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EFFICIENCY LOSS OF MIXED EQUILIBRIUM ASSOCIATED WITH ALTRUISTIC USERS AND LOGIT-BASED STOCHASTIC USERS IN TRANSPORTATION NETWORK

ABSTRACT

The efficiency loss of mixed equilibrium associated with two categories of users is investigated in this paper. The first category of users are altruistic users (AU) who have the same altruism coefficient and try to minimize their own perceived cost that assumed to be a linear combination of selfish component and altruistic component. The second category of users are Logit-based stochastic users (LSU) who choose the route according to the Logit-based stochastic user equilibrium (SUE) principle. The variational inequality (VI) model is used to formulate the mixed route choice behaviours associated with AU and LSU. The efficiency loss caused by the two categories of users is analytically derived and the relations to some network parameters are discussed. The numerical tests validate our analytical results. Our result takes the results in the existing literature as its special cases.

KEY WORDS

efficiency loss, mixed equilibrium, variational inequality, system optimum, demand ratio

1. INTRODUCTION

Since 1950s, it has been well known that the outcome of the user's selfish behaviour is generally not identical with the system optimum (SO). However, the gap was not known for a long time. In 1999 Koutsoupias and Papadimitriou presented the efficiency loss (price of anarchy) to measure the inefficiency for the user's selfish behaviour and defined it as the largest ratio between the total cost of Nash equilibrium and the total cost of an optimal solution achieved by centralized control [1]. Later, Roughgarden and Tardos introduced it into the transportation network and used it to quantify the gap between the user equilibrium (UE) and the SO [2]. After that, quantifying the efficiency loss of user's selfish behaviour in the transportation context has been an important aspect in the traffic science. The researchers have extended the above works in different aspects [3-9].

In the studies mentioned above, the authors assumed that each user has the same route-choose principle. In other words, each user tries to minimize her/his actual (perceived) travel cost. Several researchers studied the network simultaneously with heterogeneous users where different category users have different route-choose principles. Haurie and Marcotte investigated the network users belonging to some non-cooperative Cournot–Nash (CN) players, where the users belonging to the same CN player can cooperate fully with each other and different players will compete with each other. The users of one CN player aim to minimize their own total cost while competing with the users of other players [10]. Harker examined that the network users can be divided into different CN players and UE player and obtained a new network equilibrium model [11]. Recently, Yang and Zhang studied the existence of anonymous link tolls in network with UE-CN mixed equilibrium behaviours [12]. Liu et al. are concerned with the efficiency loss caused by the mixed equilibrium behaviour in a system with the advanced traveller information systems (ATIS), the users equipped with ATIS choose their route to minimize the total travel cost, while the unequipped ones make the route choice decisions on the base of minimizing their individual travel cost [13]. Guo and Yang first proved that any Pareto optimum can be decentralized into multiclass user equilibrium by positive anonymous link tolls. They further quantified the system performance gap when optimized by the two different criteria [14]. Yu and Huang investigated the efficiency loss of transportation network with UE-CN mixed equilibrium [15]. Karakostas et al. studied the effect of oblivious users using, as the measure of network performance, its price of anarchy [16].

Experiments have shown that even for simple games in controlled environments, the participants do not act selfishly; their behaviour can be either altruistic or malicious [17-18]. This paper aims to quantify the efficiency loss of a network with two categories of users. The AU choose their routes according to the UE principle by their perceived cost that is a linear combination of a selfish component and an altruistic component. The selfish component is the user's own actual travel cost, and the altruistic component is the increase in the travel cost the user causes to others (precise definition is given in Section 3). The LSU choose their routes aims to minimize their perceived costs according to the SUE principle. Section 2 introduces the notation and assumptions. Section 3 obtains the equivalent VI formulation of the mixed equilibrium associated with AU and LSU. Section 4 binds the efficiency loss of the mixed equilibrium by VI approach and investigates the relation between the upper bound and the network parameters. In section 5 a simple numerical example is provided. Finally, Section 6 gives the conclusions.

2. NOTATION AND ASSUMPTIONS

A transportation network $G=(N,A)$ is composed of a finite set of nodes N, and a finite set of directed links A. Let W be the set of all Origin-Destination (OD) pairs, R - the set of all paths in the network and R_w - the set of all paths between an OD pair $w \in W$. It is assumed that demand d_w between an OD pair $w \in W$ is a constant. Let $\mathbf{d}$ be the vector of demand in the transportation network G. Denote the flow on path $r \in R_w$, $w \in W$ as f_{rw}. Suppose that AU have the same altruism coefficient β and presume that the ratio of AU among all the users between each OD pair is identical, denoted by λ. Denote by f_{rw}^{AU} the flow of AU and by f_{rw}^{LSU} the flow of LSU on path $r \in R_w$. The vectors of path flows by AU and LSU are $\mathbf{f}^{AU} \equiv (\ldots, f_{r-1}^{AU}, f_r^{AU}, f_{r+1}^{AU}, \ldots)$ and $\mathbf{f}^{LSU} \equiv (\ldots, f_{r-1}^{LSU}, f_r^{LSU}, f_{r+1}^{LSU}, \ldots)$, respectively. By v_a^{AU} the AU flow on link a is denoted and v_a^{LSU} is the LSU flow on link a, while $\mathbf{v}^{AU} = (\ldots, v_{a-1}^{AU}, v_a^{AU}, v_{a+1}^{AU}, \ldots)$ and $\mathbf{v}^{LSU} = (\ldots, v_{a-1}^{LSU}, v_a^{LSU}, v_{a+1}^{LSU}, \ldots)$ are the vectors of the link flows by AU and LSU, respectively. Vector $\mathbf{v}_a \equiv (v_a^{AU}, v_a^{LSU})$ has components of all the flows on link a and $v_a = v_a^{AU} + v_a^{LSU}$ is the total flow on link a. We define $\mathbf{v} \equiv (\mathbf{v}^{AU}, \mathbf{v}^{LSU})$ and $\mathbf{f} \equiv (\mathbf{f}^{AU}, \mathbf{f}^{LSU})$. The link travel cost function $t_a(v_a)$, $a \in A$ is separable, differentiable, convex and monotonically increasing with the aggregate link flow v_a. Denote by $\mathbf{t}$ the vector of link travel cost in the transportation network G. The actual travel cost of the users on path $r \in R_w$ is

