X, Y]$ denote the set of homotopy classes of maps from X to Y. Similarly, for CW spectra G and H, let $[G, H]$ denote the set of homotopy classes of maps from G to H. For every category $\mathcal{C}$, if X is an object of $\mathcal{C}$, let $1_X : X \to X$ denote the identity morphism. Let $\mathbb{Z}$ denote the set of integers.

2. Preliminaries

In this section, we recall terminology of CW spectrum, shape theory, generalized cohomology and generalized cohomological dimension.

CW spectra. For more details on CW spectra, the reader is referred to [25] and [2].

Let X be a space. Let $\Sigma X = S^1 \wedge X$ and $\Sigma^k X = \Sigma(\Sigma^{k-1} X)$ for $k \geq 1$, where $\Sigma^0 X = X$.

A CW spectrum E is a sequence $\{E_n : n \in \mathbb{Z}\}$ of CW complexes with embeddings $\varepsilon_n : \Sigma E_n \to E_{n+1}$. Every CW spectrum consists of cells $e = \{e_n^d, \Sigma e_n^d, \Sigma^2 e_n^d, \ldots\}$, where e_n^d is a d-cell in the CW complex E_n and is not a suspension of any cell in E_{n-1}. The dimension of e, in notation, $\dim e$, is defined as $d - n$. The dimension of a CW spectrum E, in notation, $\dim E$, is defined as $\sup\{\dim e : e \text{ is a cell of } E\}$.

A map $f : E \to F$ between CW spectra is represented by a function $f' = \{f'_n : n \in \mathbb{Z}\} : E' \to F$, where E' is a cofinal subspectrum of E. By a function $f' = \{f'_n : n \in \mathbb{Z}\} : E' \to F$, we mean a collection $\{f'_n : n \in \mathbb{Z}\}$ of cellular maps $f'_n : E'_n \to F_n$ such that $f'_{n+1}|\Sigma E_n = \Sigma f'_n$.

The suspension spectrum $E(X)$ of a CW complex X is the spectrum defined by

$$(E(X))_n = \begin{cases} \Sigma^n X & n \geq 0, \\ * & n < 0. \end{cases}$$

For every map $f : X \to Y$ between CW complexes, let $E(f) : E(X) \to E(Y)$ denote the map represented by the function $\{f_n : n \in \mathbb{Z}\}$ such that $f_n = \Sigma^n f : \Sigma^n X \to \Sigma^n Y$ for $n \geq 0$ and f_n is the constant for $n < 0$.

Let SPEC also denote the category of CW spectra and maps, and let HSPEC denote the homotopy category of SPEC. Let Σ denote the suspension functor on SPEC, and let Σ^{-1} be its inverse. Iteratively, we define the m-th suspension functor Σ^m by $\Sigma^m = \Sigma \circ \Sigma^{m-1}$ for $m \geq 2$, where $\Sigma^1 = \Sigma$ and $\Sigma^m = \Sigma^{-1} \circ \Sigma^{m+1}$ for $m \leq -2$.

Shape theory and stable shape theory. Let Top denote the category of spaces and maps, and let CW denote the full subcategory of Top whose objects are CW complexes. Let HTop denote the homotopy category of Top, and HCW denote the full subcategory of HTop whose objects are the spaces which are homotopy equivalent to a CW complex.

We briefly recall the constructions of the shape category, the stable shape category, and the generalized stable shape theory. For more details, the reader is referred to [13], [9], and [17].

Let $\mathcal{C}$ be a category. The pro-category pro-$\mathcal{C}$ is defined as follows. The objects of pro-$\mathcal{C}$ are inverse systems in $\mathcal{C}$, and the set of morphisms from $\boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ to $\boldsymbol{Y} = (Y_\mu, q_{\mu\mu'}, M)$ is defined as

$$\text{pro-}\mathcal{C}(\boldsymbol{X}, \boldsymbol{Y}) = \lim_{\mu} \operatorname*{colim}_{\lambda} \mathcal{C}(X_\lambda, Y_\mu).$$

Here for any objects A and B in $\mathcal{C}$, $\mathcal{C}(A, B)$ denotes the set of morphisms from A to B.

If X is a space, then there exists an HCW-expansion $\boldsymbol{p} = ([p_\lambda]) : X \to \boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ in the sense of [13]. The shape category Sh is defined as follows. The objects of Sh are spaces. If X and Y are spaces, a morphism $X \to Y$ in Sh is represented by an element of pro-$\mathsf{HCW}(\boldsymbol{X}, \boldsymbol{Y})$, where $\boldsymbol{q} : Y \to \boldsymbol{Y}$ is an HCW-expansion of Y.

The stable shape category ShStab is defined as follows. The objects of ShStab are compact Hausdorff spaces. If X and Y are compact Hausdorff spaces, a morphism $X \to Y$ in ShStab is represented by an element of pro-$\mathsf{HSPEC}(E(\boldsymbol{X}), E(\boldsymbol{Y}))$, where $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ are HCW-expansions of X and Y, respectively. Here, for every inverse system $\boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ in HCW, let $E(\boldsymbol{X}) = (E(X_\lambda), [E(p_{\lambda\lambda'})], \Lambda)$ be the induced inverse system in HSPEC.

The generalized stable shape category Sh_{spec} is defined as follows. The objects of Sh_{spec} are spaces and CW spectra. If X and Y are spaces, a morphism $X \to Y$ in Sh_{spec} is represented by an element of pro-$\mathsf{HSPEC}(\boldsymbol{E}, \boldsymbol{F})$, where $\boldsymbol{\alpha} : E(\boldsymbol{X}) \to \boldsymbol{E}$ and $\boldsymbol{\beta} = E(\boldsymbol{Y}) \to \boldsymbol{F}$ are generalized HSPEC-expansions of X and Y, respectively. Here, $\boldsymbol{p} : X \to \boldsymbol{X}$ and $\boldsymbol{q} : Y \to \boldsymbol{Y}$ are HCW-expansions of X and Y, respectively. Recall that a morphism $\boldsymbol{\alpha} : E(\boldsymbol{X}) \to \boldsymbol{E}$ in pro-HSPEC is a generalized HSPEC-expansion provided whenever $\boldsymbol{\alpha}' : E(\boldsymbol{X}) \to \boldsymbol{E}'$ is a morphism in pro-HSPEC, then there exists a unique morphism $\boldsymbol{\gamma} : \boldsymbol{E} \to \boldsymbol{E}'$ in pro-HSPEC such that $\boldsymbol{\alpha}' = \boldsymbol{\gamma} \circ \boldsymbol{\alpha}$ [16]. For every HCW-expansion $\boldsymbol{p} = ([p_\lambda]) : X \to \boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ such that X_λ are CW complexex and $p_{\lambda\lambda'}$ are cellular maps, the identity induced morphism $E(\boldsymbol{X}) \to E(\boldsymbol{X})$ is a generalized HSPEC-expansion of X. If $X = E$ or $Y = F$ is a CW spectrum, a morphism $X \to Y$ in Sh_{spec} is represented by an element of pro-$\mathsf{HSPEC}((E), \boldsymbol{F})$ or pro-$\mathsf{HSPEC}(\boldsymbol{E}, (F))$. Here (E) is the rudimentary system of E.

Note that there is an embedding of Sh_{spec} into pro-HSPEC up to isomorphisms and that ShStab is a full subcategory of Sh_{spec}. For any spaces X and Y, if $\Sigma^k X$ and $\Sigma^k Y$ are equivalent in Sh for some positive integer k, then X and Y are equivalent in Sh_{spec}. For any compact Hausdorff spaces X and Y with finite shape dimension ([13, II, §1]), the converse holds.

Spanier-Whitehead duality in the category ShStab was studied by Lima ([12]), Henn ([9]) and Nowak ([18], [19]). The following duality holds in the category Sh_{spec} ([14, Theorem 4.1]): given a compact metric space X, there exist a CW-spectrum X^* and a natural isomorphism $\tau : \mathsf{Sh}_{spec}(Y \wedge X, E) \to \mathsf{Sh}_{spec}(Y, X^* \wedge E)$ for every compact Hausdorff space Y and CW-spectrum E.

Generalized cohomology theories. For every CW-spectrum E, let E^* denote the generalized cohomology theory on HCW_{spec} associated with E, which is defined by

$$E^q(G) = [G, \Sigma^q E] \text{ for every CW-spectrum } G.$$

For every CW complex X, let $E^q(X) = E^q(E(X))$. We take the Čech extension of E^q over Sh_{spec}, where the Čech extension is based on all normal open coverings of X. Equivalently, for every space X, $E^q(X) = \operatorname{colim}_{\lambda} E^q(X_\lambda) = (E^q(X_\lambda), E^q(p_{\lambda\lambda'}), \Lambda)$, where the map $\boldsymbol{p} = ([p_\lambda]) : X \to \boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ is an HCW-expansion of X.

Let ΩX be the loop space, and let $\Omega^k X = \Omega(\Omega^{k-1} X)$ for $k \geq 1$, where $\Omega^0 X = X$. Let $\Omega^\infty E_{m+\infty} = \operatorname{colim}\{\Omega^k E_{m+k},\ \tilde{\varepsilon}_k :\ k \in \mathbb{Z}\}$, where $\tilde{\varepsilon}_k : E_k \to \Omega E_{k+1}$ is the adjoint map of ε_k.

If X is a compact Hausdorff space, then we can take $\boldsymbol{X}$ so that each X_λ is a finite CW complex. Then we have

$$E^q(X) = \operatorname{colim}_{\lambda} E^q(X_\lambda) \approx \operatorname{colim}_{\lambda} [X_\lambda, \Omega^\infty E_{q+\infty}] \approx [X, \Omega^\infty E_{q+\infty}]. \tag{2.1}$$

A CW spectrum E is an Ω-spectrum provided every $\tilde{\varepsilon}_n$ is a weak homotopy equivalence. If E is an Ω-spectrum, then there exists a weak homotopy equivalence $w_q : E_q \to \Omega^\infty E_{q+\infty}$, which induces an equivalence $[X, E_q] \to [X, \Omega^\infty E_{q+\infty}]$. Thus,

$$E^q(X) \approx [X, E_q]. \tag{2.2}$$

For every map $f : E \to F$ between CW spectra, there exists an induced map $Q_q^\infty(f) : \Omega^\infty E_{q+\infty} \to \Omega^\infty F_{q+\infty}$ between ANR's. Let $Q_q^\infty(E) = \Omega^\infty E_{q+\infty}$. Then Q_q^∞ defines a functor from SPEC to HCW. If E and F are Ω-spectra, then there exists a unique (up to homotopy) map $C_q(f) : E_q \to F_q$ which makes the following diagram commute:

$$\begin{array}{ccc} E_q & \xrightarrow{w_q} & \Omega^\infty E_{q+\infty} \\ {\scriptstyle C_q(f)}\downarrow & & \downarrow{\scriptstyle Q_q^\infty(f)} \\ F_q & \xrightarrow{w'_q} & \Omega^\infty F_{q+\infty} \end{array}$$

Here the horizontal maps are weak homotopy equivalences.

