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STRONG CONVERGENCE FOR WEIGHTED SUMS OF ρ^*-MIXING RANDOM VARIABLES

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ABSTRACT. The authors discuss the strong convergence for weighted sums of sequences of ρ^*-mixing random variables. The obtained results extend and improve the corresponding theorem of Bai and Cheng [Bai, Z. D., Cheng, P. E., 2000. Marcinkiewicz strong laws for linear statistics. Statist. Probab. Lett., 46, 105-112]. The method used in this article differs from that of Bai and Cheng (2000).

1. INTRODUCTION

Let $\{X, X_n, n \geq 1\}$ be a sequence of random variables and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants. The weighted sums $\sum_{i=1}^{n} a_{ni}X_i$ are used widely in some linear statistics, such as least squares estimators, nonparametric regression function estimators and jackknife estimates. Therefore, the strong convergence for the weighted sums has been a attractive research topic in the recent literature. We refer the reader to [2, 7, 9, 10, 13, 19, 20, 22, 24, 25, 28].

Bai and Cheng ([2]) studied the following Marcinkiewicz-Zygmund strong law.

THEOREM 1.1. *Suppose* $1/p = 1/\alpha + 1/\beta$ *for* $1 < \alpha, \beta < \infty$ *and* $1 < p < 2$. *Let* $\{X, X_n, n \geq 1\}$ *be a sequence of i.i.d. random variables with mean 0, and let* $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be an array of constants satisfying*

$$(1.1) \qquad A_\alpha = \lim_{n\to\infty} \sup A_{\alpha,n} < \infty, \quad A_{\alpha,n}^{\alpha} = \sum_{i=1}^{n} |a_{ni}|^{\alpha}/n.$$

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If $E|X|^\beta < \infty$*, then*

$$n^{-1/p} \sum_{i=1}^{n} a_{ni} X_i \to 0 \qquad a.s. \tag{1.2}$$

Hsu and Robbins ([12]) introduced the following concept of the complete convergence. A sequence of random variables $\{U_n, n \geq 1\}$ is said to converge completely to a constant θ if

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \varepsilon) < \infty \text{ for all } \varepsilon > 0.$$

By the Borel-Cantelli lemma, it is easily seen that the above result implies that $U_n \to \theta$ almost surely. Therefore, the complete convergence is a very important tool in establishing almost sure convergence of summation of random variables as well as weighted sums of random variables.

Let $\{X_i, i \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Write $\mathcal{F}_\mathcal{S} = \sigma(X_i, i \in S \subset N)$. Given σ-subalgebras $\mathcal{B}$, $\mathcal{R}$ of $\mathcal{F}$, let

$$\rho(\mathcal{B}, \mathcal{R}) = \sup_{X \in L_2(\mathcal{B}), Y \in L_2(\mathcal{R})} \frac{|E[XY] - EX \cdot EY|}{(\text{Var}X \cdot \text{Var}Y)^{1/2}}.$$

Define the ρ^*-mixing coefficients by

$$\rho^*(k) = \sup\{\rho(\mathcal{F}_\mathcal{S}, \mathcal{F}_\mathcal{T}) : \text{ finite subsets } S,\, T \subset N, \text{ such that dist } (S, T) \geq k\},$$
$k \geq 0$.

Definition 1.2. *A sequence of random variables* $\{X_i, i \geq 1\}$ *is said to be a* ρ^**-mixing sequence if there exists* $k \in N$ *such that* $\rho^*(k) < 1$.

The concept of coefficient ρ^* was introduced by Moore ([15]) and Bradley ([3]) was the first who introduced the concept of ρ^*-mixing random variables to limit theorems. Since the article of Bradley ([3]) appeared, many authors studied the convergence properties for sequences or arrays of ρ^*-mixing random variables, such as Peligrad and Gut ([16]), Utev and Peligrad ([21]), Gan ([10]), Cai ([5, 6]), Kuczmaszewska ([14]), An and Yuan ([1]), Wu and Jiang ([23]), Qiu ([17]), Wu et al. ([26]), Budsaba et al. ([4]), Wang et al. ([22]), Guo and Zhu ([11]).

