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THE PROBLEM OF DIOPHANTUS FOR INTEGERS OF $\mathbb{Q}(\sqrt{-3})$

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Abstract. We solve the problem of Diophantus for integers of the quadratic field $\mathbb{Q}(\sqrt{-3})$ by finding a $D(z)$-quadruple in $\mathbb{Z}[(1+\sqrt{-3})/2]$ for each z that can be represented as a difference of two squares of integers in $\mathbb{Q}(\sqrt{-3})$, up to finitely many possible exceptions.

1. Introduction and preliminaries

Let R be a commutative ring with unity 1 and $n \in R$. The set of nonzero and distinct elements $\{a_1, a_2, a_3, a_4\}$ in R such that $a_i a_j + n$ is a perfect square in R for $1 \leq i < j \leq 4$ is called a *Diophantine quadruple with the property* $D(n)$ in R or just a $D(n)$-*quadruple*. If $n = 1$ then a quadruple with a given property is called a *Diophantine quadruple*. The problem of constructing such sets was first studied by Diophantus of Alexandria who found the rational quadruple $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$ with the property $D(1)$. Fermat found the first Diophantine quadruple in the ring of integers $\mathbb{Z}$ - the set $\{1, 3, 8, 120\}$.

The problem on the existence of $D(n)$-quadruples has been studied in different rings, but mainly in rings of integers of numbers fields. The following assertion is shown to be true in many cases: *There exists a $D(n)$-quadruple if and only if n can be represented as a difference of two squares, up to finitely many exceptions.* In the ring $\mathbb{Z}$ one part of the assertion is proved independently by several authors (Brown, Gupta, Singh, Mohanty, Ramamsamy, see [6, 25, 27]), and another by Dujella in [7]. The set of possible exceptions $S = \{-4, -3, -1, 3, 5, 8, 12, 20\}$ is still an open problem studied by many authors. The conjecture is that for $n \in S$ there does not exist a Diophantine quadruple with the property $D(n)$.

In the ring of integers $\mathbb{Z}$ well studied is the case of $n = -1$. There is a conjecture that $D(-1)$-quadruple does not exist in $\mathbb{Z}$. That is known as the $D(-1)$-*quadruple conjecture* and it was presented explicitly in [11] for the first time. While it is conjectured that $D(-1)$-quadruples do not exist in integers

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(see [11]), it is known that no $D(-1)$-quintuple exists and that if $\{a,b,c,d\}$ is a $D(-1)$-quadruple with $a<b<c<d$, then $a=1$ (see [15]). It is proved that some infinite families of $D(-1)$-triples cannot be extended to a $D(-1)$-quadruple. The non-extendibility of $\{1,b,c\}$ was confirmed for $b=2$ by Dujella in [10], for $b=5$ partially by Abu Muriefah and Al Rashed in [2], and completely by Filipin in [18]. The statement was also proved for $b=10$ by Filipin in [18], and for $b=17,26,37,50$ by Fujita in [24]. Dujella, Filipin and Fuchs in [13] proved that there are at most finitely many $D(-1)$-quadruples, by giving an upper bound of 10^{903} for the number of $D(-1)$-quadruples. This bound was improved several times: to 10^{356} by Filipin and Fujita ([19]), to $4\cdot 10^{70}$ by Bonciocat, Cipu and Mignotte ([5]) and very recently to $5\cdot 10^{60}$ by Elsholtz, Filipin and Fujita ([17]).

In the ring of Gaussian integers $\mathbb{Z}[i]$ the above assertion was proved in [9]. Namely, if $a+bi$ is not representable as a difference of the squares of two elements in $\mathbb{Z}[i]$, and in contrary if $a+bi$ is not of such form and $a+bi\notin\{\pm 2,\pm 1\pm 2i,\pm 4i\}$, then $D(a+bi)$-quadruple exists. Franušić in [20–22] found that a similar statement is true for rings of integers of some real quadratic fields, i.e. it can be seen that there exist infinitely many $D(n)$-quadruples if and only if n can be represented as a difference of two squares of integers. To be more precise, assuming the solvability of certain Pellian equation ($x^2-dy^2=\pm 2$ or $x^2-dy^2=4$ in odd numbers) we are able to obtain an effective characterization of integers that can be represented as a difference of two squares of integers in $\mathbb{Q}(\sqrt{d})$ and then apply some polynomial formulas for Diophantine quadruples in a combination with elements of a small norm. Also, in [23] the existence problem in the ring of integers of the pure cubic field $\mathbb{Q}(\sqrt[3]{2})$ has been completely solved.