$$c_{rw} = \sum_{a \in A} t_a(v_a^{AU} + v_a^{LSU})\delta_{ar}^w, \; r \in R_w, \; w \in W$$

while $\delta_{ar}^w = 1$ if the path $r \in R_w$ traverses link $a \in A$, and $\delta_{ar}^w = 0$ otherwise. The perceived travel cost of LSU on path $r \in R_w$ (this travel cost is a psychological value; it may be larger than or less than the actual travel cost) will be denoted by C_{rw}^{LSU}.

For the sake of convenience, the flow conservation conditions and nonnegative constraint conditions are summarized as follows:

$$v_a^{AU} = \sum_{w \in W} \sum_{r \in R_w} f_{rw}^{AU} \delta_{ar}^w, \; a \in A, \tag{1}$$

$$\sum_{r \in R_w} f_{rw}^{AU} = \lambda d_w, \; w \in W \quad (\mu_w^{AU}), \tag{2}$$

$$f_{rw}^{AU} \geq 0, \; r \in R_w, \; w \in W, \tag{3}$$

$$v_a^{LSU} = \sum_{w \in W} \sum_{r \in R_w} f_{rw}^{LSU} \delta_{ar}^w, \; a \in A, \tag{4}$$

$$\sum_{r \in R_w} f_{rw}^{LSU} = (1-\lambda) d_w, \; w \in W \quad (\mu_w^{LSU}), \tag{5}$$

$$f_{rw}^{LSU} \geq 0, \; r \in R_w, \; w \in W. \tag{6}$$

where μ_w^{AU}, μ_w^{LSU} is the Lagrange multiplier of equations (2) and (5), respectively. Assume that

$$\Omega_f^{AU} = \{\mathbf{f}^{AU} \,|\, \mathbf{f}^{AU} \text{ satisfying formulas (1) - (3)}\},$$

$$\Omega_v^{AU} = \{\mathbf{v}^{AU} \,|\, \exists \mathbf{f}^{AU} \text{ satisfying formulas (1) - (3)}\};$$

$$\Omega_f^{LSU} = \{\mathbf{f}^{LSU} \,|\, \mathbf{f}^{LSU} \text{ satisfying formulas (4) - (6)}\}, \text{ and}$$

$$\Omega_v^{LSU} = \{\mathbf{v}^{LSU} \,|\, \exists \mathbf{f}^{LSU} \text{ satisfying formulas (4) - (6)}\}.$$

Obviously, Ω_f^{AU}, Ω_v^{AU}, Ω_f^{LSU}, Ω_v^{LSU} are all closed and convex sets.

3. MIXED EQUILIBRIUM MODEL ASSOCIATED WITH AU AND LSU

Experiments in economics have found that the users behave not entirely selfishly, even in simple games. Based on the results of Ledyard [17], Chen and Kempe [19], and the definition of the perceived cost of altruistic user with altruism coefficient $\beta(AU)$ is as follows:

Proposition 1 - *For given* $\beta(AU)$. $(\beta \in [-1,1])$ *and for given* $w \in W$, *there is a path* $r \in R_w$, $w \in W$ *to minimize the perceived cost function*

$$C_{rw}^{AU}(\mathbf{v}) = (1-\beta)\sum_{a \in A} t_a(v_a)\delta_{ar}^w + \beta \sum_{a \in A} (t_a(v_a)v_a)'\delta_{ar}^w =$$

$$= (1-\beta)\sum_{a \in r} t_a(v_a) + \beta \sum_{a \in r} (t_a(v_a)v_a)'$$

The term

$$\sum_{a \in r} t_a(v_a)$$

is the selfish part of the cost,

$$\sum_{a \in r} (t_a(v_a)v_a)'$$

is the altruistic part. $(t_a(v_a)v_a)'$ *denotes the derivative with respect to* v_a*. Notice that we can rewrite*

$$C_{rw}^{AU}(\mathbf{v}) = \sum_{a\in r} t_a(v_a) + \beta \sum_{a\in r} v_a t_a'(v_a).$$

Thus, the perceived cost of $\beta(AU)$ on link a in the network is $t_a^{AU}(\mathbf{v}) = t_a(v_a) + \beta v_a t_a'(v_a)$. Notice that if $\beta(AU) = 0$ then this coincides with selfishness; $\beta(AU) = 1$ corresponds to complete altruism; $\beta(AU) = -1$ means the users are completely spiteful. In this paper, we have supposed that AU have the same altruism coefficient $\beta \in [0,1]$.