In this work, we study the complete convergence and Marcinkiewicz-Zygmund strong law for weighted sums of ρ^*-mixng random variables. We extend and improve Theorem 1.1 in three directions.

(i) We consider ρ^*-mixng instead of i.i.d.
(ii) Under the same conditions of Theorem 1.1 for $\alpha < \beta$, we get (3.1) which is much stronger than (1.2).
(iii) Under the same conditions of Theorem 1.1 for $\alpha \geq \beta$, we get (3.7) which is also much stronger than (1.2).

In addition, under some similar conditions of Theorem 1.1 for $\alpha \geq \beta$, we prove that (3.1) remains true.

Throughout this paper, the symbol C represents positive constants whose values may change from one place to another. For a finite set A the symbol $\sharp(A)$ denotes the number of elements in the set A.

2. Preliminaries

We present a concept of stochastic domination, which is a slight generalization of identical distribution. A sequence of random variables $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X (write $\{X_n\} \prec X$) if there exists a constant $C > 0$ such that

$$\sup_{n\geq 1} P(|X_n| > x) \leq CP(|X| > x), \quad \forall x > 0.$$

Stochastic dominance of $\{X_n, n \geq 1\}$ by the random variable X implies $E|X_n|^p \leq CE|X|^p$ if the p-moment of $|X|$ exists, i. e., if $E|X|^p < \infty$.

To prove our main results, we need the following two lemmas.

Lemma 2.1 ([21]). *Suppose N is a positive integer, $0 \leq r < 1$, and $q \geq 2$. Then there exists a positive constant $C = C(N, r, q)$ such that for a sequence $\{X_i, i \geq 1\}$ of random variables satisfying $\rho^*(N) \leq r$, $EX_i = 0$ and $E|X_i|^q < \infty$ for every $i \geq 1$, the following holds*

$$E \max_{1\leq j\leq n} \Big|\sum_{k=1}^{j} X_k\Big|^q \leq C\Big\{\sum_{k=1}^{n} E|X_k|^q + \Big(\sum_{k=1}^{n} EX_k^2\Big)^{q/2}\Big\}$$

for all $n \geq 1$.

Lemma 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of random variables with $\{X_n\} \prec X$. Then there exists a constant C such that, for all $q > 0$ and $x > 0$,*

(i) $E|X_k|^q I(|X_k| \leq x) \leq C\{E|X|^q I(|X| \leq x) + x^q P(|X| > x)\}$,
(ii) $E|X_k|^q I(|X_k| > x) \leq CE|X|^q I(|X| > x)$.

This lemma can be easily proved by using integration by parts. We omit the details.

3. Main results and proofs

In this section, we state our main results and their proofs.

Theorem 3.1. *Suppose $1/p = 1/\alpha + 1/\beta$ for $1 < \alpha$, $\beta < \infty$ and $1 < p < 2$. Let $\{X_n, n \geq 1\}$ be a sequence of ρ^*-mixing random variables with $EX_n = 0$ and $\{X_n\} \prec X$, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying* (1.1). *Then the following statements hold:*

(i) *if* $\alpha < \beta$, *then* $E|X|^\beta < \infty$ *implies*

$$\sum_{n=1}^{\infty} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > n^{1/p}\varepsilon\Big) < \infty \quad \textit{for all } \varepsilon > 0, \tag{3.1}$$

(ii) *if* $\alpha = \beta$, *then* $E|X|^\beta \log^+ |X| < \infty$ *implies* (3.1),

(iii) *if* $\alpha > \beta$, *then* $E|X|^\alpha < \infty$ *implies* (3.1).