The case of complex quadratic fields is more demanding because the set of elements with a small norm is poor (while in the real case there exist infinitely many units). A group of authors ([1, 16, 28]) worked on the problem of the existence of $D(z)$-quadruples in $\mathbb{Z}[\sqrt{-2}]$ and contributed that the problem is almost completely solved. As a prominent case, there appear the case $z=-1$, which could not be solved by the standard method via polynomial formulas. In [29] and [30] Soldo studied $D(-1)$-triples of the form $\{1,b,c\}$ and the existence of $D(-1)$-quadruples of the form $\{1,b,c,d\}$ in the ring $\mathbb{Z}[\sqrt{-t}]$, $t>0$, for $b=2,5,10,17,26,37$ or 50. He proved a more general result i.e. if positive integer b is a prime, twice prime or twice prime squared such that $\{1,b,c\}$ is a $D(-1)$-triple in the ring $\mathbb{Z}[\sqrt{-t}]$, $t>0$, then c has to be an integer. As a consequence of this result, he showed that for $t\notin\{1,4,9,16,25,36,49\}$ there does not exist a subset of $\mathbb{Z}[\sqrt{-t}]$ of the form $\{1,b,c,d\}$ with the property that the product of any two of its distinct elements diminished by 1 is a square of an element in $\mathbb{Z}[\sqrt{-t}]$. For those exceptional cases of t he showed that there exist infinitely many $D(-1)$-quadruples of the form $\{1,b,-c,d\}$, $c,d>0$ in $\mathbb{Z}[\sqrt{-t}\,]$.

In this paper, we verify assertion on the existence of $D(z)$-quadruples in complex quadratic field $\mathbb{Q}(\sqrt{-3})$, i.e. in the corresponding ring of integers $\mathbb{Z}[(1+\sqrt{-3})/2]$. In other words, we show the following theorems.

THEOREM 1.1. *There exists a $D(z)$-quadruple in the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{-3})$ if and only if z can be represented as a difference of two squares of integers in $\mathbb{Q}(\sqrt{-3})$, up to possible exceptions $z \in \{-1, 3, \frac{1}{2} - \frac{1}{2}\sqrt{-3}, \frac{1}{2} + \frac{1}{2}\sqrt{-3}\}$.*

THEOREM 1.2. *There exists a $D(z)$-quadruple in the ring $\mathbb{Z}[\sqrt{-3}]$ if and only if z can be represented as a difference of two squares of elements in $\mathbb{Z}[\sqrt{-3}]$, up to possible exceptions $z \in \{-4, -1, 3, 2-2\sqrt{-3}, 2+2\sqrt{-3}\}$.*

Although we have mentioned that the case of complex quadratic fields is rather complicated, observe that the Pellian equation $x^2 - dy^2 = 4$ is solvable for $d = -3$ in $\mathbb{Z}$ (the only solution is $1 + \sqrt{-3}$). To begin with, we will list briefly all statements that we require for the proofs of Theorem 1.1 and Theorem 1.2.

LEMMA 1.3 ([8, Theorem 1]). *Let R be a commutative ring with the unity 1 and $m, k \in R$. The set*

$$\{m, m(3k+1)^2 + 2k, m(3k+2)^2 + 2k + 2, 9m(2k+1)^2 + 8k + 4\} \tag{1.1}$$

has the $D(2m(2k+1)+1)$-property.

The set (1.1) is a $D(2m(2k+1)+1)$-quadruple if it contains no equal elements or elements equal to zero.

LEMMA 1.4. *If u is an element of a commutative ring R with the unity 1 and $\{w_1, w_2, w_3, w_4\}$ is a $D(w)$-quadruple in R, then $\{w_1u, w_2u, w_3u, w_4u\}$ is a $D(wu^2)$-quadruple in R.*

LEMMA 1.5 ([14, Theorem 1]). *An integer $z \in \mathbb{Q}(\sqrt{-3})$ can be represented as a difference of two squares of elements in $\mathbb{Z}[\sqrt{-3}]$ if and only if is one of the following forms*

$$2m + 1 + 2n\sqrt{-3},\ 4m + 4n\sqrt{-3},\ 4m + 2 + (4n+2)\sqrt{-3},$$

$m, n \in \mathbb{Z}$.