Various dimensions. Given a normal space X, one assigns the covering dimension (Čech-Lebesgue dimension) $\dim X$ which is an integer greater than or equal to -1 or ∞ by the following conditions:

(LD_1) $\dim X \leq n$, where $n = -1, 0, 1, \ldots$, if every finite open cover of X has a finite open refinement of order at most n.

(LD_2) $\dim X = n$ if $\dim X \leq n$ holds and $\dim X \leq n-1$ does not hold.

(LD_3) $\dim X = \infty$ if $\dim X \leq n$ does not hold for any $n = -1, 0, 1, \ldots$

Given a paracompact space X and an abelian group G, one assigns the cohomological dimension $\text{c-dim}_G X$, which is defined as the smallest integer n such that $\check{H}^m(X, A; G) = 0$ for every closed subset A of X and for every $m > n$. Let $\text{c-dim}_G X = \infty$ if there is no such n. It is known that $\text{c-dim}_G X \leq n$ if and only if for every closed subset A of X, the inclusion map $A \to X$ induces an epimorphism $\check{H}^m(X; G) \to \check{H}^m(A; G)$ for every $m \geq n$ (see [23]). Moreover, $\text{c-dim}_G X \leq n$ if and only if for every closed subset A of X and for every map $f : A \to K(G, n)$, there exits a continuous extension $\tilde{f} : X \to K(G, n)$ (see [10] and [24]).

Given a compact metric space X and a CW spectrum E, one assigns the generalized cohomological dimension $\text{c-dim}_E X$, which is defined as the smallest integer n such that for every closed subset A of X and for every $m \geq n$, the inclusion map $A \to X$ induces an epimorphism of $E^m(X) \to E^m(A)$. Let $\text{c-dim}_E X = \infty$ if there is no such n. Then, $\text{c-dim}_E X \leq n$ if and only if for every closed subset A of X, for every $m \geq n$ and for every map $f : A \to \Omega^\infty E_{m+\infty}$, there exits a continuous extension $\tilde{f} : X \to \Omega^\infty E_{m+\infty}$. Moreover, if E is an Ω-spectrum, then $\text{c-dim}_E X \leq n$ if and only if for every closed subset A of X and for every map $f : A \to E_n$, there exits a continuous extension $\tilde{f} : X \to E_n$.

3. Small generalized cohomological dimension

Every map $\varphi : E \to F$ between CW spectra induces a natural transformation $\varphi_q : E^q \to F^q$ between cohomology functors. If $\boldsymbol{E} = (E_\mu, \alpha_{\mu\mu'}, M)$ is an inverse system of CW spectra, for every space X we define $\boldsymbol{E}^m(X)$ as the inverse system $(E^q_\mu(X), (\alpha_{\mu\mu'})_q(X), M)$.

If $f : X \to Y$ is a map between spaces, then there is an induced morphism $\boldsymbol{E}^q(f) : \boldsymbol{E}^q(Y) \to \boldsymbol{E}^q(X)$ in pro-Ab. Here Ab denotes the category of abelian groups and homomorphisms. Indeed, for $\mu < \mu'$, there exists a commutative diagram

$$\begin{array}{ccc} E^q_\mu(X) & \xleftarrow{(\alpha_{\mu\mu'})_q(X)} & E^q_{\mu'}(X) \\ {\scriptstyle E^q_\mu(f)}\uparrow & & \uparrow{\scriptstyle E^q_{\mu'}(f)} \\ E^q_\mu(Y) & \xleftarrow{(\alpha_{\mu\mu'})_q(Y)} & E^q_{\mu'}(Y) \end{array}$$

and hence there is a level map $(E^q_\mu(f)) : \boldsymbol{E}^q(Y) \to \boldsymbol{E}^q(X)$, which defines a morphism $\boldsymbol{E}^q(f) : \boldsymbol{E}^q(Y) \to \boldsymbol{E}^q(X)$. More generally, every morphism $X \to Y$ in Sh_{spec} induces a morphism $\boldsymbol{E}^q(Y) \to \boldsymbol{E}^q(X)$.

Every morphism $\boldsymbol{\varphi} : \boldsymbol{E} \to \boldsymbol{F} = (F_\nu, \beta_{\nu\nu'}, N)$ in pro-SPEC induces a natural transformation $\boldsymbol{\varphi}_q : \boldsymbol{E}^q \to \boldsymbol{F}^q$. Indeed, if $(\varphi_\nu, \varphi) : \boldsymbol{E} \to \boldsymbol{F}$ is a system map which represents $\boldsymbol{\varphi}$, then for $\nu < \nu'$, there exists $\mu > \varphi(\nu), \varphi(\nu')$ such that

$$\varphi_\nu \circ \alpha_{\varphi(\nu)\mu} = \beta_{\nu\nu'} \circ \varphi_{\nu'} \circ \alpha_{\varphi(\nu')\mu}.$$

$$\begin{array}{ccccc} & & \overset{\alpha_{\varphi(\nu)\mu}}{\longleftarrow\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-} & & \\ E_{\varphi(\nu)} & & E_{\varphi(\nu')} & \xleftarrow[\alpha_{\varphi(\nu')\mu}]{} & E_\mu \\ \varphi_\nu \downarrow & & \downarrow \varphi_{\nu'} & & \\ F_\nu & \xleftarrow{\beta_{\nu\nu'}} & F_{\nu'} & & \end{array}$$

This implies the equality

$$(\varphi_\nu)_q(X) \circ (\alpha_{\varphi(\nu)\mu})_q(X) = (\beta_{\nu\nu'})_q(X) \circ (\varphi_{\nu'})_q(X) \circ (\alpha_{\varphi(\nu')\mu})_q(X).$$

This means that there exists a system map $((\varphi_\nu)_q(X), \varphi) : \boldsymbol{E}^q(X) \to \boldsymbol{F}^q(X)$, which represents a morphism $\boldsymbol{\varphi}_q(X) : \boldsymbol{E}^q(X) \to \boldsymbol{F}^q(X)$. Moreover, if $f : X \to Y$ is a map, then, for every ν, there exists the following commutative diagram:

$$\begin{array}{ccc} E^q_{\varphi(\nu)}(Y) & \xrightarrow{(\varphi_\nu)_q(Y)} & F^q_\nu(Y) \\ {\scriptstyle E^q_{\varphi(\nu)}(f)} \downarrow & & \downarrow {\scriptstyle F^q_\nu(f)} \\ E^q_{\varphi(\nu)}(X) & \xrightarrow{(\varphi_\nu)_q(X)} & F^q_\nu(X) \end{array}$$

Thus, the following diagram is commutative:

$$\begin{array}{ccc} \boldsymbol{E}^q(Y) & \xrightarrow{\boldsymbol{\varphi}_q(Y)} & \boldsymbol{F}^q(Y) \\ {\scriptstyle \boldsymbol{E}^q(f)} \downarrow & & \downarrow {\scriptstyle \boldsymbol{F}^q(f)} \\ \boldsymbol{E}^q(X) & \xrightarrow{\boldsymbol{\varphi}_q(X)} & \boldsymbol{F}^q(X) \end{array}$$

This shows that thus defined $\boldsymbol{\varphi}_q$ is a natural transformation. Moreover, every isomorphism $\boldsymbol{\varphi} : \boldsymbol{E} \to \boldsymbol{F}$ in pro-SPEC induces a natural equivalence $\boldsymbol{\varphi}_q : \boldsymbol{E}^q \to \boldsymbol{F}^q$.

Since homotopic maps $\varphi, \varphi' : E \to F$ between CW spectra induces the same natural transformation $\varphi_q = \varphi'_q : E^q \to F^q$, then every object $\boldsymbol{E} = (E_\mu, [\alpha_{\mu\mu'}], M)$ in pro-HSPEC induces a unique inverse system $\boldsymbol{E}^q(X) = (E^q_\mu(X), (\alpha_{\mu\mu'})_q(X), M)$ for every space X. Moreover, every morphism $\boldsymbol{\varphi} : \boldsymbol{E} \to \boldsymbol{F}$ in pro-HSPEC induces a natural transformation $\boldsymbol{\varphi}_q : \boldsymbol{E}^q \to \boldsymbol{F}^q$.

Let $\boldsymbol{E} = (E_\mu, \alpha_{\mu\mu'}, M)$ be an object in SPEC. For every space X, the *small generalized cohomological dimension* g-$\dim_{\boldsymbol{E}} X$ *with respect to* $\boldsymbol{E}$ is defined as the smallest integer n such that whenever A of X is a closed subset A of X, then the inclusion induced morphism $\boldsymbol{E}^m(X) \to \boldsymbol{E}^m(A)$ in pro-groups is an epimorphism for every $m \geq n$. Let g-$\dim_E X = \infty$ if there is no such n. Simlilarly, for every object $\boldsymbol{E}$ in HSPEC, the cohomological dimension with respect to $\boldsymbol{E}$ is well-defined.

Proposition 3.1. *Let* $\boldsymbol{E}$ *and* $\boldsymbol{F}$ *be inverse systems in* HSPEC.

1) *If* $\boldsymbol{E}$ *is dominated by* $\boldsymbol{F}$ *in* HSPEC, *then* $\text{g-dim}_{\boldsymbol{E}} X \leq \text{g-dim}_{\boldsymbol{F}} X$ *for every space* X.
2) *If* $\boldsymbol{E}$ *is isomorphic to* $\boldsymbol{F}$ *in* HSPEC, *then* $\text{g-dim}_{\boldsymbol{E}} X = \text{g-dim}_{\boldsymbol{F}} X$ *for every space* X.