Proof. We first present two inequalities which will be very useful in the following proofs. From (1.1), without loss of generality, we may assume that $\sum_{i=1}^{n} |a_{ni}|^\alpha \le n$. Then by the Hölder inequality, for any $1 \le \gamma < \alpha$,

$$\sum_{i=1}^{n} |a_{ni}|^\gamma \le \Big(\sum_{i=1}^{n} |a_{ni}|^{\gamma\frac{\alpha}{\gamma}}\Big)^{\frac{\gamma}{\alpha}} \Big(\sum_{i=1}^{n} 1\Big)^{\frac{\alpha-\gamma}{\alpha}} \le n. \tag{3.2}$$

By $\sum_{i=1}^{n} |a_{ni}|^\alpha \le n$, for any $\gamma \ge \alpha$,

$$\sum_{i=1}^{n} |a_{ni}|^\gamma = \sum_{i=1}^{n} |a_{ni}|^\alpha |a_{ni}|^{\gamma-\alpha} \le \sum_{i=1}^{n} |a_{ni}|^\alpha \Big(\sum_{i=1}^{n} |a_{ni}|^\alpha\Big)^{\frac{\gamma-\alpha}{\alpha}} \le n^{\frac{\gamma}{\alpha}}. \tag{3.3}$$

Note that $a_{ni} = a_{ni}^+ - a_{ni}^-$, where $a_{ni}^+ = \max\{a_{ni}, 0\}$ and $a_{ni}^- = \max\{-a_{ni}, 0\}$. To prove (3.1), it suffices to show that for every $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} a_{ni}^+ X_i\Big| > n^{1/p}\varepsilon\Big) < \infty$$

and

$$\sum_{n=1}^{\infty} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} a_{ni}^- X_i\Big| > n^{1/p}\varepsilon\Big) < \infty.$$

Therefore, without loss of generality, we may assume that $a_{ni} \ge 0$. For fixed $n \ge 1$, define $Y_{ni} = a_{ni}X_i I(a_{ni}|X_i| \le n^{1/p})$, $Z_{ni} = a_{ni}X_i - Y_{ni}$. Then

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > n^{1/p}\varepsilon\Big) \\
&\quad \le \sum_{n=1}^{\infty} P\Big(\max_{1\le i\le n} a_{ni}|X_i| > n^{1/p}\Big) \;+\; \sum_{n=1}^{\infty} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j} Y_{ni}\Big| > n^{1/p}\varepsilon\Big) \\
&\quad =: I_1 + I_2.
\end{aligned}$$

Let $A_n = \sum_{i=1}^n |a_{ni}|^\alpha$, then

$$\begin{aligned}
I_1 &\le \sum_{n=1}^\infty \sum_{i=1}^n P\big(a_{ni}|X_i| > n^{1/p}\big) \;\le\; C \sum_{n=1}^\infty \sum_{i=1}^n P\big(a_{ni}|X| > n^{1/p}\big) \\
&\le C \sum_{n=1}^\infty n^{-\alpha/p} \sum_{i=1}^n a_{ni}^\alpha E|X|^\alpha I\big(a_{ni}|X| > n^{1/p}\big) \\
&= C \sum_{n=1}^\infty n^{-\alpha/p} \sum_{i=1}^n a_{ni}^\alpha E|X|^\alpha I\big(|X|^\alpha > n^{\alpha/p} a_{ni}^{-\alpha}\big) \\
&\le C \sum_{n=1}^\infty n^{-\alpha/p} \sum_{i=1}^n a_{ni}^\alpha E|X|^\alpha I\big(|X|^\alpha > n^{\alpha/p} A_n^{-1}\big) \qquad (\text{since } A_n \le n\,) \\
&\le C \sum_{n=1}^\infty n^{-\alpha/\beta} E|X|^\alpha I\big(|X| > n^{1/\beta}\big) \qquad (\text{since } 1/p = 1/\alpha + 1/\beta\,) \\
&= C \sum_{n=1}^\infty n^{-\alpha/\beta} \sum_{m=n}^\infty E|X|^\alpha I\big(m < |X|^\beta \le m+1\big) \\
&= C \sum_{m=1}^\infty E|X|^\alpha I\big(m < |X|^\beta \le m+1\big) \sum_{n=1}^m n^{-\alpha/\beta}.
\end{aligned}$$

