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A NOTE ON THE NUMBER OF $D(4)$-QUINTUPLES

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Abstract. In this paper we will significantly improve the known bound on the number of $D(4)$-quintuples, illustrating the elegant use of the results the authors proved in [1] together with more efficient way of counting the number of m-tuples that was introduced in [5]. More precisely, we prove that there are at most $7 \cdot 10^{36}$ $D(4)$-quintuples.

1. Introduction

Let n be a non-zero integer. A set $\{a_1, a_2, \dots, a_m\}$ of m positive integers is called a $D(n)$-m-tuple if $a_i a_j + n$ is a perfect square for all i, j with $1 \leq i < j \leq m$. The problem of finding such sets has a long and rich history. To see all details, together with references, one should visit [2]. Here we will consider the case $n = 4$.

In the case $n = 4$ there is a conjecture, which appeared for the first time in [4], that if $\{a, b, c, d\}$ is a $D(4)$-quadruple with $a < b < c < d$, then

$$d = a + b + c + \frac{1}{2}(abc + rst),$$

where r, s and t are positive integers defined by $r^2 = ab + 4$, $s^2 = ac + 4$ and $t^2 = bc + 4$. If in a $D(4)$-quadruple the largest element is defined in such way, we call that quadruple a regular one. The conjecture is that all $D(4)$-quadruples are regular. It furthermore implies that there does not exist a $D(4)$-quintuple.

In recent years the second author [6] proved that there does not exist a $D(4)$-sextuple and that there are only finitely many quintuples. He furthermore [7] proved that irregular quadruple cannot be extended to a quintuple with a larger element and [8] that there are at most 10^{323} quintuples. Here we will significantly improve that result proving the following theorem. However, this bound is still to large to solve the problem of existence of $D(4)$-quintuples completely.

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THEOREM 1.1. *The number of $D(4)$-quintuples is less than* $7 \cdot 10^{36}$.

In the proof we will present the use of recent results proven in [1] and [5], so we will not keep this paper self-contained. However, it should not be a problem, because both papers are available. Also, here we will present the most recent results and methods in solving those kind of problems which make this paper up-to-date.

2. Improving the bounds of elements

In this section we will consider a $D(4)$-quintuple $\{a, b, c, d, e\}$ such that $a < b < c < d < e$. From [7] we first know that $\{a, b, c, d\}$ is a regular quadruple and from [8, Proposition 2] we know that there are at most 4 ways to extend a quadruple to a quintuple with a larger element. We will use both of those results later.

Here we will firstly improve the bounds from [8] of elements b and d in a $D(4)$-quintuple $\{a, b, c, d, e\}$ such that $a < b < c < d < e$. From [1, Lemma 3] it is enough to consider $b > 10^4$ which we will assume from now on. We will need the following definition.

DEFINITION 2.1. *Let $\{a, b, c\}$ be a $D(4)$-triple such that $a < b < c$.*

- *We call $\{a, b, c\}$ a triple of the first kind if $c > b^5$.*
- *We call $\{a, b, c\}$ a triple of the second kind if $b > 4a$ and $c > b^2$.*
- *We call $\{a, b, c\}$ a triple of the third kind if $b > 4a$ and $b^{5/3} < c < b^2$.*
- *We call $\{a, b, c\}$ a triple of the fourth kind if $b > 4a$ and $b^{4/3} < c < b^{5/3}$.*

If $\{a, b, c\}$ is a triple of the first, second, third or fourth kind, we call that triple a standard $D(4)$-triple.

PROPOSITION 2.2. *Every $D(4)$-quadruple $\{a, b, c, d\}$ with $a < b < c < d$ contains a standard triple $\{A, B, C\}$ such that $A < B < C = d$.*

PROOF. If $\{a, b, c, d\}$ is irregular $D(4)$-quadruple, then it easily implies $c > 4a$ and $d > c^2$, so $\{a, c, d\}$ is of the second kind.

Let now $\{a, b, c, d\}$ be a regular quadruple and $b > 4a$. Then, $d > abc > b^2$, so $\{a, b, d\}$ is of the second kind.