Proof. It suffices to show the first assertion. Let $\boldsymbol{\varphi} : \boldsymbol{E} \to \boldsymbol{F}$ and $\boldsymbol{\psi} : \boldsymbol{F} \to \boldsymbol{E}$ be morphisms in pro-HSPEC such that $\boldsymbol{\psi} \circ \boldsymbol{\varphi} = 1_{\boldsymbol{E}}$. Suppose that $\text{g-dim}_{\boldsymbol{F}} X \leq n$. Let $m \geq n$, let A be a closed subset of X, and let $i : A \to X$ be the inclusion map. Then the following diagram commutes:

$$\begin{array}{ccccc}
 & & \overset{1_{\boldsymbol{E}^n(X)}}{\curvearrowright} & & \\
\boldsymbol{E}^m(X) & \xrightarrow[\boldsymbol{\varphi}_m(X)]{} & \boldsymbol{F}^m(X) & \xrightarrow[\boldsymbol{\psi}_m(X)]{} & \boldsymbol{E}^m(X) \\
\downarrow{\scriptstyle \boldsymbol{E}^m(i)} & & \downarrow{\scriptstyle \boldsymbol{F}^m(i)} & & \downarrow{\scriptstyle \boldsymbol{E}^m(i)} \\
\boldsymbol{E}^m(A) & \xrightarrow{\boldsymbol{\varphi}_m(A)} & \boldsymbol{F}^m(A) & \xrightarrow{\boldsymbol{\psi}_m(A)} & \boldsymbol{E}^m(A) \\
 & & \underset{1_{\boldsymbol{E}^n(A)}}{\smile} & &
\end{array}$$

To show that $\boldsymbol{E}^m(i)$ is an epimorphism, let $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 : \boldsymbol{E}^m(A) \to \boldsymbol{P}$ be morphisms in pro-groups such that $\boldsymbol{\gamma}_1 \circ \boldsymbol{E}^m(i) = \boldsymbol{\gamma}_2 \circ \boldsymbol{E}^m(i)$. Then

$$\begin{aligned}
\boldsymbol{\gamma}_1 \circ \boldsymbol{\psi}_m(A) \circ \boldsymbol{F}^m(i) &= \boldsymbol{\gamma}_1 \circ \boldsymbol{E}^m(i) \circ \boldsymbol{\psi}_m(X) = \boldsymbol{\gamma}_2 \circ \boldsymbol{E}^m(i) \circ \boldsymbol{\psi}_m(X) \\
&= \boldsymbol{\gamma}_2 \circ \boldsymbol{\psi}_m(A) \circ \boldsymbol{F}^m(i).
\end{aligned}$$

Since $\boldsymbol{F}^m(i)$ is an epimorphism,

$$\boldsymbol{\gamma}_1 \circ \boldsymbol{\psi}_m(A) = \boldsymbol{\gamma}_2 \circ \boldsymbol{\psi}_m(A).$$

Since $\boldsymbol{\psi}_m(A) \circ \boldsymbol{\varphi}_m(A) = 1_{\boldsymbol{E}^m(A)}$, then

$$\boldsymbol{\gamma}_1 = \boldsymbol{\gamma}_1 \circ \boldsymbol{\psi}_m(A) \circ \boldsymbol{\varphi}_m(A) = \boldsymbol{\gamma}_2 \circ \boldsymbol{\psi}_m(A) \circ \boldsymbol{\varphi}_m(A) = \boldsymbol{\gamma}_2.$$

This shows that $\boldsymbol{E}^m(i)$ is an epimorphism, and hence $\text{g-dim}_{\boldsymbol{F}} X \leq n$. □

If Z is a space, for every space X, the *small generalized cohomological dimension* $\text{g-dim}_Z X$ *with respect to* Z is defined as $\text{g-dim}_{\boldsymbol{E}} X$, where $\boldsymbol{\alpha} = ([\alpha_\lambda]) : E(\boldsymbol{Z}) \to \boldsymbol{E}$ is a generalized HSPEC-expansion of X. Note that the definition does not depend on the choice of genralized HSPEC-expansion of Z by Proposition 3.1.

Proposition 3.1 implies the following corollary.

Corollary 3.2. *Let* Z_1 *and* Z_2 *be spaces.*

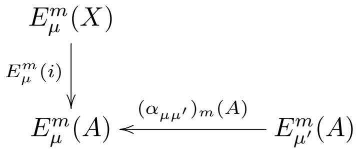
1) *If* Z_1 *is dominated by* Z_2 *in* Sh_{spec}, *then* $\text{g-dim}_{Z_1} X \leq \text{g-dim}_{Z_2} X$ *for every space* X.
2) *If* Z_1 *is isomorphic to* Z_2 *in* Sh_{spec}, *then* $\text{g-dim}_{Z_1} X = \text{g-dim}_{Z_2} X$ *for every space* X.

Since every CW spectrum is weak homotopy equivalent to an Ω-spectrum, small generalized cohomological dimension can be defined in terms of an inverse system consisting of Ω-spectra. However, we sometimes need to work on the inverse systems of CW spectra which are chosen in advance. Thus, we wish to have characterizations of small generalized cohomological dimension under a general setting.

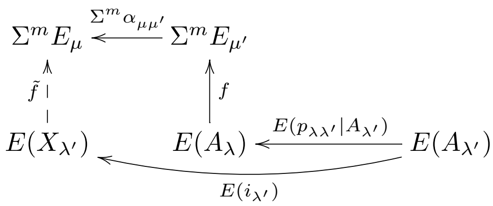
For every object $\boldsymbol{E} = (E_\mu, \alpha_{\mu\mu'}, M)$ in pro-SPEC or $\boldsymbol{E} = (E_\mu, [\alpha_{\mu\mu'}], M)$ in pro-HSPEC and for each nonnegative integer m, consider the following conditions:

$(\mathrm{S}_1)_m$ If A is a closed subset of X and if $i : A \to X$ is the inclusion map, then every $\mu \in M$ admits $\mu' > \mu$ such that

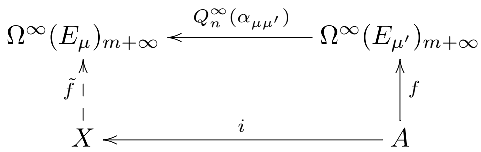
$$\mathrm{Im}((\alpha_{\mu\mu'})_m(A)) \subseteq \mathrm{Im}(E^m_\mu(i)).$$



$(\mathrm{S}_2)_m$ If A is a closed subset of X, and if $\boldsymbol{p} = ([p_\lambda]) : X \to \boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ is an HCW-expansion of X such that the induced morphism $\boldsymbol{p}|A = ([p_\lambda|A]) : A \to \boldsymbol{A} = (A_\lambda, [p_{\lambda\lambda'}|A_{\lambda'}], \Lambda)$ is an HCW-expansion of A, then every μ admits $\mu' > \mu$ such that for every $\lambda \in \Lambda$ and for every map $f : E(A_\lambda) \to \Sigma^m E_{\mu'}$, there exist $\lambda' > \lambda$ and a map $\tilde{f} : E(X_{\lambda'}) \to \Sigma^m E_\mu$ with $\tilde{f} \circ E(i_{\lambda'}) = \Sigma^m \alpha_{\mu\mu'} \circ f \circ E(p_{\lambda\lambda'}|A_{\lambda'})$, where $i_{\lambda'} : A_{\lambda'} \to X_{\lambda'}$ is the inclusion map.



$(\mathrm{S}_3)_m$ For every closed subset A of X, every μ admits $\mu' > \mu$ such that for every map $f : A \to \Omega^\infty(E_{\mu'})_{m+\infty}$, there exists a map $\tilde{f} : X \to \Omega^\infty(E_\mu)_{m+\infty}$ with $\tilde{f}|A = Q^\infty_n(\alpha_{\mu\mu'}) \circ f$.



$(\mathrm{S}_4)_m$ For every closed subset A of X, every μ admits $\mu' > \mu$ such that for every map $f : A \to (E_{\mu'})_m$, there exists a map $\tilde{f} : X \to (E_\mu)_m$ with $\tilde{f}|A = C_m(\alpha_{\mu\mu'}) \circ f$.

$$\begin{array}{ccc} (E_\mu)_m & \xleftarrow{C_m(\alpha_{\mu\mu'})} & (E_{\mu'})_m \\ {\scriptstyle \tilde{f}} \uparrow & & \uparrow {\scriptstyle f} \\ X & \xleftarrow{\ i\ } & A \end{array}$$

THEOREM 3.3. *Let* $\boldsymbol{E}$ *be an object* $(E_\mu, \alpha_{\mu\mu'}, M)$ *in* pro-SPEC *or* $(E_\mu, [\alpha_{\mu\mu'}], M)$ *in* pro-HSPEC.

1) g-$\dim_{\boldsymbol{E}} X \le n$ *iff* $(\mathrm{S}_1)_m$ *holds for* $m \ge n$.
2) g-$\dim_{\boldsymbol{E}} X \le n$ *iff* $(\mathrm{S}_2)_m$ *holds for* $m \ge n$.
3) *If* X *is a compact Hausdorff space, then* g-$\dim_{\boldsymbol{E}} X \le n$ *iff* $(\mathrm{S}_3)_m$ *holds for* $m \ge n$.
4) *If* X *is a compact Hausdorff space and if each* E_μ *is an* Ω*-spectrum, then* g-$\dim_{\boldsymbol{E}} X \le n$ *iff* $(\mathrm{S}_4)_m$ *holds for* $m \ge n$.

PROOF. The first assertion follows from the definition of g-$\dim_{\boldsymbol{E}} X$ and [13, Theorem 3, p. 109]. To show the second assertion, first, suppose that g-$\dim_{\boldsymbol{E}} X \le n$. Let $m \ge n$, let $\lambda \in \Lambda$, and let $f : E(A_\lambda) \to \Sigma^m E_{\mu'}$ be a map. Since f represents an element of $E^m_{\mu'}(A)$, then, by $(\mathrm{S}_1)_m$, there exists a map $\tilde{f}' : E(X_{\lambda'}) \to \Sigma^m E_\mu$ for some $\lambda' > \lambda$ which represents an element of $E^m_\mu(X)$ and safisfies

$$\tilde{f}' \circ E(i_{\lambda'}) \simeq (\alpha_{\mu\mu'})_m \circ f \circ E(p_{\lambda\lambda'}|A_{\lambda'}).$$

By the homotopy extension therem for CW spectra ([25, 8.20]) there exists a map $\tilde{f} : E(X_{\lambda'}) \to \Sigma^m E_\mu$ such that

$$\tilde{f} \circ E(i_{\lambda'}) = (\alpha_{\mu\mu'})_m \circ f \circ E(p_{\lambda\lambda'}|A_{\lambda'}),$$

as required. Reversing the argument, we can show the converse.

Similarly, we can also show the third and the fourth assertions, using the natural equivalences (2.1) and (2.2) (see Section 2). ∎

Theorem 3.3 immediately implies

COROLLARY 3.4. *Let* Z *be a space, and let* $\boldsymbol{\alpha} = ([\alpha_\mu]) : E(\boldsymbol{Z}) \to \boldsymbol{E} = (E_\mu, [\alpha_{\mu\mu'}], M)$ *be a generalized* HSPEC*-expansion of* Z*. Then we have the following:*

1) g-$\dim_Z X \le n$ *iff* $(\mathrm{S}_1)_m$ *holds for* $m \ge n$.
2) g-$\dim_Z X \le n$ *iff* $(\mathrm{S}_2)_m$ *holds for* $m \ge n$.
3) *If* X *is a compact Hausdorff space, then* g-$\dim_Z X \le n$ *iff* $(\mathrm{S}_3)_m$ *holds for* $m \ge n$.
4) *If* X *is a compact Hausdorff space and if each* E_μ *is an* Ω*-spectrum, then* g-$\dim_Z X \le n$ *iff* $(\mathrm{S}_4)_m$ *holds for* $m \ge n$.