The AU aim to minimize their personal perceived travel cost under the current routing decisions of the LSU, which is equivalent to solve

$$\min_{\mathbf{f}^{AU}\in\Omega^{AU}} \sum_{a\in A} \int_0^{v_a^{AU}} t_a^{AU}(v_a^{LSU} + x)dx. \qquad (7)$$

where the variables v_a^{LSU}, $a \in A$ are taken as fixed. If the function of $t_a^{AU}(v_a)$ is strictly increasing by link travel cost function $t_a(v_a)$ defined before, then the minimization problem (7) has a unique solution.

The LSU in the transportation network are considered now. In the SUE state, each utility-maximized user always chooses the minimum perceived travel cost path for travel [20]. The given path utility U_{rw}^{LSU} is related to its travel cost, then U_{rw}^{LSU} is given by

$$U_{rw}^{LSU} = -\theta C_{rw}^{LSU} = -\theta c_{rw}^{LSU} + \xi_{rw},\ r\in R_w,\ w\in W. \qquad (8)$$

where C_{rw}^{LSU} is the random perceived travel cost along the path, c_{rw}^{LSU} is the actual travel cost along the path as defined before, ξ_{rw} is a random term associated with the path under consideration and can be considered to represent the unobservable or unmeasurable factors of utility. A positive unit scaling parameter θ is related to the standard deviation of the random term and measures the sensitivity of path choices to travel cost, so $-\theta c_{rw}^{LSU}$ is the measure utility. If the P_{rw}^{LSU} denotes the probability of the LSU choosing path $r \in R_w$, then the utility maximization (perceived travel cost minimization) principle implies that

$$P_{rw}^{LSU} = \Pr(U_{rw}^{LSU} \geq U_{kw}^{LSU}, \forall k\in R_w),\ r\in R_w,\ w\in W. \qquad (9)$$

This choice probability has the following properties:

$$0 \leq P_{rw}^{LSU} \leq 1,\ r\in R_w,\ w\in W, \qquad (10)$$

$$\sum_{r\in R_w} P_{rw}^{LSU} = 1,\ w\in W. \qquad (11)$$

If the Logit-based model assumes that the random terms ξ_{rw} in (8) are independently and identically distributed Gumbel random variables, then the choice probability can be given by

$$P_{rw}^{LSU} = \frac{\exp(-\theta c_{rw}^{LSU})}{\sum_{l\in R_w}\exp(-\theta c_{lw}^{LSU})},\ r\in R_w,\ w\in W. \qquad (12)$$

and the path flow assignment can be given by

$$f_{rw}^{LSU} = (1-\lambda)d_w P_{rw}^{LSU},\ r\in R_w,\ w\in W. \qquad (13)$$

The AU choose their paths aiming to minimize their perceived cost according to the UE principle and the LSU choose their paths aiming to minimize their perceived cost according to the SUE principle. Then, at the state of mixed equilibrium associated with AU and LSU in the transportation network, the AU (LSU) travel cannot reduce their perceived cost by unilaterally changing their choice at the equilibrium. The condition of mixed equilibrium associated with AU and LSU in transportation network can be formulated as follows [20]:

$$(C_{rw}^{AU} - \mu_w^{AU})f_{rw}^{AU} = 0,\ C_{rw}^{AU} - \mu_w^{AU} \geq 0,\ r\in R_w,\ w\in W, \qquad (14)$$

$$f_{rw}^{LSU} = (1-\lambda)d_w \frac{\exp(-\theta c_{rw}^{LSU})}{\sum_{l\in R_w}\exp(-\theta c_{lw}^{LSU})},\ r\in R_w,\ w\in W. \qquad (15)$$

satisfying equations (1)-(6), where μ_w^{AU} is the minimal path perceived cost of OD pair w at mixed equilibrium for the AU. The mixed equilibrium can be formulated as VI by the following [21].

Lemma 1 - *Let* $(G,\mathbf{d},\mathbf{t},\beta,\lambda)$ *be a mixed instance associated with AU and LSU. If the separable link travel cost function* $t_a(v_a)$*,* $a\in A$ *is strictly increasing and convex, then mixed equilibrium of the instance* $(G,\mathbf{d},\mathbf{t},\beta,\lambda)$ *is equivalent with finding* $\bar{\mathbf{f}} = (\bar{\mathbf{f}}^{AU},\bar{\mathbf{f}}^{LSU})$*, such that for each* $f_{rw}^{AU}\in\Omega_f^{AU}$*,* $f_{rw}^{LSU}\in\Omega_f^{LSU}$

$$\begin{aligned}&\sum_{w\in W}\sum_{r\in R_w} C_{rw}^{AU}(\bar{\mathbf{f}})(f_{rw}^{AU} - \bar{f}_{rw}^{AU}) + \\ &\sum_{w\in W}\sum_{r\in R_w}\left(c_{rw}^{LSU}(\bar{\mathbf{f}}) + \frac{1}{\theta}\ln\frac{\bar{f}_{rw}^{LSU}}{(1-\lambda)d_w}\right)(f_{rw}^{LSU} - \bar{f}_{rw}^{LSU}) \geq 0\end{aligned} \qquad (16)$$

Proof: If $\bar{\mathbf{f}} = (\bar{\mathbf{f}}^{AU},\bar{\mathbf{f}}^{LSU})$ is the mixed equilibrium of the instance $(G,\mathbf{d},\mathbf{t},\beta,\lambda)$, then

$$(C_{rw}^{AU}(\bar{\mathbf{f}}) - \mu_w^{AU})\bar{f}_{rw}^{AU} = 0,$$
$$C_{rw}^{AU}(\bar{\mathbf{f}}) - \mu_w^{AU} \geq 0,\ r\in R_w,\ w\in W, \qquad (17)$$

$$\bar{f}_{rw}^{LSU} = (1-\lambda)d_w\frac{\exp(-\theta c_{rw}^{LSU})}{\sum_{l\in R_w}\exp(-\theta c_{rw}^{LSU})},\ r\in R_w,\ w\in W. \qquad (18)$$