If $\{a, b, c, d\}$ is a regular quadruple and $b \leq 4a < 5a$, then from [1, Lemma 1] we know that $c = c_\nu^\pm$ where

$$c_0^+ = 0,\ c_1^+ = a + b + 2r,\ c_{\nu+2}^+ = (ab + 2)c_{\nu+1}^+ - c_\nu^+ + 2(a + b),$$

$$c_0^- = 0,\ c_1^- = a + b - 2r,\ c_{\nu+2}^- = (ab + 2)c_{\nu+1}^- - c_\nu^- + 2(a + b).$$

Here $r = \sqrt{ab + 4}$. Now if $c > 4b^3$, then $d > abc > b^5$ and $\{a, b, d\}$ is of the first kind. If $c \leq 4b^3$, then it is easy to check that $c = c_1^+$, $c = c_2^-$ or $c = c_2^+$. If $c = c_1^+$ we have $c = a + b + 2r$ and $4a < c < 4b$ which implies $d > abc > c^2$. So $\{a, c, d\}$ is of the second kind. If $c = c_2^-$, then $d = c_3^-$ which implies $c^{5/3} < d < c^2$, so $\{a, c, d\}$ is of the third kind. In the case $c = c_2^+$ we have $d = c_3^+$ which implies $c^{4/3} < d < c^{5/3}$, so $\{a, c, d\}$ is of the fourth kind. □

We are now ready to prove the main result of this section.

PROPOSITION 2.3. *Let $\{a, b, c, d, e\}$ be a $D(4)$-quintuple with $a < b < c < d < e$. Then $\{a, b, c, d\}$ is a regular quadruple and it contains the standard triple $\{A, B, C\}$ such that $A < B < C = d$. Moreover,*

(i) *$\{a, b, c, d\}$ cannot contain the triple $\{A, B, d\}$ of the first kind.*
(ii) *if $\{a, b, c, d\}$ contains the triple $\{A, B, d\}$ of the second kind, then $d < 10^{89}$ and $b < 3.17 \cdot 10^{44}$.*
(iii) *if $\{a, b, c, d\}$ contains the triple $\{A, B, d\}$ of the third kind, then $d < 10^{66}$ and $b < 6.33 \cdot 10^{16}$.*
(iv) *if $\{a, b, c, d\}$ contains the triple $\{A, B, d\}$ of the fourth kind, then $d < 10^{59}$ and $b < 1.1 \cdot 10^{12}$.*

PROOF. We will firstly use congruence method more carefully, similarly as it was done in [10]. In [8] the second author proved that if we have the triple $\{A, B, C = d\}$ in a quintuple, such that $A < B < C$, then we have to solve finitely many equations of the form $V_{2m} = W_{2n}$, where (V_m) and (W_n) are binary recurrence sequences. More precisely, the second author proved there, that we have to consider only the case with even indices when we have an extension to a quintuple. Considering congruences modulo C^2 we get the lower bound on m. Precisely, $V_{2m} = W_{2n}$ for $n > 2$ implies $m > 0.495B^{-0.5}C^{0.5}$. Here we use $B \geq b > 10^4$ and $n < m < 2n$. So we have

$$Am^2 \pm Sm \equiv Bn^2 \pm Tn \pmod{C},$$

where $S = \sqrt{AC + 4}$ and $T = \sqrt{BC + 4}$. Let us assume $m \leq 0.495B^{-0.5}C^{0.5}$. Then it is easy to see that absolute value of both sides of the congruence are less than C and have the same sign (for example $Bn^2 < B \cdot 0.495^2 B^{-1} C < C/4$) so we actually have an equation here, i.e.

$$Am^2 - Bn^2 = \pm(Tn - Sm).$$

That implies

$$4m^2 - 4n^2 = (C \pm (Tn + Sm))(Bn^2 - Am^2).$$

If $Bn^2 = Am^2$, then $4m^2 - 4n^2 = 0$, or $m = n$ which is contradiction. So we must have

$$4(m^2 - n^2) \geq |C \pm (Tn + Sm)|.$$

The case with sign '+' gives us $4m^2 > C$ which is contradiction to $m \leq 0.495B^{-0.5}C^{0.5}$. Let us now consider the case with sign '-'. We have

$$4(m^2 - n^2) \geq C - (Tn + Sm)$$

which yields

$$C \leq Tn + Sm + 4(m^2 - n^2) \leq 2Tm + 4 \cdot 0.75m^2 <$$

$$< 0.99(BC+4)^{0.5}B^{-0.5}C^{0.5} + 0.736B^{-1}C < C$$

which gives us contradiction again, so we must have $m > 0.495B^{-0.5}C^{0.5}$.