4. Large generalized cohomological dimension

In condition $(\mathrm{S}_1)_m$, note that the index μ' depends on the choice of the closed subset A of X. In this section, we consider a stronger version of the condition so that μ' does not depend on A. This stronger condition gives another version of generalized cohomological dimension, which is shown to possess better properties. We call it large generalized cohomological dimension. Consider the following condition:

$(\mathrm{L}_1)_m$ For each $\mu \in M$, there exists $\mu' > \mu$ such that for every closed subset A of X,

$$\mathrm{Im}((\alpha_{\mu\mu'})_m(A)) \subseteq \mathrm{Im}(E^m_\mu(i)),$$

where $i : A \to X$ is the inclusion map.

Let $\boldsymbol{E} = (E_\mu, \alpha_{\mu\mu'}, M)$ be an object in SPEC. For every space X, the *large generalized cohomological dimension* $\text{G-dim}_{\boldsymbol{E}} X$ *with respect to* $\boldsymbol{E}$ is defined as the smallest integer n such that $(\mathrm{L}_1)_m$ holds for $m \geq n$. Let $\text{G-dim}_E X = \infty$ if there is no such n. Simlilarly, for every object $\boldsymbol{E}$ in pro-HSPEC, the large generalized cohomological dimension with respect to $\boldsymbol{E}$ is well-defined. Note that for every CW spectrum E, if (E) denotes the rudimentary system, then $\text{g-dim}_{(E)} X = \text{G-dim}_{(E)} X$ for every space X, and it equals $\text{c-dim}_E X$ for every compact metric space X.

Consider the following conditions which are analogous to $(\mathrm{S}_2)_m$, $(\mathrm{S}_3)_m$ and $(\mathrm{S}_4)_m$:

$(\mathrm{L}_2)_m$ every μ admits $\mu' > \mu$ such that whenever A is a closed subset of X and $\boldsymbol{p} = ([p_\lambda]) : X \to \boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ is an HCW-expansion of X with the induced morphism $\boldsymbol{p}|A = ([p_\lambda|A]) : A \to \boldsymbol{A} = (A_\lambda, [p_{\lambda\lambda'}|A_{\lambda'}], \Lambda)$ being an HCW-expansion of A, then for every $\lambda \in \Lambda$ and for every map $f : E(A_\lambda) \to \Sigma^m E_{\mu'}$, there exist $\lambda' > \lambda$ and a map $\tilde{f} : E(X_{\lambda'}) \to \Sigma^m E_\mu$ with $\tilde{f} \circ E(i_{\lambda'}) = \Sigma^m \alpha_{\mu\mu'} \circ f \circ E(p_{\lambda\lambda'}|A_{\lambda'})$, where $i_{\lambda'} : A_{\lambda'} \to X_{\lambda'}$ is the inclusion map.

$(\mathrm{L}_3)_m$ every μ admits $\mu' > \mu$ such that for every closed subset A of X and for every map $f : A \to \Omega^\infty(E_{\mu'})_{m+\infty}$, there exists a map $\tilde{f} : X \to \Omega^\infty(E_\mu)_{m+\infty}$ with $\tilde{f}|A = Q^\infty_m(\alpha_{\mu\mu'}) \circ f$.

$(\mathrm{L}_4)_m$ every μ admits $\mu' > \mu$ such that for every closed subset A of X and for every map $f : A \to (E_{\mu'})_m$, there exists a map $\tilde{f} : X \to (E_\mu)_m$ with $\tilde{f}|A = C_m(\alpha_{\mu\mu'}) \circ f$.

By $(\mathrm{L}_1)_m$, (2.1) and (2.2) (see Section 2), we have

Theorem 4.1. *Let* $\boldsymbol{E}$ *be an object* $(E_\mu, \alpha_{\mu\mu'}, M)$ *in* pro-SPEC *or an object* $(E_\mu, [\alpha_{\mu\mu'}], M)$ *in* pro-HSPEC.

1) $\text{G-dim}_{\boldsymbol{E}} X \leq n$ *iff* $(\mathrm{L}_2)_m$ *holds for* $m \geq n$.
2) *If* X *is a compact Hausdorff space, then* $\text{G-dim}_{\boldsymbol{E}} X \leq n$ *iff* $(\mathrm{L}_3)_m$ *holds for* $m \geq n$.

3) *If X is a compact Hausdorff space and if each E_μ is an Ω-spectrum, then* G-$\dim_{\boldsymbol{E}} X \leq n$ *iff* $(\mathrm{L}_4)_m$ *holds for* $m \geq n$.

An object $(E_\mu, \alpha_{\mu\mu'}, M)$ in pro-SPEC or an object $(E_\mu, [\alpha_{\mu\mu'}], M)$ in pro-HSPEC is said to be finite dimensional if there exists a nonnegative integer N such that $\dim E_\mu \leq N$ for every $\mu \in M$.

THEOREM 4.2. *Let $\boldsymbol{E} = (E_\mu, [\alpha_{\mu\mu'}], M)$ be an object in* pro-SPEC *such that every E_μ is an Ω-spectrum and $\boldsymbol{E}$ is finite dimensional. For every compact Hausdorff space X and for every nonnegative integer m, the implication $(\mathrm{L}_4)_m \Rightarrow (\mathrm{L}_4)_{m+1}$ holds.*

PROOF. We follow the idea by S. Ferry for proving the implication c-$\dim_{\mathbb{Z}} X \leq n \Rightarrow$ c-$\dim_{\mathbb{Z}} X \leq n+1$ (see J. Walsh [26, Appendix A]).

Suppose that condition $(\mathrm{L}_4)_m$ holds, and let $\mu \in M$. Without loss of generality, we can assume that each CW spectrum E_μ consists of locally finite simplicial complexes $(E_\mu)_k$, $k \in \mathbb{Z}$, and that every induced map $C_k(\alpha_{\mu\mu'})$ is a simplicial map. For each $\mu \in M$, let $P((E_\mu)_{m+1}, e^\mu_{0,m+1})$ be the path space consisting of all paths $\omega : I = [0,1] \to (E_\mu)_{m+1}$ with $\omega(0) = e^\mu_{0,m+1}$, a base point of $(E_\mu)_{m+1}$, and let $((E_\mu)_{m+1}, e^\mu_{0,m+1})$ possess the compact-open topology. The map $\varphi^\mu_{m+1} : P((E_\mu)_{m+1}, e^\mu_{0,m+1}) \to (E_\mu)_{m+1}$ defined by $\varphi^\mu_{m+1}(\omega) = \omega(1)$ is a Hurewicz fibration with fibres being $(E_\mu)_m$ since $(\varphi^\mu_{m+1})^{-1}(e^\mu_{0,m+1}) = \Omega(E_\mu)_{m+1}$ is weak homotopy equivalent and hence homotopy equivalent to $(E_\mu)_m$.

To verify $(\mathrm{L}_4)_{m+1}$, let $\mu \in M$. Since $\boldsymbol{E}$ is finite dimensional, there exists a nonnegative integer d such that $\dim E_\mu \leq d$ for $\mu \in M$. Let $n = d + m$, and choose $\mu_0 = \mu < \mu_1 < \cdots < \mu_n = \mu'$ in M such that for every closed subset B of X, every map $f : B \to (E_{\mu_{i+1}})_m$ admits a map $\tilde{f} : X \to (E_{\mu_i})_m$ such that $\tilde{f}|B = C_m(\alpha_{\mu_i\mu_{i+1}}) \circ f$. Let A be a closed subset of a space X, and let $f : A \to (E_{\mu'})_{m+1}$ be a map. We wish to show that there exists a map $\tilde{f} : X \to (E_\mu)_{m+1}$ such that $\tilde{f}|A = C_{m+1}(\alpha_{\mu\mu'}) \circ f$. For that, it suffices to construct a map $g : A \to P((E_\mu)_{m+1}, e^\mu_{0,m+1})$ such that $\varphi^\mu_{m+1} \circ g = C_{m+1}(\alpha_{\mu\mu'}) \circ f$. For, since $P((E_\mu)_{m+1}, e^\mu_{0,m+1})$ is contractible, then there will exist an extension $\tilde{g} : X \to P((E_\mu)_{m+1}, e^\mu_{0,m+1})$ of g, and hence $\tilde{f} = \varphi^\mu_{m+1} \circ \tilde{g} : X \to (E_\mu)_{m+1}$ satisfies $\tilde{f}|A = C_{m+1}(\alpha_{\mu\mu'}) \circ f$.

Let $l = \dim(E_{\mu'})_{m+1}$ $(\leq d+m)$. For $q = 0, 1, \ldots, l$, let

$$M_q = \{f^{-1}(\sigma) : \ \sigma \text{ is an } q\text{-simplex of } (E_{\mu'})_{m+1}\}.$$

We inductively define maps $g_i : \bigcup_{q \leq i} M_q \to P((E_{\mu_{n-i}})_{m+1}, e^{\mu_{n-i}}_{0,m+1})$ for $i = 0, \ldots, l$ such that

$$\varphi^{\mu_i}_{m+1} \circ g_i = C_{m+1}(\alpha_{\mu_i\mu_n}) \circ f| \bigcup_{q \leq i} M_q.$$

For each 0-simplex σ of $(E_{\mu'})_{m+1}$, choose a point $p_\sigma \in (\varphi^{\mu'}_{m+1})^{-1}(\sigma)$. Then we define a map $g_0 : M_0 \to P((E_{\mu'})_{m+1}, e^{\mu'}_{0,m+1})$ by $g_0|f^{-1}(\sigma) = p_\sigma$.