By the relation of complementarity problem and VI, (17) can be rewritten as

$$C_{rw}^{AU}(\bar{\mathbf{f}})(f_{rw}^{AU} - \bar{f}_{rw}^{AU}) \geq 0,\ r\in R_w,\ w\in W. \qquad (19)$$

From (18) next follows [20]:

$$c_{rw}^{LSU}(\bar{\mathbf{f}}) + \frac{1}{\theta}\ln\frac{\bar{f}_{rw}^{LSU}}{(1-\lambda)d_w} - S(c^w) = 0 \qquad (20)$$

where c^w is the path travel cost in OD pair $w\in W$, $S(c^w)$ is the desired minimum perceived travel cost [20]. Then $\bar{\mathbf{f}}_{rw}^{LSU}$ is the solution of the following VI:

$$\begin{aligned}&\sum_{w\in W}\sum_{r\in R_w}\left(c_{rw}^{LSU}(\bar{\mathbf{f}}) + \frac{1}{\theta}\ln\frac{\bar{f}_{rw}^{LSU}}{(1-\lambda)d_w} - S(c^w)\right)\cdot \\ &\cdot(f_{rw}^{LSU} - \bar{f}_{rw}^{LSU}) \geq 0,\ \forall \bar{\mathbf{f}}^{LSU}\in\Omega_f^{LSU}\end{aligned} \qquad (21)$$

i.e.,

$$\begin{aligned}&\sum_{w\in W}\sum_{r\in R_w}\left(c_{rw}^{LSU}(\bar{\mathbf{f}}) + \frac{1}{\theta}\ln\frac{\bar{f}_{rw}^{LSU}}{(1-\lambda)d_w}\right)(f_{rw}^{LSU} - \bar{f}_{rw}^{LSU}) - \\ &-\sum_{w\in W}\sum_{r\in R_w} S(c^w)(f_{rw}^{LSU} - \bar{f}_{rw}^{LSU}) \geq 0,\ \forall \bar{\mathbf{f}}^{LSU}\in\Omega_f^{LSU}.\end{aligned} \qquad (22)$$

The second term of (22) is zero due to the flow conservation condition and the OD demand is constant. Then,

$$\sum_{w\in W}\sum_{r\in R_w}\left(c_{rw}^{LSU}(\bar{\mathbf{f}})+\frac{1}{\theta}\ln\frac{\bar{f}_{rw}^{LSU}}{(1-\lambda)d_w}\right)(f_{rw}^{LSU}-\bar{f}_{rw}^{LSU})\geq 0,$$
$$\forall \bar{\mathbf{f}}^{LSU}\in\Omega_f^{LSU}. \quad (23)$$

Thus, in view of (19) and (23), we can obtain that $\bar{\mathbf{f}}$ is the solution of (16).

If $\bar{\mathbf{f}}=(\bar{\mathbf{f}}^{AU},\bar{\mathbf{f}}^{LSU})$ is the solution of (16), by the Karush-Kuhn-Tucker conditions of VI, we can obtain:

$$(C_{rw}^{AU}(\bar{\mathbf{f}})-\mu_w^{AU})\bar{f}_{rw}^{AU}=0,\ C_{rw}^{AU}(\bar{\mathbf{f}})-\mu_w^{AU}\geq 0,$$
$$r\in R_w,\ w\in W \quad (24)$$

$$\left(c_{rw}^{LSU}(\bar{\mathbf{f}})+\frac{1}{\theta}\ln\frac{\bar{f}_{rw}^{LSU}}{(1-\lambda)d_w}-\mu_w^{LSU}\right)\bar{f}_{rw}^{LSU}=0,$$
$$r\in R_w,\ w\in W. \quad (25)$$

$$c_{rw}^{LSU}(\bar{\mathbf{f}})+\frac{1}{\theta}\ln\frac{\bar{f}_{rw}^{LSU}}{(1-\lambda)d_w}-\mu_w^{LSU}\geq 0,\ r\in R_w,\ w\in W. \quad (26)$$

where μ_w^{AU}, μ_w^{LSU} is the Lagrange multiplier of equation (2) and (5), respectively. For all $\bar{f}_{rw}^{LSU}>0$, $r\in R_w$, $w\in W$, according to (25), then

$$c_{rw}^{LSU}(\bar{\mathbf{f}})+\frac{1}{\theta}\ln\frac{\bar{f}_{rw}^{LSU}}{(1-\lambda)d_w}-\mu_w^{LSU}=0,\ r\in R_w,\ w\in W. \quad (27)$$

Thus,

$$\bar{f}_{rw}^{LSU}=(1-\lambda)d_w\exp(\theta\mu_w^{LSU}-\theta c_{rw}^{AU}(\bar{\mathbf{f}})),$$
$$r\in R_w,\ w\in W. \quad (28)$$

Sum up equation (28) and according to the flow conservation condition, we have

$$\mu_w^{LSU}=\frac{1}{\theta}\ln\frac{1}{\sum_{l\in R_w}\exp(-\theta c_{lw}^{LSU})}. \quad (29)$$

Substituting (29) into (27) yields

$$f_{rw}^{LSU}=(1-\lambda)d_w\frac{\exp(-\theta c_{rw}^{LSU})}{\sum_{l\in R_w}\exp(-\theta c_{rw}^{LSU})},\ r\in R_w,\ w\in W. \quad (30)$$

This completes the proof.