Note that

$$\sum_{n=1}^m n^{-\alpha/\beta} \;\le\; \begin{cases} C\, m^{1-\alpha/\beta} & \text{for } \alpha < \beta, \\ C\, \log m & \text{for } \alpha = \beta, \\ C & \text{for } \alpha > \beta. \end{cases}$$

Hence we get

$$I_1 \;\le\; \begin{cases} C\, E|X|^\beta & \text{for } \alpha < \beta, \\ C\, E|X|^\beta \log^+ |X| & \text{for } \alpha = \beta, \\ C\, E|X|^\alpha & \text{for } \alpha > \beta. \end{cases}$$

Taking into account the conditions of Theorem 3.1 we obtain $I_1 < \infty$.

Next we prove $I_2 < \infty$. We first prove that

$$n^{-1/p} \max_{1\le j\le n} \left| \sum_{i=1}^j EY_{ni} \right| \to 0 \quad \text{as } n \to \infty. \tag{3.4}$$

Note that we always have $E|X|^\alpha < \infty$ and $E|X|^\beta < \infty$ under the assumptions of Theorem 3.1 By $EX_n = 0$, Lemma 2.2, $1/p = 1/\alpha + 1/\beta$ and $E|X|^\beta < \infty$, we get

$$
\begin{aligned}
n^{-1/p} \max_{1\le j\le n}\Big|\sum_{i=1}^{j} EY_{ni}\Big| \;&=\; n^{-1/p} \max_{1\le j\le n}\Big|\sum_{i=1}^{j} EZ_{ni}\Big| \\
&\le n^{-1/p}\sum_{i=1}^{n} a_{ni}E|X_i|I(a_{ni}|X_i|>n^{1/p}) \\
&= C\,n^{-1/p}\sum_{i=1}^{n} a_{ni}E|X|I(a_{ni}|X|>n^{1/p}) \qquad (\text{similar to } I_1<\infty\,) \\
&\le C\,n^{-1/p}\sum_{i=1}^{n} a_{ni}E|X|I(|X|>n^{1/\beta}) \qquad (\text{by (3.2) and } \beta>1\,) \\
&\le C\,n^{-1/p+1/\beta}E|X|^{\beta}I(|X|>n^{1/\beta}) \\
&= C\,n^{-1/\alpha}E|X|^{\beta}I(|X|>n^{1/\beta})\to 0 \quad \text{as } n\to\infty.
\end{aligned}
$$

Then it follows by (3.4) that for n large enough

$$\max_{1\le j\le n}\Big|\sum_{i=1}^{j} EY_{ni}\Big| < n^{1/p}\varepsilon/2.$$

Let $q>\max\{\alpha,\beta,2p/(2-p)\}$. Then the Markov inequality and Lemma 2.1 yield

$$
\begin{aligned}
I_2 &\le C\sum_{n=1}^{\infty} P\Big(\max_{1\le j\le n}\Big|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\Big| > n^{1/p}\varepsilon/2\Big) \\
&\le C\sum_{n=1}^{\infty} n^{-q/p}E\max_{1\le j\le n}\Big|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\Big|^{q} \\
&\le C\sum_{n=1}^{\infty} n^{-q/p}\sum_{i=1}^{n}E|Y_{ni}|^{q} + C\sum_{n=1}^{\infty} n^{-q/p}\Big(\sum_{i=1}^{n}EY_{ni}^{2}\Big)^{q/2} \\
&=: I_3+I_4.
\end{aligned}
$$

By Lemma 2.2, we have

$$
\begin{aligned}
I_3 &\le C\sum_{n=1}^{\infty} n^{-q/p}\sum_{i=1}^{n} a_{ni}^{q}E|X|^{q}I(a_{ni}|X|\le n^{1/p}) \\
&\quad + C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P(a_{ni}|X|>n^{1/p}) \\
&=: I_5+I_6.
\end{aligned}
$$