LEMMA 1.6 ([14, Theorem 2]). *An integer $z \in \mathbb{Q}(\sqrt{-3})$ can be represented as a difference of two squares of elements in $\mathbb{Z}[(1+\sqrt{-3})/2]$ if and only if is one of the following forms*

$$2m+1+2n\sqrt{-3},\ 2m+(2n+1)\sqrt{-3},\ 4m+4n\sqrt{-3},\ 4m+2+(4n+2)\sqrt{-3},$$
$$\frac{2m+1}{2} + \frac{2n+1}{2}\sqrt{-3},$$

$m, n \in \mathbb{Z}$.

LEMMA 1.7 ([22, Lemma 5]). *For each $M, N \in \mathbb{Z}$, there exist $k \in \mathbb{Z}[(1+\sqrt{-3})/2]$ such that*

1. $2M+1+2N\sqrt{-3} = 2m(k+1)+1$, *where* $m = \frac{1}{2}+\frac{1}{2}\sqrt{-3}$,
2. $4M+3+(4N+2)\sqrt{-3} = 2m(2k+1)+1$, *where* $m = 1+\sqrt{-3}$,
3. $2M+(2N+1)\sqrt{-3} = m(2k+1)+1$, *where* $m = 1+\sqrt{-3}$,
4. $2M+1+(2N+1)\sqrt{-3} = m(2k+1)+1$, *where* $m = \frac{1}{2}+\frac{1}{2}\sqrt{-3}$,
5. $\frac{2M+1}{2}+\frac{2N+1}{2}\sqrt{-3} = \frac{m}{2}(2k+1)+1$, *where* $m = 1+\sqrt{-3}$.

By using Lemmas 1.3, 1.4 and 1.7, we effectively construct Diophantine quadruples for integers of the forms given in Lemmas 1.5 and 1.6. The following assertion gives the nonexistence of a $D(z)$-quadruple in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$ if z cannot be represented as a difference of two squares in $\mathbb{Z}[(1+\sqrt{-3})/2]$, i.e. if and only if z is of the form $4m+2+4n\sqrt{-3}$, $4m+(4n+2)\sqrt{-3}$, $2m+1+(2n+1)\sqrt{-3}$.

LEMMA 1.8 ([22, Theorem 2]). *If z has one of the forms*

$$4m+2+4n\sqrt{-3},\ 4m+(4n+2)\sqrt{-3},\ 2m+1+(2n+1)\sqrt{-3},$$

where $m, n \in \mathbb{Z}$, then a $D(z)$-quadruple in $\mathbb{Z}[(1+\sqrt{-3})/2]$ does not exist.

The nonexistence of a $D(z)$-quadruple in $\mathbb{Z}[\sqrt{-3}]$ if z cannot be represented as a difference of two squares in $\mathbb{Z}[\sqrt{-3}]$ follows partially from Lemma 1.8 (if $z = 4m+2+4n\sqrt{-3}$ or $z = 4m+(4n+2)\sqrt{-3}$) and from the following assertion.

LEMMA 1.9. *Let $d \in \mathbb{Z}$ is not a perfect square. Then there is no $D(m+(2n+1)\sqrt{d})$-quadruple in the ring $\mathbb{Z}[\sqrt{d}]$.*

PROOF. The proof of Proposition 1 in [1] given for $d = 2$ can be immediately rewritten for an arbitrary d. ∎

2. $D(z)$-QUADRUPLES IN $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$

Let us denote the set

$$D_4 = \{mu, (m(3k+1)^2+2k)u, (m(3k+2)^2+2k+2)u, (9m(2k+1)^2+8k+4)u\}.$$

According to Lemmas 1.3 and 1.4, D_4 is $D((2m(2k+1)+1)u^2)$-quadruple if it contains no equal elements or elements equal to zero. This polynomial formula combining with specific values of m and u solves our problem, up to finitely many cases. Our results are listed in the tables of the following subsections.