4. EFFICIENCY LOSS OF MIXED EQUILIBRIUM ASSOCIATED WITH AU AND LSU

Let $\mathbf{v}^{so}=(\mathbf{v}^{so,AU},\mathbf{v}^{so,LSU})$ and $\mathbf{v}_A^{so}=(\mathbf{v}_a^{so})$, $a\in A$ be the solution and the aggregate link flow of the following optimization problem, respectively:

$$\min_{\mathbf{v}\in\Omega}\sum_{a\in A}t_a(v_a)v_a, \quad (31)$$

where $\Omega=\Omega_v^{AU}\times\Omega_v^{LSU}$. Let $T(\mathbf{v}^{so})$ measure the minimal total travel cost of the transportation network, i.e., $T(\mathbf{v}^{so})$ is the total travel cost at system optimum. $\mathbf{f}^{so}$, $\mathbf{v}^{so}$ are the path flow and link flow at system optimum, respectively. Following the definition by Koutsoupias and Papadimitriou [1], the efficiency loss of the mixed equilibrium behaviour above is formulated as:

$$\rho(G,\mathbf{d},\mathbf{t},\beta,\lambda)=\frac{T^{mix}}{T^{SO}}=\frac{T(\bar{\mathbf{f}})}{T(\mathbf{f}^{SO})}=\frac{T(\bar{\mathbf{v}})}{T(\mathbf{v}^{SO})}. \quad (32)$$

where $T^{mix}=T(\bar{\mathbf{f}})=\sum_{w\in W}\sum_{r\in R_w}c_{rw}(\bar{\mathbf{f}})\bar{f}_{rw}$, or

$$T^{mix}=T(\bar{\mathbf{v}})=\sum_{a\in A}t_a(\bar{v}_a^{AU}+\bar{v}_a^{LSU})(\bar{v}_a^{AU}+\bar{v}_a^{LSU})=$$
$$=\sum_{a\in A}t_a(\bar{v}_a)\bar{v}_a.$$
$$T^{SO}=T(\mathbf{f}^{SO})=\sum_{w\in W}\sum_{r\in R_w}c_{rw}(\mathbf{f}^{SO})f_{rw}^{SO},\text{ or}$$
$$T^{SO}=T(\mathbf{v}^{SO})=\sum_{a\in A}t_a(v_a^{SO})v_a^{SO}.$$

Hence, $\rho(G,\mathbf{d},\mathbf{t},\beta,\lambda)\geq 1$. According to the original composition of each OD pair demand, the SO path flow $\mathbf{f}^{SO}=(\ldots,f_{r-1}^{SO},f_r^{SO},f_{r+1}^{SO},\ldots)$ can be decomposed into $\mathbf{f}^{SO,AU}=\lambda\mathbf{f}^{SO}$, $\mathbf{f}^{SO,LSU}=(1-\lambda)\mathbf{f}^{SO}$ and the SO link flow can be decomposed into $\mathbf{v}^{SO,AU}=\lambda\mathbf{v}^{SO}$, $\mathbf{v}^{SO,LSU}=(1-\lambda)\mathbf{v}^{SO}$. Replace f_{rw}^{AU} by $f_{rw}^{SO,AU}$ and replace f_{rw}^{LSU} by $f_{rw}^{SO,LSU}$ in (16), respectively. Then it can be concluded that

$$\sum_{w\in W}\sum_{r\in R_w}C_{rw}^{AU}(\bar{\mathbf{f}})(f_{rw}^{SO,AU}-\bar{f}_{rw}^{AU})+$$
$$+\sum_{w\in W}\sum_{r\in R_w}\left(c_{rw}^{LSU}(\bar{\mathbf{f}}^{AU},\bar{\mathbf{f}}^{LSU})+\frac{1}{\theta}\ln\frac{\bar{f}_{rw}^{LSU}}{(1-\lambda)d_w}\right)\cdot \quad (33)$$
$$\cdot(f_{rw}^{SO,LSU}-\bar{f}_{rw}^{LSU})\geq 0.$$

This leads to

$$\sum_{a\in A}(t_a(\bar{v}_a)+\beta\bar{v}_at'_a(\bar{v}_a))(v_a^{SO,AU}-\bar{v}_a^{AU})+$$
$$+\sum_{a\in A}t_a(\bar{v}_a)(v_a^{SO,LSU}-\bar{v}_a^{LSU})+ \quad (34)$$
$$+\frac{1}{\theta}\sum_{w\in W}\sum_{r\in R_w}\ln\frac{\bar{f}_{rw}^{LSU}}{(1-\lambda)d_w}(f_{rw}^{SO,LSU}-\bar{f}_{rw}^{LSU})\geq 0.$$

Thus,

$$T^{mix}\leq T^{SO}+\sum_{a\in A}v_a^{SO}(t_a(\bar{v}_a)-t_a(v_a^{SO}))+$$
$$+\beta\sum_{a\in A}\bar{v}_at'_a(\bar{v}_a)(\lambda v_a^{SO}-\bar{v}_a^{AU})+ \quad . \quad (35)$$
$$+\frac{1}{\theta}\sum_{w\in W}\sum_{r\in R_w}\ln\frac{\bar{f}_{rw}^{LSU}}{(1-\lambda)d_w}((1-\lambda)f_{rw}^{SO}-\bar{f}_{rw}^{LSU}).$$