By a similar argument as in the proof of $I_1 < \infty$, we can get $I_6 < \infty$. Then we prove $I_5 < \infty$. For $j \geq 1$ and $n \geq 1$, let

$$I_{nj} = \big\{1 \leq i \leq n:\ n^{1/\alpha}(j+1)^{-1/\alpha} < a_{ni} \leq n^{1/\alpha} j^{-1/\alpha}\big\}.$$

Then $\{I_{nj}, j \geq 1\}$ are disjoint, $\bigcup_{j\geq 1} I_{nj} \subseteq \{1, 2, \cdots, n\}$ for all $n \geq 1$ from $\sum_{i=1}^n a_{ni}^\alpha \leq n$. Note that for all $k \geq 1$, we have

$$\begin{aligned} n &\geq \sum_{i=1}^n a_{ni}^\alpha \ = \ \sum_{j=1}^\infty \sum_{i\in I_{nj}} a_{ni}^\alpha \ \geq \ \sum_{j=1}^\infty \sharp(I_{nj})\, n\,(j+1)^{-1} \\ &\geq \sum_{j=k}^\infty \sharp(I_{nj})\, n\,(j+1)^{-1} \ = \ \sum_{j=k}^\infty \sharp(I_{nj})\, n\,(j+1)^{-q/\alpha}(j+1)^{q/\alpha-1} \\ &\geq \sum_{j=k}^\infty \sharp(I_{nj})\, n\,(j+1)^{-q/\alpha}(k+1)^{q/\alpha-1}. \end{aligned}$$

Hence for all $k \geq 1$, we have

$$\sum_{j=k}^\infty \sharp(I_{nj})\, j^{-q/\alpha} \ \leq \ C\,(k+1)^{1-q/\alpha}. \tag{3.5}$$

Then

$$\begin{aligned} I_5 &= C\sum_{n=1}^\infty n^{-q/p} \sum_{j=1}^\infty \sum_{i\in I_{nj}} a_{ni}^q E|X|^q I(a_{ni}|X| \leq n^{1/p}) \\ &\leq C\sum_{n=1}^\infty n^{-q/p} \sum_{j=1}^\infty \sharp(I_{nj}) n^{q/\alpha} j^{-q/\alpha} E|X|^q I(|X| \leq n^{1/p-1/\alpha}(j+1)^{1/\alpha}) \\ &= C\sum_{n=1}^\infty n^{-q/\beta} \sum_{j=1}^\infty \sharp(I_{nj}) j^{-q/\alpha} E|X|^q I(|X| \leq n^{1/\beta}(j+1)^{1/\alpha}) \\ &= C\sum_{n=1}^\infty n^{-q/\beta} \sum_{j=1}^\infty \sharp(I_{nj}) j^{-q/\alpha} E|X|^q I(|X| \leq n^{1/\beta}) \\ &\quad + C\sum_{n=1}^\infty n^{-q/\beta} \sum_{j=1}^\infty \sharp(I_{nj}) j^{-q/\alpha} \\ &\qquad \cdot \sum_{k=1}^j E|X|^q I(n^{1/\beta}k^{1/\alpha} < |X| \leq n^{1/\beta}(k+1)^{1/\alpha}) \\ &=: I_5^* + I_5^{**}. \end{aligned}$$

By (3.5) and $q > \beta$ we have

$$\begin{aligned}
I_5^* &\le C\sum_{n=1}^{\infty} n^{-q/\beta} E|X|^q I(|X| \le n^{1/\beta}) \\
&= C\sum_{n=1}^{\infty} n^{-q/\beta} \sum_{m=1}^{n} E|X|^q I(m-1 < |X|^\beta \le m) \\
&= C\sum_{m=1}^{\infty} E|X|^q I(m-1 < |X|^\beta \le m) \sum_{n=m}^{\infty} n^{-q/\beta} \\
&\le C\sum_{m=1}^{\infty} m^{1-q/\beta} E|X|^q I(m-1 < |X|^\beta \le m) \\
&\le CE|X|^\beta < \infty.
\end{aligned}$$