2.1. $D(2m+1+2n\sqrt{-3})$-*quadruples.*

In this subsection, for integers A and B, we will separate the cases of $z = 4A+3+(4B+2)\sqrt{-3}$ and $z = 4A+1+4B\sqrt{-3}$ to corresponding subcases.

z	k	m	u	D_4 in	exceptions of z
$4A+3+4B\sqrt{-3}$	$A+B\sqrt{-3}$	1	1	$\mathbb{Z}[\sqrt{-3}]$	$-1,\ 3$
$4A+1+(4B+2)\sqrt{-3}$	$\frac{-1+2A}{2}+\frac{1+2B}{2}\sqrt{-3}$	1	1	$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$	–
$8A+3+(8B+2)\sqrt{-3}$	$\frac{A+3B}{2}+\frac{-A+B}{2}\sqrt{-3}$	$1+\sqrt{-3}$	1	$\mathbb{Z}[\sqrt{-3}]$	$3+2\sqrt{-3}$
$8A+7+(8B+6)\sqrt{-3}$	$\frac{A+3B+2}{2}+\frac{-A+B}{2}\sqrt{-3}$	$1+\sqrt{-3}$	1	$\mathbb{Z}[\sqrt{-3}]$	$-1-2\sqrt{-3}$
$8A+7+(8B+2)\sqrt{-3}$	$\frac{A-3B-1}{2}+\frac{A+B+1}{2}\sqrt{-3}$	$1-\sqrt{-3}$	1	$\mathbb{Z}[\sqrt{-3}]$	$-1+2\sqrt{-3}$
$8A+3+(8B+6)\sqrt{-3}$:	$\frac{A-3B-3}{2}+\frac{A+B+1}{2}\sqrt{-3}$	$1-\sqrt{-3}$	1	$\mathbb{Z}[\sqrt{-3}]$	$3-2\sqrt{-3}$
$8A+5+8B\sqrt{-3}$	$A+B\sqrt{-3}$	2	1	$\mathbb{Z}[\sqrt{-3}]$	$5,\ -3$
$8A+1+(8B+4)\sqrt{-3}$	$\frac{2A-1}{2}+\frac{2B+1}{2}\sqrt{-3}$	2	1	$\mathbb{Z}[\sqrt{-3}]$	–
$8A+1+8B\sqrt{-3}$	$A-1+B\sqrt{-3}$	4	1	$\mathbb{Z}[\sqrt{-3}]$	$1,\ 9,\ -7$
$8A+5+(8B+4)\sqrt{-3}$	$\frac{4A-2B-3}{4}+\frac{2B+1}{4}\sqrt{-3}$	4	1	$\mathbb{Z}[\sqrt{-3}]$	–

TABLE 1

It is easy to check that for those exceptions of z in Table 1, the polynomial formula D_4 gives the set with two equal elements, or some element is equal to zero. Therefore, in those exceptions of z (and all further exceptions), we used the method for the first time described in [8] (but only for quadruples in $\mathbb{Z}$), to construct $D(z)$-quadruples with all distinct elements, of the form $\{u, v, u+v+2r, u+4v+4r\}$, for some $u, v, r \in \mathbb{Z}[\sqrt{-3}]$, or $u, v, r \in \mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$, respectively. Except in cases of $z = -1, 3$, we found the following $D(z)$-quadruples in $\mathbb{Z}[\sqrt{-3}]$:

- $\{3+\sqrt{-3}, 1-\sqrt{-3}, -2, -5-3\sqrt{-3}\}$ is the $D(3+2\sqrt{-3})$-quadruple,
- $\{3-\sqrt{-3}, 1+\sqrt{-3}, -2, -5+3\sqrt{-3}\}$ is the $D(3-2\sqrt{-3})$-quadruple,
- $\{1+3\sqrt{-3}, -1+\sqrt{-3}, 2, 1-\sqrt{-3}\}$ is the $D(-1-2\sqrt{-3})$-quadruple,
- $\{1-3\sqrt{-3}, -1-\sqrt{-3}, 2, 1+\sqrt{-3}\}$ is the $D(-1+2\sqrt{-3})$-quadruple,
- $\{8, 1+\sqrt{-3}, 1-\sqrt{-3}, -4\}$ is the $D(5)$-quadruple,
- $\{\sqrt{-3}, 3\sqrt{-3}, 8\sqrt{-3}, 120\sqrt{-3}\}$ is the $D(-3)$-quadruple,
- $\{1, 3, 8, 120\}$ is the $D(1)$-quadruple,
- $\{6, -2-2\sqrt{-3}, -2+2\sqrt{-3}, -14\}$ is the $D(9)$-quadruple,
- $\{2+2\sqrt{-3}, 1+\sqrt{-3}, 1-\sqrt{-3}, 2-2\sqrt{-3}\}$ is the $D(-7)$-quadruple.