If an upper bound for the sum of the last three terms on the right hand side (RHS) of (35) can be found, then we bind the overall cost inefficiency of the mixed equilibrium associated with AU and LSU in the transportation network. For the sum of the second and the three terms on RHS of (35), its upper bound can be obtained by the following. A parameter is defined:

$$\varphi_a(t_a,\bar{v}_a^{AU},\bar{v}_a^{LSU},\beta,\lambda)=$$
$$=\max_{v_a\geq 0}\frac{(t_a(\bar{v}_a)-t_a(v_a))v_a+\beta\bar{v}_at'_a(\bar{v}_a)(\lambda v_a-\bar{v}_a^{AU})}{t_a(\bar{v}_a)\bar{v}_a}. \quad (36)$$

Note that the denominator on RHS of (36) is given and fixed. This indicates that we only need to obtain the maximum of the numerator on RHS of (36). Let

$$F(v_a)=(t_a(\bar{v}_a)-t_a(v_a))v_a+\beta\bar{v}_at'_a(\bar{v}_a)(\lambda v_a-\bar{v}_a^{AU}),$$
$$v_a\in[0,+\infty).$$

Obviously, $F(v_a)$ is continuous in domain, so $F(v_a)$ has the maximum within $v_a\in[0,\bar{v}_a]$ as long as we can obtain $F'(v_a)\leq 0$ under the condition $v_a\geq\bar{v}_a$. It is easy to obtain $F'(v_a)=t_a(\bar{v}_a)+\beta\lambda\bar{v}_at'_a(\bar{v}_a)-t_a(v_a)-v_at'_a(v_a)$ and $F''(v_a)=-2t'_a(v_a)-v_at''_a(v_a)$. If $t_a(v_a)$ is convex, monotone increasing function, then we have $F''(v_a) \leq 0$, when $v_a \geq \bar{v}_a \geq 0$. This means $F'(a)$ decreases with the variable $v_a \in [\bar{v}_a, +\infty)$. Since $F'(\bar{v}_a) = (\beta\lambda - 1)\bar{v}_a t'_a(\bar{v}_a) \leq 0$, we get $F'(v_a) \leq F'(\bar{v}_a) \leq 0$ under the condition $v_a \geq \bar{v}_a$. Thus we conclude that in (36) the maximum is obtained within $[0, \bar{v}_a]$.

For each given class of link travel cost function L (a family of linear cost functions or polynomials of a certain degree), let

$$\varphi(L,\beta,\lambda) = \max_{t_a \in C, a \in A} \varphi_a(t_a, \bar{v}_a^{AU}, \bar{v}_a^{LSU}, \beta, \lambda). \quad (37)$$

with the definitions (36) and (37), we have

$$\sum_{a \in A} v_a^{SO}(t_a(\bar{v}_a) - t_a(v_a^{SO})) + \beta \sum_{a \in A} \bar{v}_a t'_a(\bar{v}_a)(\lambda v_a^{SO} - \bar{v}_a^{AU}) \leq \varphi(L,\beta,\lambda) T^{mix}. \quad (38)$$

The upper bound of the fourth term on RHS of (35) can be obtained by solving the following maximization problem using the method in [6].

Lemma 2 - *Consider the following maximization problem*

$$\max Z(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} (y_i - x_i) \ln \frac{x_i}{C}, \quad (39)$$

subject to

$$\sum_{i=1}^{n} x_i = C, \; \sum_{i=1}^{n} y_i = C, \; x_i, y_i \geq 0, \; i = 1,2,\ldots,n, \quad (40)$$

where $C > 0$ is a constant. The optimal value of this problem is $Z_{\max} = kC$, where k solves equation $ke^{k+1} = (n-1)$, with e being the base of natural logarithm.

From Lemma 2, we can obtain

$$\sum_{r \in R_w} \ln \frac{\bar{f}_{rw}^{LSU}}{(1-\lambda)d_w} (f_{rw}^{SO,LSU} - \bar{f}_{rw}^{LSU}) \leq k_w (1-\lambda) d_w, \quad (41)$$

where k_w solves $k_w e^{k_w+1} = |R_w| - 1$, $w \in W$. Substituting (38) and (41) into (35), it yields

$$T^{mix} \leq T^{SO} + \varphi(L,\beta,\lambda) T^{mix} + \frac{1}{\theta} \sum_{w \in W} k_w (1-\lambda) d_w. \quad (42)$$

Let

$$D = \sum_{w \in W} (1-\lambda) d_w$$

be the total stochastic traffic demand in network and

$$\bar{k} = \sum_{w \in W} \frac{(1-\lambda)d_w}{D} k_w,$$

then (42) can be rewritten as

$$T^{mix} \leq T^{SO} + \varphi(L,\beta,\lambda) T^{mix} + \frac{1}{\theta} \bar{k} D. \quad (43)$$

If we define

$$\bar{c} = \frac{T^{SO}}{\sum_{w \in W} d_w}$$

as the actual average travel cost of all network users at system optimum, then we have the following theorem.