By (3.5) and $q > \alpha$ we obtain

$$\begin{aligned}
I_5^{**} &= C\sum_{n=1}^{\infty} n^{-q/\beta} \sum_{k=1}^{\infty} E|X|^q I(n^{1/\beta}k^{1/\alpha} < |X| \le n^{1/\beta}(k+1)^{1/\alpha}) \\
&\quad \cdot \sum_{j=k}^{\infty} \sharp(I_{nj}) j^{-q/\alpha} \\
&\le C\sum_{n=1}^{\infty} n^{-q/\beta} \sum_{k=1}^{\infty} (k+1)^{1-q/\alpha} E|X|^q I(n^{1/\beta}k^{1/\alpha} < |X| \le n^{1/\beta}(k+1)^{1/\alpha}) \\
&\le C\sum_{n=1}^{\infty} n^{-\alpha/\beta} \sum_{k=1}^{\infty} E|X|^\alpha I(n^{1/\beta}k^{1/\alpha} < |X| \le n^{1/\beta}(k+1)^{1/\alpha}) \\
&= C\sum_{n=1}^{\infty} n^{-\alpha/\beta} E|X|^\alpha I(|X| > n^{1/\beta}) \qquad (\text{similar to } I_1 < \infty) \\
&\le \begin{cases} C\,E|X|^\beta < \infty & \text{for } \alpha < \beta, \\ C\,E|X|^\beta \log^+|X| < \infty & \text{for } \alpha = \beta, \\ C\,E|X|^\alpha < \infty & \text{for } \alpha > \beta. \end{cases}
\end{aligned}$$

Next we prove $I_4 < \infty$. By Lemma 2.2 and $q > 2p/(2-p) > 2$, we have

$$\begin{aligned}
I_4 &\le C\sum_{n=1}^{\infty} n^{-q/p} \Big(\sum_{i=1}^{n} a_{ni}^2 EX^2 I(a_{ni}|X| \le n^{1/p}) \\
&\quad + n^{2/p} \sum_{i=1}^{n} P(a_{ni}|X| > n^{1/p}) \Big)^{q/2} \\
&\le C\sum_{n=1}^{\infty} n^{-q/p} \Big(\sum_{i=1}^{n} a_{ni}^2 EX^2 I(a_{ni}|X| \le n^{1/p}) \Big)^{q/2}
\end{aligned}$$

$$+ C\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n} P(a_{ni}|X| > n^{1/p})\right)^{q/2}$$
$$=: I_7 + I_8.$$

By a similar argument as in the proof of $I_1 < \infty$ and $E|X|^\beta < \infty$, we have

$$\sum_{i=1}^{n} P(a_{ni}|X| > n^{1/p}) \le C\sum_{i=1}^{n} P(|X| > n^{1/p-1/\alpha})$$
$$\le CE|X|^\beta I(|X| > n^{1/\beta}) \to 0 \quad \text{as } n \to \infty.$$

Hence, for n large enough $\sum_{i=1}^{n} P(a_{ni}|X| > n^{1/p}) < 1$ holds. Therefore, similarly to the proof of $I_1 < \infty$, we can get

$$I_8 \le C\sum_{n=1}^{\infty}\sum_{i=1}^{n} P(a_{ni}|X| > n^{1/p}) < \infty.$$

Finally, we prove $I_7 < \infty$. From $1/p = 1/\alpha + 1/\beta$ and $1 < p < 2$, we know that $\alpha \le 2$ and $\beta \le 2$ can not hold simultaneously. Hence we need only to consider the following three cases.