2.2. $D(2m+(2n+1)\sqrt{-3})$-*quadruples.*

z	k	m	u	D_4 in	exceptions of z
$4A+(4B+3)\sqrt{-3}$	$\frac{A+3B+1}{2}+\frac{-A+B+1}{2}\sqrt{-3}$	$\frac{1+\sqrt{-3}}{2}$	1	$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$	$-\sqrt{-3}$
$4A+2+(4B+1)\sqrt{-3}$	$\frac{A+3B}{2}+\frac{-A+B}{2}\sqrt{-3}$	$\frac{1+\sqrt{-3}}{2}$	1	$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$	$2+\sqrt{-3}$
$4A+(4B+1)\sqrt{-3}$	$\frac{A-3B}{2}-1+\frac{A+B}{2}\sqrt{-3}$	$\frac{1-\sqrt{-3}}{2}$	1	$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$	$\sqrt{-3}$
$4A+2+(4B+3)\sqrt{-3}$	$\frac{A-3B-3}{2}+\frac{A+B+1}{2}\sqrt{-3}$	$\frac{1-\sqrt{-3}}{2}$	1	$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$	$2-\sqrt{-3}$

TABLE 2

For the exceptions noted in Table 2, we found the following $D(z)$-quadruples in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$:

- $\{\frac{1}{2}-\frac{3}{2}\sqrt{-3}, -1, \frac{1}{2}-\frac{1}{2}\sqrt{-3}, -\frac{3}{2}+\frac{1}{2}\sqrt{-3}\}$ is the $D(-\sqrt{-3})$-quadruple,
- $\{\frac{1}{2}+\frac{1}{2}\sqrt{-3}, -\frac{7}{2}+\frac{1}{2}\sqrt{-3}, -2, -\frac{23}{2}+\frac{1}{2}\sqrt{-3}\}$ is the $D(2+\sqrt{-3})$-quadruple,
- $\{\frac{1}{2}+\frac{3}{2}\sqrt{-3}, -1, \frac{1}{2}+\frac{1}{2}\sqrt{-3}, -\frac{3}{2}-\frac{1}{2}\sqrt{-3}\}$ is the $D(\sqrt{-3})$-quadruple,
- $\{\frac{1}{2}-\frac{1}{2}\sqrt{-3}, -\frac{7}{2}-\frac{1}{2}\sqrt{-3}, -2, -\frac{23}{2}-\frac{1}{2}\sqrt{-3}\}$ is the $D(2-\sqrt{-3})$-quadruple.

2.3. $D(4m+4n\sqrt{-3})$-*quadruples.*

z	k	m	u	D_4 in	exceptions of z
$8A+8B\sqrt{-3}$	$\frac{-A+3B-2}{2}-\frac{A+B}{2}\sqrt{-3}$	$\frac{1}{2}$	$1+\sqrt{-3}$	$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$	0
$8A+4+(8B+4)\sqrt{-3}$	$\frac{-A+3B-1}{2}-\frac{A+B+1}{2}\sqrt{-3}$	$\frac{1}{2}$	$1+\sqrt{-3}$	$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$	$-4+4\sqrt{-3}$

TABLE 3

The set $\{1, 2-2\sqrt{-3}, 5, 13-4\sqrt{-3}\}$ is the $D(-4+4\sqrt{-3})$-quadruple in $\mathbb{Z}[\sqrt{-3}]$ and it is easy to see that there exits infinitely many $D(0)$-quadruples.

We obtain a $D(8A+4+8B\sqrt{-3})$-quadruple by multiplying elements of a $D(2m+1+2n\sqrt{-3})$-quadruple by $u=2$ except for $z=-4, 12$, but

$$\{1, \frac{7}{2}+\frac{1}{2}\sqrt{-3}, \frac{7}{2}-\frac{1}{2}\sqrt{-3}, 13\}$$

is the $D(-4)$-quadruple, and

$$\{-2, 7+\sqrt{-3}, 7-\sqrt{-3}, 30\}$$

is the $D(12)$-quadruple. Also, a $D(8A+(8B+4)\sqrt{-3})$-quadruple is obtained by multiplying elements of a $D(2m+(2n+1)\sqrt{-3})$-quadruple by $u=2$. Obviously, the resulting sets are subsets of $\mathbb{Z}[\sqrt{-3}]$ (except for $z=-4$).