Suppose that we have defined a map $g_i : \bigcup_{q\leq i} M_q \to P((E_{\mu_{n-i}})_{m+1}, e^{\mu_{n-i}}_{0,m+1})$ for $i < l$ such that

$$\varphi^{\mu_i}_{m+1} \circ g_i = C_{m+1}(\alpha_{\mu_i \mu'}) \circ f| \bigcup_{q\leq i} M_q.$$

We define a map $g_{i+1} : \bigcup_{q\leq i+1} M_q \to P((E_{\mu_{n-i-1}})_{m+1}, e^{\mu_{n-i-1}}_{0,m+1})$ as follows. Let σ be an $(i+1)$-simplex of $(E_{\mu'})_{m+1}$. Since $C_{m+1}(\alpha_{\mu_{n-i},\mu'})$ is a simplicial map, it follows that $C_{m+1}(\alpha_{\mu_{n-i},\mu'})(\sigma)$ is a simplex of $(E_{\mu_{n-1}})_{m+1}$. So, $(\varphi^{\mu_{n-i}}_{m+1})^{-1}(C_{m+1}(\alpha_{\mu_{n-i},\mu'}))(\sigma)$ is homotopy equivalent to $(E_{\mu_{n-i}})_m$, and there exists the following commutative diagram:

$$\begin{array}{ccc}
(E_{\mu_{n-i-1}})_m & \xleftarrow{C_m(\alpha_{\mu_{n-i-1}\mu_{n-i}})} & (E_{\mu_{n-i}})_m \\
\simeq \downarrow & & \downarrow \simeq \\
(\varphi^{\mu_{n-i-1}}_{m+1})^{-1}(\alpha_{\mu_{n-i-1},\mu'})_{m+1}(\sigma) & \longleftarrow & (\varphi^{\mu_{n-i}}_{m+1})^{-1}(\alpha_{\mu_{n-i},\mu'})_{m+1}(\sigma) \\
\subseteq \downarrow & & \downarrow \subseteq \\
P((E_{\mu_{n-i-1}})_{m+1}, e^{\mu_{n-i-1}}_{0,m+1}) & \xleftarrow{(C_{m+1}(\alpha_{\mu_{n-i-1}\mu_{n-i}}))_*} & P((E_{\mu_{n-i}})_{m+1}, e^{\mu_{n-i}}_{0,m+1}) \\
\varphi^{\mu_{n-i-1}}_{m+1} \downarrow & & \downarrow \varphi^{\mu_{n-i}}_{m+1} \\
(E_{\mu_{n-i-1}})_{m+1} & \xleftarrow{C_{m+1}(\alpha_{\mu_{n-i-1}\mu_{n-i}})} & (E_{\mu_{n-i}})_{m+1}
\end{array}$$

So, the map

$$g_i|(\bigcup_{q\leq i} M_q) \cap f^{-1}(\sigma) : (\bigcup_{q\leq i} M_q) \cap f^{-1}(\sigma) \to (\varphi^{\mu_{n-i}}_{m+1})^{-1}((\alpha_{\mu_{n-i},\mu'})_{m+1}(\sigma))$$

admits a map

$$g_{i+1}|f^{-1}(\sigma) : f^{-1}(\sigma) \to (\varphi^{\mu_{n-i-1}}_{m+1})^{-1}(C_{m+1}(\alpha_{\mu_{n-i-1},\mu'})(\sigma))$$

such that

$$g_{i+1}|(\bigcup_{q\leq i} M_q) \cap f^{-1}(\sigma) = (C_{m+1}(\alpha_{\mu_{n-i-1}\mu_{n-i}}))_* \circ g_i|(\bigcup_{q\leq i} M_q) \cap f^{-1}(\sigma).$$

Then,

$$\varphi^{\mu_{n-i-1}}_{m+1} \circ g_{i+1}(f^{-1}(\sigma)) \subseteq C_{m+1}(\alpha_{\mu_{n-i-1}\mu'})(\sigma).$$

Since $C_{m+1}(\alpha_{\mu_{n-i-1}\mu'})$ is a simplicial map, maps $g_{i+1}|f^{-1}(\sigma)$ induce a map $g'_{i+1} : \bigcup_{q\leq i+1} M_q \to P((E_{\mu_{n-i-1}})_{m+1}, e^{\mu_{n-i-1}}_{0,m+1})$ such that

$$\varphi^{\mu_{n-i-1}}_{m+1} \circ g'_{i+1} \simeq C_{m+1}(\alpha_{\mu_{n-i-1}\mu'}) \circ f|(\bigcup_{q\leq i+1} M_q).$$

Hence, we obtain a map $g_{i+1} : \underset{q\leq i+1}{\cup} M_q \to P((E_{\mu_{n-i-1}})_{m+1}, e^{\mu_{n-i-1}}_{0,m+1})$ such that

$$\varphi^{\mu_{n-i-1}}_{m+1} \circ g_{i+1} = C_{m+1}(\alpha_{\mu_{n-i-1}\mu'}) \circ f|(\underset{q\leq i+1}{\cup} M_q), \text{ and}$$
$$g_{i+1} \simeq g'_{i+1}.$$

The map $g = (C_{m+1}(\alpha_{\mu\mu_{n-l}})) \circ g_l : A = \underset{q\leq l}{\cup} M_q \to P((E_\mu)_{m+1}, e^{\mu}_{0,m+1})$ satisfies $\varphi^{\mu}_{m+1} \circ g = C_{m+1}(\alpha_{\mu\mu'}) \circ f$ as required. □

THEOREM 4.3. *Let $\boldsymbol{E} = (E_\mu, [\alpha_{\mu\mu'}], M)$ be an object in* pro-SPEC *such that every E_μ is an Ω-spectrum and $\boldsymbol{E}$ is finite dimensional, and let X be a compact Hausdorff space. Then the following are equivalent.*

1) G-$\dim_{\boldsymbol{E}} X \leq n$,
2) $(\mathrm{L}_4)_n$ *holds, and*
3) $\boldsymbol{E}^{n+1}(X \cup CA) = 0$ *for every closed subset A of X.*

PROOF. 1) ⇔ 2) follows from Theorems 4.1 3) and 4.2. 3) ⇒ 2) follows from the following long exact sequence in pro-Ab:

$$\cdots \to \boldsymbol{E}^n(X \cup CA) \to \boldsymbol{E}^n(X) \to \boldsymbol{E}^n(A) \to \boldsymbol{E}^{n+1}(X \cup CA) \to \cdots$$

It remains to show that 2) ⇒ 3)). Let A be a closed subset of X, and let d be an integer such that $\dim E_\mu \leq d$ for $\mu \in M$. Following the process in the proof of Theorem 4.2, for $\mu \in M$, choose $\mu_0 = \mu < \mu_1 < \cdots < \mu_{d+n} = \mu'$ in M such that for every closed subset B of X, every map $f : B \to (E_{\mu_{i+1}})_n$ admits a map $\tilde{f} : X \to (E_{\mu_i})_n$ such that $\tilde{f}|A = C_n(\alpha_{\mu_i\mu_{i+1}}) \circ f$. Suppose that $f : X \cup CA \to (E_{\mu'})_{n+1}$ is a map. This defines a map $f' : X \to (E_{\mu'})_{n+1}$ such that $f|A \simeq *$. As in the proof of Theorem 4.2, there exists a map $\tilde{f}' : X \to (E_\mu)_n$ such that $\varphi^{\mu}_{n+1} \circ \tilde{f}' = C_{n+1}(\alpha_{\mu\mu'}) \circ f'$. Since $\varphi^{\mu}_{n+1} \circ \tilde{f}' \simeq *$, then $C_{n+1}(\alpha_{\mu\mu'}) \circ f \simeq *$. This shows that $\boldsymbol{E}^{n+1}(X \cup CA) = 0$. □

PROPOSITION 4.4. *Let $\boldsymbol{E}$ and $\boldsymbol{F}$ be objects in* pro-HSPEC.

1) *If $\boldsymbol{E}$ is dominated by $\boldsymbol{F}$ in* pro-HSPEC*, then* G-$\dim_{\boldsymbol{E}} X \leq$ G-$\dim_{\boldsymbol{F}} X$ *for every space X.*
2) *If $\boldsymbol{E}$ is isomorphic to $\boldsymbol{F}$ in* pro-HSPEC*, then* G-$\dim_{\boldsymbol{E}} X =$ G-$\dim_{\boldsymbol{F}} X$ *for every space X.*

PROOF. It suffices to show the first assertion. Suppose that $\boldsymbol{E} = (E_\mu, [\alpha_{\mu\mu'}], M)$ is dominated by $\boldsymbol{F} = (F_\nu, [\beta_{\mu\mu'}], M)$ in pro-HSPEC. Then there exist system maps $([f_\nu], f) : \boldsymbol{E} \to \boldsymbol{F}$ and $([g_\mu], g) : \boldsymbol{F} \to \boldsymbol{E}$ which respectively induce morphisms F and G in pro-HSPEC such that $G \circ F = 1_{\boldsymbol{E}}$. Let $\mu \in M$. Then there exists $\mu' > \mu$ such that

$$g_\mu \circ f_{g(\mu)} \circ \alpha_{fg(\mu)\mu'} \simeq \alpha_{\mu\mu'}.$$

Suppose that G-$\dim_{\boldsymbol{F}} X \leq n$, and let $m \geq n$. Then there exists $\nu > g(\mu)$ such that

$$\mathrm{Im}((\beta_{g(\mu)\nu})_m(A)) \subseteq \mathrm{Im}(E^m_{g(\mu)}(i))$$

for every closed subset A of X, where $i : A \to X$ is the inclusion map. For the μ' and $f(\nu)$ in M, there exists $\mu'' > \mu', f(\nu)$ such that

$$f_{g(\mu)} \circ \alpha_{fg(\mu)\mu''} \simeq \beta_{g(\mu)\nu} \circ f_\nu \circ \alpha_{f(\nu)\mu''}.$$

To show $\text{G-dim}_{\boldsymbol{E}} X \leq n$, we verify the condition in $(\mathrm{L}_2)_m$ with μ' replaced by μ''. For that, let A is a closed subset of X, let $\boldsymbol{p} = ([p_\lambda]) : X \to \boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ be an HCW-expansion of X with the induced morphism $\boldsymbol{p}|A = ([p_\lambda|A]) : A \to \boldsymbol{A} = (A_\lambda, [p_{\lambda\lambda'}|A_{\lambda'}], \Lambda)$ being an HCW-expansion of A, and let $f : E(A_\lambda) \to \Sigma^m E_{\mu''}$ be a map, where $\lambda \in \Lambda$. Then there exists a map $\tilde{f}' : E_{\lambda'} \to \Sigma^m F_{g(\mu)}$ for some $\lambda' > \lambda$ such that

$$\tilde{f}' \circ E(i_{\lambda'}) = \Sigma^m \beta_{g(\mu)\nu} \circ \Sigma^m f_\nu \circ \Sigma^m \alpha_{f(\nu)\mu''} \circ f \circ E(p_{\lambda\lambda'}|A).$$

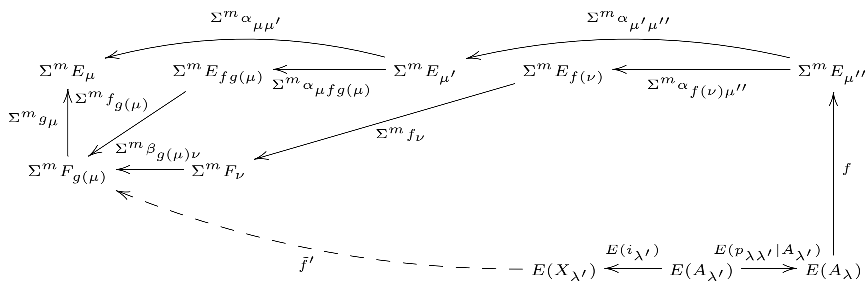
Here $i_{\lambda'} : A_{\lambda'} \to X_{\lambda'}$ is the inclusion map. So,

$$\Sigma^m g_\mu \circ \tilde{f}' \circ E(i_{\lambda'}) \simeq \Sigma^m \alpha_{\mu\mu''} \circ f \circ E(p_{\lambda\lambda'}|A).$$

By the homotopy extension theorem for CW spectra, there exists a map $\tilde{f} : E(X_{\lambda'}) \to \Sigma^m E_\mu$ such that

$$\tilde{f} \circ E(i_{\lambda'}) = \Sigma^m \alpha_{\mu\mu''} \circ f \circ E(p_{\lambda\lambda'}|A).$$

This shows that $\text{G-dim}_{\boldsymbol{E}} X \leq n$.