(*i*) Here we combine this lower bound on m together with Lemma 6, Lemma 7 and the proof of Proposition 1 from [1]. The only difference is that we have

$$0.00325a(a')^{-1}b^{-1}(b-a)^{-2}c > 0.00325a(4b)^{-1}b^{-3}c > 0.0008125b^{-4}c.$$

It implies

$$\frac{0.495b^{-0.5}d^{0.5}}{2} < \frac{\log(128.08b^6d^2)\log(0.00041b^2d^2)}{\log(bd)\log(0.0008125b^{-4}d)}.$$

From $d > b^5$, and using that the right hand side is decreasing in d for $d > b^5$, we get

$$0.495b^2 < \frac{2\log(128.08b^{16})\log(0.00041b^{12})}{\log(b^6)\log(0.0008125b)}$$

which cannot be satisfied for $b > 10^4$, so this case cannot appear.

In the cases (*ii*)-(*iv*) we get the upper bound

$$\frac{2m}{\log(2m+1)} < 6.543\cdot 10^{15}\log^2 C$$

as it was done in [8, Lemma 7].

(*ii*) Here we have $d = C > B^2$ which implies

$$2m > 0.99\cdot C^{-0.25}C^{0.5} = 0.99C^{0.25}.$$

Then we have

$$\frac{0.99d^{0.25}}{\log(0.99d^{0.25}+1)} < 6.543\cdot 10^{15}\log^2 d$$

or $d < 10^{89}$. That also yields, using $b^2 < d$, $b < 3.17\cdot 10^{44}$.

In (*iii*) it is easy to see that $d = c_3^- > b^4/16 > b^3$ which implies $2m > 0.99d^{1/3}$. Then we have

$$\frac{0.99d^{1/3}}{\log(0.99d^{1/3}+1)} < 6.543\cdot 10^{15}\log^2 d$$

which gives us $d < 10^{66}$ and $b < 6.33\cdot 10^{16}$.

In the case of (*iv*) from $d = c_3^+ > b^5/16 > b^4$ we have $2m > 0.99d^{3/8}$. Then we have

$$\frac{0.99d^{3/8}}{\log(0.99d^{3/8}+1)} < 6.543\cdot 10^{15}\log^2 d$$

which gives us $d < 10^{59}$ and $b < 1.1\cdot 10^{12}$. ∎

3. Counting the number of quintuples

In this section we will prove Theorem 1.1. To prove this we will use mostly the methods from [5] where it was introduced the more efficient way of counting using divisor sums. Here we can use lemmas exactly the same (Lemma 3.1) or similar to Lemma 3.5 from [5] because there are at most $2^{\omega(n)+2}$ solutions of the congruence $x^2 \equiv 4 \pmod{n}$ satisfying $0 < x < n$ (see [9, Chapter 5, 4g]). Here the $\omega(n)$ denotes the number of distinct prime factors of n.

Lemma 3.1. *If $N \geq 3$, then*

$$\sum_{n=1}^{N} 2^{\omega(n)} < N(\log N + 1).$$

Lemma 3.2.

$$\sum_{n=3}^{N} d(n^2 - 4) < 4N(\log^2 N + 4\log N + 2),$$

where $d(n)$ is the number of positive divisors of n.

The last Lemma can be used in the case when we get an upper bound N on $r = \sqrt{ab+4}$, because it implies that the number of $D(4)$-pairs $\{a, b\}$ with $a < b$ is less than $2N(\log^2 N + 4\log N + 2)$.

We will count the number of possible quintuples $\{a, b, c, d, e\}$ such that $a < b < c < d < e$ considering the cases of standard triples from the previous section. As we said before, we will use that $\{a, b, c, d\}$ is a regular quadruple, that there are at most 4 ways to extend it to a quintuple with a larger element and that $b > 10^4$.

(ii) Let now $\{a, b, c, d\}$ contains a triple of the second kind. Then we know that $d < 10^{89}$ and $b < 3.17 \cdot 10^{44}$. Here we will consider two possibilities from [3, Lemma 1] where authors proved that $c = a + b + 2r$ or $c > ab$ but we will also consider the subcases $ab < c \leq a^2b^2$ and $c > a^2b^2$.