2.4. $D(4m+2+(4n+2)\sqrt{-3})$-*quadruples.*

z	k	m	u	D_4 in	exceptions of z
$8A+2+(8B+2)\sqrt{-3}$	$-A-1-B\sqrt{-3}$	$\frac{1+\sqrt{-3}}{4}$	$1+\sqrt{-3}$	$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$	$-6+2\sqrt{-3}$, $2+2\sqrt{-3}$
$8A+6+(8B+6)\sqrt{-3}$	$-\frac{2A+3}{2}-\frac{2B+1}{2}\sqrt{-3}$	$\frac{1+\sqrt{-3}}{4}$	$1+\sqrt{-3}$	$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$	–
$8A+2+(8B+6)\sqrt{-3}$	$-(A+1)-(B+1)\sqrt{-3}$	$\frac{1-\sqrt{-3}}{4}$	$1-\sqrt{-3}$	$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$	$-6-2\sqrt{-3}$, $2-2\sqrt{-3}$
$8A+6+(8B+2)\sqrt{-3}$	$-\frac{2A+3}{2}-\frac{2B+1}{2}\sqrt{-3}$	$\frac{1-\sqrt{-3}}{4}$	$1-\sqrt{-3}$	$\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$	–

TABLE 4

While the polynomial formula D_4 gives sets with two equal elements, for those exceptions of z of Table 4, we found the following $D(z)$-quadruples in $\mathbb{Z}[\frac{1+\sqrt{-3}}{2}]$:

- $\{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}, -9-2\sqrt{-3}, -\frac{25}{2}-\frac{1}{2}\sqrt{-3}, -\frac{85}{2}-\frac{11}{2}\sqrt{-3}\}$ is the $D(-6+2\sqrt{-3})$-quadruple,
- $\{-\frac{1}{2}+\frac{1}{2}\sqrt{-3}, -\frac{5}{2}+\frac{3}{2}\sqrt{-3}, -1+2\sqrt{-3}, -\frac{13}{2}+\frac{13}{2}\sqrt{-3}\}$ is the $D(2+2\sqrt{-3})$-quadruple,
- $\{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}, -9+2\sqrt{-3}, -\frac{25}{2}+\frac{1}{2}\sqrt{-3}, -\frac{85}{2}+\frac{11}{2}\sqrt{-3}\}$ is the $D(-6-2\sqrt{-3})$-quadruple,
- $\{-\frac{1}{2}-\frac{1}{2}\sqrt{-3}, -\frac{5}{2}-\frac{3}{2}\sqrt{-3}, -1-2\sqrt{-3}, -\frac{13}{2}-\frac{13}{2}\sqrt{-3}\}$ is the $D(2-2\sqrt{-3})$-quadruple.

2.5. $D(\frac{2m+1}{2}+\frac{2n+1}{2}\sqrt{-3})$-*quadruples.*

We derive $D(\frac{2m+1}{2}+\frac{2n+1}{2}\sqrt{-3})$-quadruples from $D(2m+1+2n\sqrt{-3})$ and $D(2m+(2n+1)\sqrt{-3})$ -quadruples by multiplying them by $\frac{1+\sqrt{-3}}{2}$ and $\frac{1-\sqrt{-3}}{2}$.

- Multiplying the elements of a $D(2m+1+2n\sqrt{-3})$-quadruple by $u=\frac{1+\sqrt{-3}}{2}$ we obtain a $D((2m+1+2n\sqrt{-3})u^2)$-quadruple except for $z=\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, $-\frac{3}{2}+\frac{3}{2}\sqrt{-3}$. The number $(2m+1+2n\sqrt{-3})u^2$ is of the form $\frac{2A+1}{2}+\frac{2B+1}{2}\sqrt{-3}$ and for given $A,B\in\mathbb{Z}$ the equation

$$(2m+1+2n\sqrt{-3})u^2=\frac{2A+1}{2}+\frac{2B+1}{2}\sqrt{-3} \tag{2.1}$$

  has an integer solution ($m,n\in\mathbb{Z}$) if and only if $-A+3B\equiv 1\,(\text{mod } 4)$ and $A+B\equiv 3\,(\text{mod } 4)$, i.e. $(A,B) \bmod 4\in\{(0,3),(1,2),(2,1),(3,0)\}$. Concerning exceptions, the set