Theorem 1 - *For a given separable link travel cost function class L. Let each instance $t_a(v_a) \in L$ being a differentiable, convex and monotonically increasing function of the aggregate link flow v_a. Let $(G, \mathbf{d}, \mathbf{t}, \beta, \lambda)$ be a mixed instance associated with AU and LSU. If T^{mix} is the total actual travel cost at the mixed equilibrium, and T^{SO} is the minimum total actual travel cost, then*

$$\rho(G, \mathbf{d}, \mathbf{t}, \beta, \lambda) = \frac{T^{mix}}{T^{SO}} \leq \left(\frac{1}{1 - \varphi(L,\beta,\lambda)}\right)\left(1 + \frac{(1-\lambda)\bar{k}}{\theta \bar{c}}\right). \quad (44)$$

Theorem 1 states that the upper bound of the efficiency loss for the mixed equilibrium associated with AU and LSU in transportation network with fixed demand depends on six parameters, namely $\varphi(L,\beta,\lambda)$, θ, $\bar{k}$, $\bar{c}$, β, λ. Parameter $\varphi(L,\beta,\lambda) \leq 1$ is a dimensionless number of the efficiency loss depending on the link travel cost functions, the altruism coefficient and the demand ratio. The upper bound of the efficiency loss is a monotonically increasing function of $\varphi(L,\beta,\lambda)$. Parameter θ in its original meaning is related to the standard error of the distribution of the perceived path travel costs [20], and the Logit-based model assumes that all paths in the network have the same standard error. Specifically,

$$\theta = \frac{\pi}{\sqrt{6}\sigma},$$

where σ is the common standard deviation of the perceived path travel costs. The upper bound of the inefficiency is a monotonically decreasing function of θ. When $\theta \to +\infty$,

$$\rho(G, \mathbf{d}, \mathbf{t}, \beta, \lambda) \leq \frac{1}{1 - \varphi(L,\beta,\lambda)}$$

then the model becomes the partly uniform altruism traffic assignment problem. Parameter $\bar{k}$ is a dimensionless coefficient increasing with the number of feasible paths and thus reflects the degree of network complexity. Equation (44) also states that the upper bound of the efficiency loss is increasing with network complexity. Since

$$\bar{c} = \frac{T^{SO}}{\sum_{w \in W} d_w},$$

then the upper bound of the efficiency loss decreases with the actual average travel cost and increases with the total traffic demand. Parameter λ is the demand ratio of the altruism users in the transportation network. The upper bound of the efficiency loss is a monotonically decreasing function of λ. When $\lambda = 0$,

$$\rho \leq \left(\frac{1}{1 - \varphi(L)}\right)\left(1 + \frac{\bar{k}}{\theta \bar{c}}\right),$$

which is the result in [6], when $\lambda = 1$,

$$\rho \leq \frac{1}{1 - \varphi(L,\beta)},$$

thus the model becomes a completely uniform altruism traffic assignment, which is the result in [22].

The efficiency loss bound given in Theorem 1 is the worst-case measure for the mixed equilibrium model, taking over all the possible instances. In fact, the actual ratio of the total cost at equilibrium to the SO total cost can be substantially small. Indeed, in transportation network, the free-flow travel cost is usually not a negligible fraction. Considering this, we can present a parameterized and improved bound on the equilibrium inefficiency, as done by Correa et al. in [3].

Theorem 2 - *For a given separable link travel cost function class L. Let each instance $t_a(v_a) \in L$ being a differentiable, convex and monotonically increasing function of the aggregate link flow v_a and $t_a^0 = t_a^0(0) \geq \eta(\bar{\mathbf{v}}) t_a(\bar{v}_a)$ for all $a \in A$ with constant $0 \leq \eta(\bar{\mathbf{v}}) \leq 1$. Let $(G, \mathbf{d}, \mathbf{t}, \beta, \lambda)$ be a mixed instance associated with AU and LSU. If T^{mix} is the total actual travel cost at the mixed equilibrium, and T^{SO} is the minimum total actual travel cost, then*

$$\rho(G, \mathbf{d}, \mathbf{t}, \beta, \lambda) = \frac{T^{mix}}{T^{SO}} \leq \leq \left(\frac{1}{1 - (\eta(\bar{\mathbf{v}}))\varphi(L, \beta, \lambda)}\right)\left(1 + \frac{(1-\lambda)\bar{k}}{\theta \bar{c}}\right). \quad (45)$$

5. EFFICIENCY LOSS EXAMPLE OF MIXED EQUILIBRIUM ASSOCIATED WITH AU AND LSU

Consider a directed graph consisting of two nodes and two links (*Figure 1*).

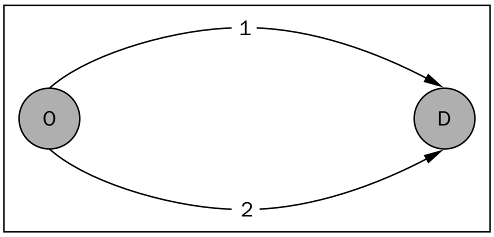

Figure 1 - The network used in the example

The link travel cost functions are defined as $t_1 = v_1$, $t_2 = 1$, respectively. There is one OD pair with fixed demand $d = 1$. Supposing demand ratio $\lambda = 0.8$, altruism coefficient $\beta = 0.1$, parameter $\theta = 1$.

According to (31), the SO link flow solution can be obtained by solving the following minimization problem:

$$\min (v_1)^2 + v_2$$
$$\text{s.t. } v_1 + v_2 = 1$$
$$v_1, v_2 \geq 0$$

The optimal solution is $v_1^{SO} = 0.5$, $v_2^{SO} = 0.5$ and the total travel cost is 0.75.