□

If Z is a space, for every space X, the *large generalized cohomological dimension* $\text{g-dim}_Z X$ *with respect to* Z is defined as $\text{G-dim}_{\boldsymbol{E}} X$, where $\boldsymbol{\alpha} = ([\alpha_\lambda]) : E(\boldsymbol{Z}) \to \boldsymbol{E}$ is a generalized HSPEC-expansion of X. Note that the definition does not depend on the choice of the genralized HSPEC-expansion of X by Proposition 4.4.

Proposition 4.4 immediately implies

COROLLARY 4.5. *Let Z_1 and Z_2 be spaces.*

1) *If Z_1 is dominated by Z_2 in Sh_{spec}, then* $\text{G-dim}_{Z_1} X \leq \text{G-dim}_{Z_2} X$ *for every space X.*
2) *If Z_1 is isomorphic to Z_2 in Sh_{spec}, then* $\text{G-dim}_{Z_1} X = \text{G-dim}_{Z_2} X$ *for every space X.*

The following is obvious from the definition.

PROPOSITION 4.6.

1) *If* $\boldsymbol{E}$ *is an object in* pro-HSPEC, *then* $\text{g-dim}_{\boldsymbol{E}} X \leq \text{G-dim}_{\boldsymbol{E}} X$ *for every space* X.
2) *If* Z *is a space, then* $\text{g-dim}_Z X \leq \text{G-dim}_Z X$ *for every space* X.

Theorem 4.3 immediately implies

COROLLARY 4.7. *Let* Z *be a space. Then the following are equivalent.*

1) $\text{G-dim}_Z X \leq n$,
2) $(\mathrm{L}_4)_n$ *holds for any generalized* HSPEC-*expansion* $\boldsymbol{\alpha} : E(\boldsymbol{Z}) \to \boldsymbol{E}$ *of* Z, *and*
3) $\boldsymbol{E}^{n+1}(X \cup CA) = 0$ *for every closed subset* A *of* X *and for every (some) generalized* HSPEC-*expansion* $\boldsymbol{\alpha} : E(\boldsymbol{Z}) \to \boldsymbol{E}$ *of* Z.

5. Properties of generalized cohomological dimensions

Let $\boldsymbol{E} = (E_\mu, \alpha_{\mu\mu'}, M)$ and $\boldsymbol{F} = (F_\mu, \beta_{\mu\mu'}, M)$ be objects in pro-SPEC. Let $(f_\mu) : \boldsymbol{E} \to \boldsymbol{F}$ be a level mapping. Levelwise special cofibre sequences $E_\mu \overset{f_\mu}{\to} F_\mu \overset{j_\mu}{\to} F_\mu \cup_{f_\mu} CE_\mu$ induce the sequence $\boldsymbol{E} \overset{(f_\mu)}{\to} \boldsymbol{F} \overset{(j_\mu)}{\to} \boldsymbol{D} = (F_\mu \cup_{f_\mu} CE_\mu, \gamma_{\mu\mu'}, M)$, where $\gamma_{\mu\mu'} : F_{\mu'} \cup_{f_{\mu'}} CE_{\mu'} \to F_\mu \cup_{f_\mu} CE_\mu$ is the map determined by $\alpha_{\mu\mu'}$ and $\beta_{\mu\mu'}$. The sequence $\boldsymbol{G} \to \boldsymbol{H} \to \boldsymbol{K}$ in pro-SPEC is called a *cofibre sequence* if there is the following commutative diagram in pro-HSPEC:

$$\begin{array}{ccccc} \boldsymbol{G} & \longrightarrow & \boldsymbol{H} & \longrightarrow & \boldsymbol{K} \\ \downarrow & & \downarrow & & \downarrow \\ \boldsymbol{E} & \longrightarrow & \boldsymbol{F} & \longrightarrow & \boldsymbol{D} \end{array}$$

Here the bottom row is the sequence represented by some levelwise special cofibre sequences, and the vertical arrows are isomorphisms in pro-HSPEC.

THEOREM 5.1. *Let* $\boldsymbol{G} \to \boldsymbol{H} \to \boldsymbol{K}$ *be a cofibre sequence in* pro-HSPEC *such that* $\boldsymbol{G}$, $\boldsymbol{H}$, *and* $\boldsymbol{K}$ *are finite dimensional. Then, for every compact Hausdorff space* X, *the following inequalities hold:*

1) $\text{G-dim}_{\boldsymbol{H}} X \leq \max\{\text{G-dim}_{\boldsymbol{G}} X, \text{G-dim}_{\boldsymbol{K}} X\}$,
2) $\text{G-dim}_{\boldsymbol{G}} X \leq \max\{\text{G-dim}_{\boldsymbol{H}} X, \text{G-dim}_{\boldsymbol{K}} X + 1\}$,
3) $\text{G-dim}_{\boldsymbol{K}} X \leq \max\{\text{G-dim}_{\boldsymbol{G}} X - 1, \text{G-dim}_{\boldsymbol{H}} X\}$.

PROOF. For every closed subset A of X, there is an exact sequence in pro-Ab:

$$\boldsymbol{G}^n(X \cup CA) \to \boldsymbol{H}^n(X \cup CA) \to \boldsymbol{K}^n(X \cup CA)$$
$$\to \boldsymbol{G}^{n+1}(X \cup CA) \to \boldsymbol{H}^{n+1}(X \cup CA) \to \boldsymbol{K}^{n+1}(X \cup CA).$$

The inequalities in 1), 2) and 3) follow from the exactness of this sequence and Theorem 4.3. □

Proposition 5.2. *Let Z be a space.*

1) *If* G-dim$_Z X \geq k$, *then* G-dim$_{\Sigma^k Z} X =$ G-dim$_Z X - k$ *for every space* X.
2) *If* g-dim$_Z X \geq k$, *then* g-dim$_{\Sigma^k Z} X =$ g-dim$_Z X - k$ *for every space* X.

Proof. The two equalities follow from the fact that if $\boldsymbol{\alpha} = ([\alpha_\mu]) : E(\boldsymbol{Z}) \to \boldsymbol{E} = (E_\mu, [\alpha_{\mu\mu'}], M)$ is a generalized HSPEC-expansion of Z, then $\Sigma^k \boldsymbol{\alpha} = ([\Sigma^k \alpha_\mu]) : E(\Sigma^k \boldsymbol{Z}) \to \Sigma^k \boldsymbol{E} = (\Sigma^k E_\mu, [\Sigma^k \alpha_{\mu\mu'}], M)$ is a generalized HSPEC-expansion of $\Sigma^k Z$, where $\boldsymbol{r} : Z \to \boldsymbol{Z}$ is an HCW-expansion of Z. Here note $(\Sigma^k E_\mu)_{m-k} = (E_\mu)_m$ for $m \in \mathbb{Z}$ and $k \in \mathbb{Z}$. □

Proposition 5.3. *Let Z be a space, X a compact Hausdorff space, and Y a compact metric space. Then we have*

1) g-dim$_{Y^* \wedge Z} X \leq$ g-dim$_Z X \wedge Y$,
2) G-dim$_{Y^* \wedge Z} X \leq$ G-dim$_Z X \wedge Y$.

Here Y^* is the CW spectrum dual to Y in Sh_{spec}.

Proof. Let A be a closed subset of X. Let $\boldsymbol{\alpha} = ([\alpha_\mu]) : E(\boldsymbol{Z}) \to \boldsymbol{E} = (E_\mu, [\alpha_{\mu\mu'}], M)$ is a generalized HSPEC-expansion of Z, and let $\boldsymbol{p} = ([p_\lambda]) : X \to \boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ be an HSPEC-expansion of X such that $\boldsymbol{p}|A = ([p_\lambda|A]) : A \to \boldsymbol{A} = (A_\lambda, [p_{\lambda\lambda'}|A_{\lambda'}], \Lambda)$ is an HSPEC-expansion of A. Also let $\boldsymbol{q} = ([q_i]) : Y \to \boldsymbol{Y} = (Y_i, [q_{i,i+1}])$ be an HSPEC-expansion of Y.

For each positive integer i and $\mu \in M$, there exist a finite CW spectrum Y_i^* and the following commutative diagram for each $\lambda \in \Lambda$:

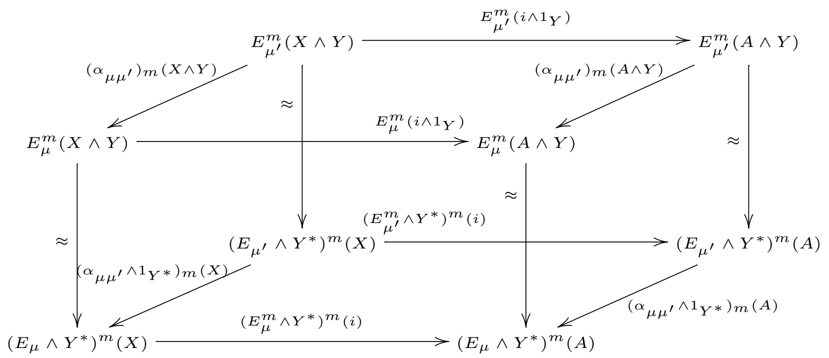
$$\begin{array}{ccccccc} E_\mu^m(X_\lambda \wedge Y_i) & = & [E(X_\lambda \wedge Y_i), \Sigma^m E_\mu] & \longrightarrow & [E(A_\lambda \wedge Y_i), \Sigma^m E_\mu] & = & E_\mu^m(A_\lambda \wedge Y_i) \\ & & \downarrow \approx & & \downarrow \approx & & \\ & & [E(X_\lambda), \Sigma^m E_\mu \wedge Y_i^*] & \longrightarrow & [E(A_\lambda), \Sigma^m E_\mu \wedge Y_i^*] & & \end{array}$$

The horizontal maps are induced by the inclusion map $i_\lambda : A_\lambda \to X_\lambda$, and the vertical maps are isomorphisms. All those maps are natural with repect to maps $p_{\lambda\lambda'} \wedge q_{\lambda\lambda'} : X_{\lambda'} \wedge Y_{\lambda'} \to X_\lambda \wedge Y_\lambda$ and $\alpha_{\mu\mu'} : E_{\mu'} \to E_\mu$. For each $\mu \in M$, there exists a CW spectrum Y^*, which is the union of increasing sequence of finite CW spectra which are homotopy equivalent to Y_i^* (see [14, Theorem