Case 1: $\alpha < 2 < \beta$
By (3.3), $q > \beta$, $E|X|^\beta < \infty$ and $1/p = 1/\alpha + 1/\beta$, we have

$$I_7 \le C\sum_{n=1}^{\infty} n^{-q/p+q/\alpha}\big(EX^2\big)^{q/2} = C\sum_{n=1}^{\infty} n^{-q/\beta}\big(EX^2\big)^{q/2} < \infty.$$

Case 2: $\beta < 2 < \alpha$
By (3.2), $E|X|^\alpha < \infty$ and $q > 2p/(2-p)$, we have

$$I_7 \le C\sum_{n=1}^{\infty} n^{-q/p+q/2}\big(EX^2\big)^{q/2} < \infty.$$

Case 3: $\alpha \ge 2$ and $\beta \ge 2$
By (3.2), $E|X|^\alpha < \infty$ and $q > 2p/(2-p)$, we have

$$I_7 \le C\sum_{n=1}^{\infty} n^{-q/p+q/2}\big(EX^2\big)^{q/2} < \infty.$$

The proof is completed. □

By the Borel-Cantelli lemma, we get directly the following corollary.

Corollary 3.2. *Under the conditions of Theorem 3.1,*

$$n^{-1/p}\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| \to 0 \qquad a.s. \tag{3.6}$$

Remark 3.3. Since (3.1) implies (3.6) and (3.6) is much stronger than (1.2), Theorem 3.1 and Corollary 3.2 extend and improve Theorem 1.1 for the case $\alpha < \beta$.

Theorem 3.4. *Suppose* $1/p = 1/\alpha + 1/\beta$ *for* $1 < \beta \leq \alpha < \infty$ *and* $1 < p < 2$. *Let* $\{X_n, n \geq 1\}$ *be a sequence of* ρ^**-mixing random variables with* $EX_n = 0$ *and* $\{X_n\} \prec X$, *and let* $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be an array of constants satisfying* (1.1). *If* $E|X|^\beta < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > n^{1/p}\varepsilon\Big) < \infty \quad \textit{for all } \varepsilon > 0. \tag{3.7}$$

Proof. Following the notations of Y_{ni} and Z_{ni} we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} a_{ni}X_i\Big| > n^{1/p}\varepsilon\Big)\\
&\quad\leq \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq i\leq n} a_{ni}|X_i| > n^{1/p}\Big)\\
&\quad\quad+ \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j} Y_{ni}\Big| > n^{1/p}\varepsilon\Big)\\
&\quad =: I_9 + I_{10}.
\end{aligned}$$

Note that $E|X|^\beta < \infty$ implies $\sum_{n=1}^{\infty} P(|X| > n^{1/\beta}) < \infty$. Then by a similar argument as in the proof of $I_1 < \infty$, we have

$$\begin{aligned}
I_9 &\leq C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P\big(a_{ni}|X| > n^{1/p}\big) \leq C\sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n} P\big(|X| > n^{1/p-1/\alpha}\big)\\
&= C\sum_{n=1}^{\infty} P\big(|X| > n^{1/\beta}\big) < \infty.
\end{aligned}$$

Let $q > \max\{\beta, 2\}$. Note that (3.4) still holds. Then by the Markov inequality and Lemma 2.1, it follows that

$$\begin{aligned}
I_{10} &\leq \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j}(Y_{ni} - EY_{ni})\Big| > n^{1/p}\varepsilon/2\Big)\\
&\leq C\sum_{n=1}^{\infty} n^{-1-q/p} E\max_{1\leq j\leq n}\Big|\sum_{i=1}^{j}(Y_{ni} - EY_{ni})\Big|^q\\
&\leq C\sum_{n=1}^{\infty} n^{-1-q/p}\sum_{i=1}^{n} E|Y_{ni}|^q + C\sum_{n=1}^{\infty} n^{-1-q/p}\Big(\sum_{i=1}^{n} EY_{ni}^2\Big)^{q/2}\\
&=: I_{11} + I_{12}.
\end{aligned}$$

By $q > \beta > p$, $\beta \leq \alpha$ and (3.2), we have

$$\begin{aligned} I_{10} &= C \sum_{n=1}^{\infty} n^{-1-q/p} \sum_{i=1}^{n} a_{ni}^q E|X_i|^q I(a_{ni}|X_i| \leq n^{1/p}) \\ &\leq C \sum_{n=1}^{\infty} n^{-1-\beta/p} \sum_{i=1}^{n} a_{ni}^{\beta} E|X_i|^{\beta} I(a_{ni}|X_i| \leq n^{1/p}) \\ &\leq C \sum_{n=1}^{\infty} n^{-\beta/p} E|X|^{\beta} < \infty. \end{aligned}$$