First if $c > a^2b^2$ we have $d > abc > a^3b^3 > 0.99r^6$ which yields $r < 6.83 \cdot 10^{14} = N_1$. Then the number of pairs $\{a, b\}$ with $a < b$ is less than $2N_1(\log^2 N_1 + 4\log N_1 + 2)$ from Lemma 3.2. For a fixed pair $\{a, b\}$, the element c which extends it to a triple $\{a, b, c\}$ belongs to the union of finitely many binary recurrent sequences, and the number of those sequences is less than or equal to the number of solutions of the congruence $t_0^2 \equiv 4 \pmod{b}$ with $|t_0| < b$. The number of this is less than $8 \cdot 2^{\omega(b)}$. In every sequence we have $t_\nu = \sqrt{bc_\nu + 4} > 2(r-1)^{\nu-1}$ which with the upper bound on $c_\nu < d < 10^{89}$ and $b > 10^4$ gives us $\nu \leq 24$. So in each sequence we can have at most 24 elements. Because the product of the first 30 primes is greater than $3.17 \cdot 10^{44}$ we have that the number of sequences is less than $8 \cdot 2^2 9 < 4.3 \cdot 10^9$. So the number of possible quintuples in this case is less than

$$2N_1(\log^2 N_1 + 4\log N_1 + 2) \cdot 4.3 \cdot 10^9 \cdot 24 \cdot 4 < 7.37 \cdot 10^{29}.$$

Secondly, if $ab < ab \leq a^2b^2$, we have $d > abc > a^2b^2 > 0.99r^4$ which yields $r < 1.79 \cdot 10^{22} = N_2$. Then the number of pairs $\{a,b\}$ with $a < b$ is less than $2N_2(\log^2 N_2 + 4\log N_2 + 2)$. For a fixed pair $\{a,b\}$ as before we have at most $8 \cdot 2^{\omega(b)}$ sequences. Again the product of the first 30 primes exceeds $3.17 \cdot 10^{44}$ and therefore the number of sequences is less than $8 \cdot 2^2 9 < 4.3 \cdot 10^9$. But now, from $c \leq a^2b^2$ we get that in every sequence we can have at most 4 elements. So the number of possible quintuples in this case is less than

$$2N_2(\log^2 N_2 + 4\log N_2 + 2) \cdot 4.3 \cdot 10^9 \cdot 4 \cdot 4 < 6.98 \cdot 10^{36}.$$

In the last subcase when $c = a + b + 2r$, we have $d > abc > (r^2 - 4)(3r + 1)$ which implies $r < 3.22 \cdot 10^{29} = N_3$. Because c and d are unique here we have that the number of quintuples is less than

$$2N_3(\log^2 N_3 + 4\log N_3 + 2) \cdot 4 < 1.26 \cdot 10^{34}.$$

(iii) If $\{a,b,c,d\}$ contains a triple of the third kind we have from Proposition 2.3 that $b < 6.33 \cdot 10^{16} = N_4$. Since c and d are unique here, we have from Lemma 3.1 that the number of quintuples is less than

$$4N_4(\log N_4 + 1) \cdot 4 < 4.02 \cdot 10^{19}.$$

(iv) Finally, if $\{a,b,c,d\}$ contains a triple of the third kind we have $b < 1.1 \cdot 10^{12} = N_5$. Since c and d are again unique in this case, from Lemma 3.1 we have that the number of quintuples is less than

$$4N_5(\log N_5 + 1) \cdot 4 < 5.06 \cdot 10^{14}.$$

If we sum up everything, we have just proved that the number of $D(4)$-quintuples is less than

$$7.37 \cdot 10^{29} + 6.98 \cdot 10^{36} + 1.26 \cdot 10^{34} + 4.02 \cdot 10^{19} + 5.06 \cdot 10^{14} < 7 \cdot 10^{36}$$

which finishes the proof of Theorem 1.1.

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O broju $D(4)$-petorki

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SAŽETAK. Elegantnom primjenom rezultata iz [1] i efikasnijom metodom prebrojavanja m-torki predstavljenoj u [5], u ovom članku značajno smo poboljšali najbolju prethodno poznatu ogradu za broj $D(4)$-petorki, Točnije, dokazali smo da postoji najviše $7 \cdot 10^{36}$ $D(4)$-petorki.

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