4.1]), and there exists the following commutative diagram:

$$
\begin{array}{ccc}
E^m_\mu(X \wedge Y) & & E^m_\mu(A \wedge Y) \\
\| & & \| \\
\operatorname*{colim}_{(\lambda,i)} E^m_\mu(X_\lambda \wedge Y_i) & \longrightarrow & \operatorname*{colim}_{(\lambda,i)} E^m_\mu(A_\lambda \wedge Y_i) \\
\approx \downarrow & & \downarrow \approx \\
\operatorname*{colim}_{(\lambda,i)} [E(X_\lambda), (\Sigma^m E_\mu) \wedge Y_i^*] & \longrightarrow & \operatorname*{colim}_{(\lambda,i)} [E(A_\lambda), (\Sigma^m E_\mu) \wedge Y_i^*] \\
\approx \downarrow & & \downarrow \approx \\
\operatorname*{colim}_{\lambda} [E(X_\lambda), (\Sigma^m E_\mu) \wedge Y^*] & \longrightarrow & \operatorname*{colim}_{\lambda} [E(A_\lambda), (\Sigma^m E_\mu) \wedge Y^*] \\
\| & & \| \\
(E_\mu \wedge Y^*)^m(X) & & (E_\mu \wedge Y^*)^m(A)
\end{array}
$$

Moreover, for $\mu < \mu'$, there exists the following commutative diagram:



Here $i : A \to X$ is the inclusion map. By tracing the diagram, we see that

$$\operatorname{Im}((\alpha_{\mu\mu'})_m(A \wedge Y)) \subseteq \operatorname{Im}(E^m_\mu(i \wedge 1_Y)) \tag{5.1}$$

implies

$$\operatorname{Im}((\alpha_{\mu\mu'} \wedge 1_{Y^*})_m(A)) \subseteq \operatorname{Im}((E^m_\mu \wedge Y^*)^m(i)). \tag{5.2}$$

Now suppose that g-$\dim_Z X \wedge Y \leq n$. Let A be a closed subset of X, and $m \geq n$. Also let $\mu \in M$. Then, there exists $\mu' > \mu$ satisfying (5.1). So, (5.2) holds for the same μ'. Hence g-$\dim_{Y^* \wedge Z} X \leq n$.

Suppose that G-$\dim_Z X \wedge Y \leq n$. Let $m \geq n$, and let $\mu \in M$. Then there exists $\mu' > \mu$ satisfying (5.1) for every closed subset A of X, which implies that (5.2) holds with the same μ and μ' for every closed subset A of X. Hence G-$\dim_{Y^*\wedge Z} X \leq n$. □

Generalized cohomological dimensions g-$\dim_{\boldsymbol{E}}$ and G-$\dim_{\boldsymbol{E}}$ have the continuity property with respect to their coefficients.

THEOREM 5.4. *Let* $\boldsymbol{E} = (E_\mu, [\alpha_{\mu\mu'}], M)$ *be an object in* pro-HSPEC. *If for every* $\mu \in M$, *there exists* $\mu' > \mu$ *such that* G-$\dim_{E_{\mu'}} X (=$ g-$\dim_{E_{\mu'}} X) \leq n$, *then* g-$\dim_{\boldsymbol{E}} X \leq$ G-$\dim_{\boldsymbol{E}} X \leq n$.

PROOF. It suffices to show G-$\dim_{\boldsymbol{E}} X \leq n$. Let $\mu \in M$. Suppose that $\mu' > \mu$ and g-$\dim_{E_{\mu'}} \leq n$. If A is a closed subset of X and if $m \geq n$, then $E^m_{\mu'}(i) : E^m_{\mu'}(X) \to E^m_{\mu'}(A)$ is surjective. Since the diagram

$$\begin{array}{ccc} E^m_\mu(X) & \xleftarrow{(\alpha_{\mu\mu'})_m(X)} & E^m_{\mu'}(X) \\ {\scriptstyle E^m_\mu(i)}\downarrow & & \downarrow{\scriptstyle E^m_{\mu'}(i)} \\ E^m_\mu(A) & \xleftarrow{(\alpha_{\mu\mu'})_m(A)} & E^m_{\mu'}(A) \end{array}$$

commutes, then

$$\mathrm{Im}((\alpha_{\mu\mu'})_m(A)) \subseteq \mathrm{Im}(E^m_\mu(i)).$$

Since μ' does not depend of the choice of A, we have G-$\dim_{\boldsymbol{E}} \leq n$. □

For inverse systems of CW spectra with some additional condition, the converse of Theorem 5.4 holds.

THEOREM 5.5. *Let* $\boldsymbol{E} = (E_\mu, [\alpha_{\mu\mu'}], M)$ *be an object in* pro-HSPEC *with the following property:*

(M) *For every* $\mu \in M$, *there exists* $\mu_1 > \mu$ *such that for each* $\mu' > \mu_1$, *there exists a map* $r : E_\mu \to E_{\mu'}$ *with* $\alpha_{\mu\mu'} \circ r \simeq 1_\mu$.

Then G-$\dim_{E_\mu} X =$ g-$\dim_{E_\mu} X \leq$ g-$\dim_{\boldsymbol{E}} X \leq$ G-$\dim_{\boldsymbol{E}} X$ *for every* $\mu \in M$ *and for every compact metric space* X.

PROOF. Let $\mu \in M$, and choose $\mu_1 > \mu$ as in (M). It suffices to show that g-$\dim_{\boldsymbol{E}} X \leq n$ implies g-$\dim_{E_\mu} X \leq n$. Suppose g-$\dim_{\boldsymbol{E}} X \leq n$. Let A be a closed subset of X, and let $m \geq n$. There exist $\mu' > \mu_1$ and a map $r : E_\mu \to E_{\mu'}$ such that

$$\alpha_{\mu\mu'} \circ r \simeq 1_{E_\mu}, \text{ and } \mathrm{Im}((\alpha_{\mu\mu'})^m(A)) \subseteq \mathrm{Im}((E_\mu)^m(i)).$$

Then $(\alpha_{\mu\mu'})^m(A) \circ r^m(A) = 1_{(E_{\mu'})^m(A)}$, and hence $(E_\mu)^m(i) : (E_\mu)^m(X) \to (E_\mu)^m(A)$ is an epimorphism. This shows that $\text{g-dim}_{E_\mu} X \leq n$ as required.

$$\begin{array}{ccc}
(E_\mu)^m(X) & \xleftarrow{(\alpha_{\mu\mu'})_m(X)} & (E_{\mu'})^m(X) \\
{\scriptstyle (E_\mu)^m(i)}\downarrow & & \downarrow{\scriptstyle (E_{\mu'})^m(i)} \\
(E_{\mu'})^m(A) & \underset{(\alpha_{\mu\mu'})_m(A)}{\overset{r_m(A)}{\rightleftarrows}} & (E_{\mu'})^m(A)
\end{array}$$

□

Here we note that property (M) is not invariant under isomorphisms in pro-HSPEC. It is easily seen that property (M) in Theorem 5.5 implies the stably movability in the sense of [17].

A space Z is said to *have property (M)* if it admits a generalized HSPEC-expansion $\boldsymbol{\alpha} : E(\boldsymbol{Z}) \to \boldsymbol{E}$ such that $\boldsymbol{E}$ satisfies (M).

Example 5.6. The Hawaiian earing Z has property (M). Let $\boldsymbol{q} = (q_i) : Z \to \boldsymbol{Z} = (Z_i, q_{i,i+1})$ be the limit of Z, where Z_i is defined as

$$Z_i = C_1 \vee \cdots \vee C_i,$$

where

$$C_j = \left\{ (x_1, x_2) \in \mathbb{R}^2 \ : \ \left(x_1 - \frac{1}{j} \right)^2 + x_2^2 = \left(\frac{1}{j} \right)^2 \right\} \ (j = 1, 2, \dots, i),$$

and $q_{i,i+1} : Z_{i+1} \to Z_i$ is defined as

$$q_{i,i+1} | C_1 \vee \cdots \vee C_i = \mathrm{id}_{C_1 \vee \cdots \vee C_i} \ \text{ and,}$$
$$q_{i,i+1} | C_{i+1} \text{ is a homeomorphism of } C_{i+1} \text{ onto } C_i.$$

Then, $E(\boldsymbol{Z}) = (E(Z_i), [E(q_{i,i+1})])$ has property (M). Indeed, if $r : Z_i \to Z_{i+1}$ is the inclusion map, then $q_{i,i+1} \circ r = 1_{Z_i}$. This also implies that the induced inverse sequence $E(\boldsymbol{Z})$ satisfies property (M). Here note that this implies that $\boldsymbol{Z}$ is movable.

Lemma 5.7. *Let K and L be finite simplicial complexes, and let X be a compact Hausdorff space. Suppose that $h : K \to L$ is a simplicial map such that for every simplex σ of L,* $\text{g-dim}_{E(h^{-1}(\sigma))} X \leq n$*. Then, if* $\text{g-dim}_{E(L)} X \leq n$*, then* $\text{g-dim}_{E(K)} X \leq n$*.*

Proof. Note that $\text{g-dim}_{E(K)} \leq n$ iff for each $m \geq n$, every map $f : \Sigma^k A \to \Sigma^{k+m} K$ from a suspension of any closed subset A of X admits a map $\tilde{f} : \Sigma^{k+l} X \to \Sigma^{k+l+m} K$ for some nonnegative integer l such that $\tilde{f} | \Sigma^{k+l} A = \Sigma^l f$. Then we can modify the proof of [7, Proposition 1.2] to prove our assertion. □

LEMMA 5.8. *Let $\{K_i : i \in J\}$ be an arbitrary family of finite CW complexes. Then, for every compact Hausdorff space X,* $\text{g-dim}_{\bigvee_{i\in J} E(K_i)} X \leq n$ *if and only if* $\text{g-dim}_{E(K_i)} X \leq n$ *for $i \in J$.*

PROOF. We can prove the assertion similarly to [7, Corollary 1.11], using Lemma 5.7. ∎

THEOREM 5.9. *Let Z be the Hawaiian earing, and let $S = E(S^0)$, the sphere spectrum. Then* $\text{G-dim}_S X = \text{g-dim}_S X = \text{g-dim}_Z X + 1 = \text{G-dim}_Z X + 1$ *for every compact metric space X.*

PROOF. Suppose that $\text{g-dim}_Z X \leq n$. Then, by Thereom 5.5 we obtain that $\text{g-dim}_{E(Z_i)} X \leq n$ for some i. This together with Proposition 5.8 implies $\text{g-dim}_{E(C_i)} X \leq n$. Since $E(C_i) = \Sigma S$, $\text{g-dim}_{\Sigma S} X \leq n$. Thus $\text{G-dim}_{\Sigma S} X = \text{g-dim}_{\Sigma S} X \leq \text{g-dim}_Z X \leq \text{G-dim}_Z X$. Conversely, suppose that $\text{G-dim}_{\Sigma S} X \leq n$. Then, $\text{G-dim}_{E(C_j)} X \leq n$ for every j. By Lemma 5.8, $\text{G-dim}_{E(Z_i)} X \leq n$ for every i. This together with Theorem 5.4 implies $\text{G-dim}_Z X \leq n$. Thus $\text{G-dim}_{\Sigma S} X = \text{g-dim}_{\Sigma S} X = \text{g-dim}_Z X = \text{G-dim}_Z X$. Since $\text{g-dim}_S X - 1 = \text{g-dim}_{\Sigma S} X$ and $\text{G-dim}_S X - 1 = \text{G-dim}_{\Sigma S} X$ (Proposition 5.2), we have the equalities in the assertion. ∎

6. GENERALIZED COHOMOLOGICAL DIMENSION WITH RESPECT TO APPROXIMATELY PERFECTLY CONNECTED SPACES

A CW spectrum E is said to be *connected* provided for every $n \in \mathbb{Z}$, $\lim_{k\to\infty} \pi_{i+k}(E_{n+k}) = 0$ for $i < n$. A connected CW spectrum is *perfectly connected* provided for every $n \geq 0$, the $(n+1)$-skeleton $E_n^{(n+1)}$ of the CW complex E_n is an n-dimensional sphere and for every $n < 0$, E_n is a singleton [6].