Finally, we prove $I_{12} < \infty$. As mentioned in the proof of $I_7 < \infty$, we know that $\alpha \leq 2$ and $\beta \leq 2$ can not hold simultaneously. Hence we need only consider the following two cases. If $2 \leq \beta < \alpha$, by (3.2) and $p < 2$, we have

$$I_{12} \leq C \sum_{n=1}^{\infty} n^{-1-q/p+q/2} \big(EX^2\big)^{q/2} < \infty.$$

If $\beta < 2 < \alpha$, by (3.2) and $\beta > p$, we have

$$\begin{aligned} I_{12} &= C \sum_{n=1}^{\infty} n^{-1-q/p} \bigg(\sum_{i=1}^{n} a_{ni}^2 E|X_i|^2 I(a_{ni}|X_i| \leq n^{1/p}) \bigg)^{q/2} \\ &\leq C \sum_{n=1}^{\infty} n^{-1-(\beta-p)q/(2p)} \big(E|X|^{\beta}\big)^{q/2} < \infty. \end{aligned}$$

The proof is completed. ∎

COROLLARY 3.5. *Under the conditions of Theroem 3.4,* (1.2) *holds.*

PROOF. Let $S_j = \sum_{i=1}^{j} a_{ni} X_i$. From (3.7) we have

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} n^{-1} P\Big(\max_{1 \leq j \leq n} \big|S_j\big| > n^{\frac{1}{p}} \varepsilon \Big) \\ &= \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} n^{-1} P\Big(\max_{1 \leq j \leq n} \big|S_j\big| > n^{\frac{1}{p}} \varepsilon \Big) \\ &\geq \frac{1}{2} \sum_{k=1}^{\infty} P\Big(\max_{1 \leq j \leq 2^k} \big|S_j\big| > 2^{\frac{k+1}{p}} \varepsilon \Big). \end{aligned}$$

Then by the Borel-Cantelli Lemma,

$$\lim_{k \to \infty} 2^{-\frac{k+1}{p}} \max_{1 \leq j \leq 2^k} \big|S_j\big| = 0 \qquad a.s.$$

For every positive integers n, there exists a positive integer k_0 such that $2^{k_0-1} \leq n < 2^{k_0}$. Then

$$n^{-\frac{1}{p}}\left|S_n\right| \leq \max_{2^{k_0-1}\leq n<2^{k_0}} n^{-\frac{1}{p}}\left|S_n\right|$$
$$\leq 2^{\frac{2}{p}}2^{-\frac{(k_0+1)}{p}} \max_{1\leq j<2^{k_0}}\left|S_j\right| \to 0 \ \ a.s. \quad \text{as } \ k_0 \to \infty,$$

which implies (1.2). The proof is completed. □

Remark 3.6. Since (3.7) implies (1.2) which is shown in the proof of Corollary 3.5, Theroem 3.4 and Corollary 3.5 extend and improve Theorem 1.1 for the case $\alpha \geq \beta$.

Conjecture 3.7. *The author conjectures that* (3.7) *in Theroem 3.4 can be replaced into* (3.1). *Despite our efforts to prove this conjecture, it is still an open problem.*

Remark 3.8. Compared with the results of Cai ([6, Theorem 2.1]) and Budsaba et al. ([4, Theorem 1 and 2]), our main results and those of Cai ([6]) and Budsaba et al. ([4]) do not completely overlap with each other, though the conditions of our results have some similarities to those of Cai ([6]) and Budsaba et al. ([4]).

Remark 3.9. The crucial tool used in this paper is the Rosenthal-type inequality (Lemma 2.1) for maximum partial sums of ρ^*-mixing sequence. As we know, for NA random variables and φ-mixing random variables, the above inequality also holds (see [18, 27] respectively). Therefore, the results in this paper also remain true for them.

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