The perceived cost of AU on link *a* in the network is $t_a^{AU} = t_a(v_a) + \beta v_a t'_a(v_a)$. In this example the altruism coefficient $\beta = 0.1$ and $t_1 = v_1$, $t_2 = 1$. Path 1 is the link 1 and path 2 is the link 2. So the perceived cost of AU on path *i*, $i = 1,2$ is the perceived cost on link *i*, $i = 1,2$, i.e., $t_1^{AU} = v_1^{AU} + v_1^{LSU} + 0.1 \cdot (v_1^{AU} + v_1^{LSU}) \cdot 1 = 1.1 v_1^{AU} + 1.1 v_1^{LSU}$, $t_2^{AU} = 1 + 0.1 \cdot (v_2^{AU} + v_2^{LSU}) \cdot 0 = 1$. According to (14) and (15), the mixed equilibrium is obtained by solving the following equations simultaneously:

$$(1.1 v_1^{AU} + 1.1 v_1^{LSU} - \mu^{AU}) v_1^{AU} = 0,$$
$$1.1 v_1^{AU} + 1.1 v_1^{LSU} - \mu^{AU} \geq 0.$$
$$(1 - \mu^{AU}) v_2^{AU} = 0, \ 1 - \mu^{AU} \geq 0.$$
$$v_1^{LSU} = 0.2 \frac{\exp(-v_1^{AU} - v_1^{LSU})}{\exp(-v_1^{AU} - v_1^{LSU}) + \exp(-1)}.$$
$$v_2^{LSU} = 0.2 \frac{\exp(-1)}{\exp(-v_1^{AU} - v_1^{LSU}) + \exp(-1)}.$$

The mixed equilibrium solution is $\bar{v}_1^{AU} = 0.8$, $\bar{v}_2^{AU} = 0$, $\bar{v}_1^{LSU} = 0.1048$, $\bar{v}_2^{LSU} = 0.0952$, $\mu^{AU} = 0.9953$. The aggregate link flow is $\bar{v}_1 = 0.9048$, $\bar{v}_2 = 0.0952$ which generates the system total travel cost 0.9139. Thus, the efficiency loss is

$$\rho = \frac{0.9139}{0.75} = 1.2185.$$

Based on definition

$$\varphi_a(t_a, \bar{v}_a^{AU}, \bar{v}_a^{LSU}, \beta, \lambda) = = \max_{v_a \geq 0} \frac{(t_a(\bar{v}_a) - t_a(v_a))v_a + \beta \bar{v}_a t'_a(\bar{v}_a)(\lambda v_a - \bar{v}_a^{AU})}{t_a(\bar{v}_a)\bar{v}_a}$$

and $t_1 = v_1$, $t_2 = 1$, we can obtain

$$\varphi_1(t_1, \bar{v}_1^{AU}, \bar{v}_1^{LSU}, \beta, \lambda) = = \max_{v_a \geq 0} \frac{(0.9048 - v_1)v_1 + 0.1 \cdot 0.9048 \cdot 1 \cdot (0.8 v_1 - 0.8)}{0.9048 \cdot 0.9048} \quad (46)$$

while $v_1 = 0.4886$ the problem (46) reaches at the maximum $\varphi_1 = 0.2032$. It is easy to obtain that $\varphi_2 = 0$. Then, $\varphi(L, \beta, \lambda) = \max\{\varphi_1, \varphi_2\} = 0.2032$. Because k_w satisfies equation $k_w e^{k_w + 1} = |R_w| - 1 = 2 - 1 = 1$, then $k_w = 0.2785$. It is easy to obtain that $\bar{k} = 0.2785$ by

$$\bar{k} = \sum_{w \in W} \frac{(1-\lambda)d_w}{D} k_w$$

and

$$D = \sum_{w \in W} (1-\lambda) d_w, \ \bar{c} = 0.75$$

by

$$\bar{c} = \frac{T^{SO}}{\sum_{w \in W} d_w}.$$

Thus, the bound becomes

$$\rho < \left(\frac{1}{1 - \varphi(L, \beta, \lambda)}\right)\left(1 + \frac{(1-\lambda)\bar{k}}{\theta \bar{c}}\right) = 1.3482$$

according to Theorem 1.

6. CONCLUSION

This paper has investigated the efficiency loss in a transportation network associated with AU and LSU. A variational inequality model is presented to formulate the route choice behaviours associated with AU and LSU with fixed demand. The analytical results show that the upper bounds of the efficiency loss of mixed equilibrium depend on the type of link travel cost function, the altruism coefficient, the demand ratio, the network complexity, the travel demand and the degree of travel perception error on travel cost. It is shown that our result takes the results in [6] and [22] as its special cases. Our ongoing work is to explore the efficiency loss of mixed equilibrium associated with AU and LSU in transportation network with elastic demand.

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摘要

利他用户和LOGIT型随机用户混合均衡交通网络的效率损失

研究了两类不同用户组成的混合均衡交通网络的效率损失问题·第一类用户是按照最小化理解成本出行的利他用户，所有利他用户具有相同利他系数且理解成本是自私项和利他项的线性组合.第二类用户是按照Logit随机用户均衡原则选择出行路径的Logit型随机用户. 构建了刻画这类混合行为的变分不等式模型. 运用解析推导方法得到了这两类用户构成的交通网络的效率损失，并讨论效率损失和网络参数之间的关系. 给出了数值算例验证了我们的结论. 我们的研究结果以现有文献的结论为特例.

关键词

效率损失，混合均衡，变分不等式，系统最优，需求比例

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