A space X is said to be *approximately perfectly connected* provided X admits a generalized HSPEC-expansion $\boldsymbol{\alpha} = ([\alpha_\lambda]) : E(\boldsymbol{X}) \to \boldsymbol{E} = (E_\lambda, [p_{\lambda\lambda'}], \Lambda)$ such that every E_λ is a perfectly connected CW spectrum.

EXAMPLE 6.1. Every compact metric space which admits an inverse limit $\boldsymbol{p} = (p_\lambda) : X \to \boldsymbol{X} = (X_\lambda, p_{\lambda\lambda'}, \Lambda)$ with $X_\lambda = S^0$ is approximately perfectly connected.

Generalizing the result that the generalized cohomological dimension with respect to a perfectly connected CW spectrum is bounded above by the covering dimension [6, Lemma 1], we show that this holds for the generalized cohomological dimension with respect to approximately perfectly connected spaces.

PROPOSITION 6.2. *If Z is an approximately perfectly connected space, then*

$$\text{g-dim}_Z X \leq \text{G-dim}_Z X \leq \dim X.$$

PROOF. It suffices to show $\text{G-dim}_Z X \leq \dim X$. Z admits a generalized HSPEC-expansion $\boldsymbol{\alpha} = ([\alpha_\mu]) : E(\boldsymbol{Z}) \to \boldsymbol{E} = (E_\mu, [\alpha_{\mu\mu'}], M)$ such that each E_μ is a perfectly connected CW spectrum. By [5, Lemma 1], $\text{G-dim}_{E_\mu} X \leq \dim X$. By Theorem 5.4, we have $\text{G-dim}_Z X \leq \dim X$. ∎

The generalized cohomological dimension with respect to a perfectly connected CW spectrum coincides with the covering dimension if the space has finite covering dimension ([6, Theorem 1]). We show in the following that this holds for the generalized cohomological dimension with respect to approximately perfectly connected spaces.

THEOREM 6.3. *If Z is approximately perfectly connected, then for every finite-dimensional compact Hausdorff space X,*

$$\text{g-dim}_Z X = \text{G-dim}_Z X = \dim X.$$

We can generalize the technique used for [6, Theorem 1], which is based on the following lemma ([6, Lemma 2]).

LEMMA 6.4. *Let $i : S^n \to \Omega^k \Sigma^k S^n$ be the natural embedding, i.e., i is the map induced by the map $j : S^n \to C(S^k, S^k \wedge S^n)$ defined by $j(x)(t) = [(t, x)]$, the equivalence class represented by $(t, x) \in S^k \times S^n$. For every map $f : X \to \Omega^k \Sigma^k S^n$ from a compact metric space X with $\dim X = n + 1$, there exists a homotopy $H : X \times I \to \Omega^k \Sigma^k S^n$ such that*

$$H_0 = f, \qquad \operatorname{Im} H_1 \subseteq i(S^n), \qquad f|f^{-1}(i(S^n)) = H_1|f^{-1}(i(S^n)).$$

PROOF OF THEOREM 6.3. Suppose that $\dim X = n+1$ and $\text{g-dim}_Z X \leq n$. Fix $\mu \in M$, and let A be a closed subset of X. Then, there exists $\mu' > \mu$ such that every map $f : A \to \Omega^\infty (E_{\mu'})_{n+\infty}$ admits a map $\tilde{f} : X \to \Omega^\infty (E_\mu)_{n+\infty}$ with $\tilde{f}|A = Q_n^\infty(\alpha_{\mu\mu'}) \circ f$. Now let $f : A \to S^n = (E_{\mu'})_n^{(n+1)}$ be a map. Then f representes a map $A \to \Omega^\infty (E_{\mu'})_{n+\infty}$. So, there exists a map $\tilde{f} : X \to \Omega^\infty (E_{\mu'})_{n+\infty}$ such that $\tilde{f}|A = Q_n^\infty(\alpha_{\mu\mu'}) \circ f$ and $\tilde{f}(X) \subseteq \Omega^k (E_\mu)_{n+k}$ for some $k \geq 0$. Then there is an associated map $\varphi : X \times S^k \to (E_\mu)_{n+k}$. But, since $\dim X = n+1$, there exists a map $\psi : X \times S^k \to (E_\mu)_{n+k}^{(n+k+1)} = S^{n+k} = \Sigma^k S^n$ such that $\varphi \simeq \psi$ as maps $(X \times S^k, A \times S^k) \to (\Sigma^k S^n, S^n)$. Then ψ induces a map $\xi : X \to \Omega^k \Sigma^k S^n$ such that $\xi \simeq \tilde{f}$ as maps into $\Omega^k (E_\mu)_{n+k}$, and $\xi|A \simeq \tilde{f}|A = Q_n^\infty(\alpha_{\mu\mu'}) \circ f$. By Lemma 6.4, there exists a map $\eta : X \to S^n$ such that $\eta \simeq \xi$ rel. A. The map $\eta : X \to S^n$ satisfies $\eta|A = f$. This shows $\dim X \leq n$, contradicting to the assumption that $\dim X = n + 1$. ∎

The generalized cohomological dimension with respect to a perfectly connected Ω-spectrum is bounded below by the integral cohomological covering dimension ([6, Theorem 2]). We show in the following theorem that an analogous result holds for the generalized cohomological dimension with respect to approximately perfectly connected spaces with some additional condition.

THEOREM 6.5. *If Z is an approximately perfectly connected space with property (M) (see Theorem 5.5), then* $\text{c-dim}_{\mathbb{Z}} X \leq \text{g-dim}_Z X$ *for every compact metric space X.*

PROOF. By Theorem 5.5, there exists a perfectly connected CW spectrum E such that $\text{g-dim}_E X \leq \text{g-dim}_Z X$. By [5, Theorem 2], $\text{c-dim}_{\mathbb{Z}} X \leq \text{g-dim}_E X$. The two inequalities imply that $\text{c-dim}_{\mathbb{Z}} X \leq \text{g-dim}_Z X$ as required. □

Recall the construction of Kahn continuum ([11], see also [13, Example 1, p. 153]). J. F. Adams ([1]) constructed a finite polyhedron P and a map $\gamma : \Sigma^r P \to P$ for some positive integer r such that for every positive integer m, the composition

$$\gamma \circ (\Sigma^r \gamma) \circ (\Sigma^{2r} \gamma) \circ \cdots \circ (\Sigma^{(m-1)r} \gamma) : \Sigma^{mr} P \to P$$

is essential. The Kahn continuum Z is the limit of the inverse sequence

$$P \xleftarrow{\gamma} \Sigma^r P \xleftarrow{\Sigma^r \gamma} \Sigma^{2r} P \xleftarrow{\Sigma^{2r} \gamma} \cdots .$$

Then Z is not stable shape equivalent to a point.

The following shows that the condition in Theorem 6.5 that Z is approximately perfectly connected is essential.

THEOREM 6.6. *Let Z be the Kahn continuum. Then, for every finite dimensional compact Hausdorff space X,* $\text{g-dim}_Z X = 0$.

PROOF. Indeed, let $n = \dim X$, and let m be a positive integer. Take $m' > m$ such that $m'r > n$. Let A be a closed subset of X. Then there exists an HCW-expansion $\boldsymbol{p} = ([p_\lambda]) : X \to \boldsymbol{X} = (X_\lambda, [p_{\lambda\lambda'}], \Lambda)$ of X such that each X_λ is a finite polyhedron with $\dim X_\lambda \leq n$, and the restriction $\boldsymbol{p}|A = ([p_\lambda|A]) : A \to \boldsymbol{A} = (A_\lambda : [p_{\lambda\lambda'}|A_{\lambda'}], \Lambda)$ is an HCW-expansion of A, where every A_λ is a finite polyhedron. Let $\lambda \in \Lambda$, and let $f : E(A_\lambda) \to E(\Sigma^{m'r} P)$ be a map between CW spectra. Since A_λ is a finite polyhedron, f is represented by a map $f' : \Sigma^k A_\lambda \to \Sigma^{m'r+k} P$ for some nonnegative integer k. Since $\dim \Sigma^k A_\lambda \leq n + k < m'r + k$, f' is inessential, and there exists a map $\tilde{f}' : \Sigma^k X_\lambda \to \Sigma^{m'r+k} P$ such that $\tilde{f}'|\Sigma^k A_\lambda = f'$. Then the composition

$$\Sigma^{mr+k} \gamma \circ \Sigma^{(m+1)r+k} \gamma \circ \cdots \circ \Sigma^{(m'-1)r+k} \gamma \circ \tilde{f}'$$

represents a map $\tilde{f} : E(X_\lambda) \to E(\Sigma^{mr} P)$ such that

$$\tilde{f} \circ E(i_\lambda) = E(\Sigma^{mr+k} \gamma \circ \Sigma^{(m+1)r+k} \gamma \circ \cdots \circ \Sigma^{(m'-1)r+k} \gamma) \circ f,$$

where $i_\lambda : A_\lambda \to X_\lambda$ is the inclusion map. This shows that $\text{g-dim}_Z X = 0$. □

REMARK 6.7. Note that each coordinate CW spectrum $E(\Sigma^{nr} P)$ is connected, i.e., $\lim_{k \to \infty} \pi_{i+k}(\Sigma^{(n+k)r} P) = 0$ for each $i < n$, but not perfectly connected.

The Kahn continuum Z is weak stable shape equivalent to a point, i.e., pro-$\pi_n(Z) = 0$ for every n. In general, any space that is weak stable shape equivalent to a point satisfies the assertion of Theorem 6.6.

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T. Miyata
Department of Mathematics and Informatics
Graduate School of Human Development and Environment
Kobe University
Kobe 657-8501
Japan
E-mail: tmiyata@kobe-u.ac.jp

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