$$\{\frac{1}{2}+\frac{1}{2}\sqrt{-3}, -\frac{5}{2}-\frac{3}{2}\sqrt{-3}, -1-2\sqrt{-3}, -\frac{15}{2}-\frac{15}{2}\sqrt{-3}\}$$

  represents the $D(-\frac{3}{2}+\frac{3}{2}\sqrt{-3})$-quadruple, while we could not find the $D(\frac{1}{2}-\frac{1}{2}\sqrt{-3})$-quadruple.
- Multiplying the elements of a $D(2m+1+2n\sqrt{-3})$-quadruple by $u=\frac{1-\sqrt{-3}}{2}$ we obtain a $D((2m+1+2n\sqrt{-3})u^2)$-quadruple except for $z=\frac{1}{2}+\frac{1}{2}\sqrt{-3}$, $-\frac{3}{2}-\frac{3}{2}\sqrt{-3}$. For given $A,B\in\mathbb{Z}$ the equation (2.1) has an integer solution if and only if $A+3B\equiv 0 \pmod 4$ and $A-B\equiv 0 \pmod 4$, i.e. $(A,B) \bmod 4\in\{(0,0),(1,1),(2,2),(3,3)\}$. The set

$$\{\frac{1}{2}-\frac{1}{2}\sqrt{-3}, -\frac{5}{2}+\frac{3}{2}\sqrt{-3}, -1+2\sqrt{-3}, -\frac{15}{2}+\frac{15}{2}\sqrt{-3}\}$$

  is the $D(-\frac{3}{2}-\frac{3}{2}\sqrt{-3})$-quadruple and we have not detected a $D(\frac{1}{2}+\frac{1}{2}\sqrt{-3})$-quadruple.

- Multiplying the elements of a $D(2m+(2n+1)\sqrt{-3})$-quadruple by $u=\frac{1+\sqrt{-3}}{2}$ we obtain a $D(\frac{2A+1}{2}+\frac{2B+1}{2}\sqrt{-3})$-quadruple. For given $A,B\in\mathbb{Z}$ the equation

$$(2m+(2n+1)\sqrt{-3})u^2=\frac{2A+1}{2}+\frac{2B+1}{2}\sqrt{-3} \tag{2.2}$$

  has an integer solution if and only if $-A+3B\equiv 3\pmod 4$ and $A+B\equiv 1\pmod 4$, i.e. $(A,B)\mod 4\in\{(0,1),(1,0),(3,2),(2,3)\}$.
- Multiplying the elements of a $D(2m+(2n+1)\sqrt{-3})$-quadruple by $u=\frac{1-\sqrt{-3}}{2}$ we obtain a $D(\frac{2A+1}{2}+\frac{2B+1}{2}\sqrt{-3})$-quadruple. For given $A,B\in\mathbb{Z}$ the equation (2.2) has an integer solution if and only if $A+3B\equiv 2\pmod 4$ and $-A+B\equiv 2\pmod 4$, i.e. $(A,B)\mod 4\in\{(0,2),(2,0),(1,3),(3,1)\}$.

3. $D(z)$ quadruples in $\mathbb{Z}[\sqrt{-3}]$

In the previous section we see that some $D(z)$-quadruples that have been constructed already lie in $\mathbb{Z}[\sqrt{-3}]$ but some of them do not although z can be represented as a difference of squares in $\mathbb{Z}[\sqrt{-3}]$. Here we show that this can be improved.

3.1. $D(2m+1+2n\sqrt{-3})$-*quadruples.*

z	k	m	u	D_4 in	exceptions of z
$4A+1+(4B+2)\sqrt{-3}$	$\frac{2A-2B+1}{2}+\frac{A+1}{2}\sqrt{-3}$	$\sqrt{-3}/3$	$\sqrt{-3}$	$\mathbb{Z}[\sqrt{-3}]$	$-3-2\sqrt{-3}$, $-3+2\sqrt{-3}$

Table 5

The set $\{-\sqrt{-3},-2+\sqrt{-3},-2,-8+3\sqrt{-3}\}$ is a $D(-3-2\sqrt{-3})$, while the set $\{\sqrt{-3},-2-\sqrt{-3},-2,-8-3\sqrt{-3}\}$ is a $D(-3+2\sqrt{-3})$-quadruple in $\mathbb{Z}[\sqrt{-3}]$.

3.2. $D(4m+2+(4n+2)\sqrt{-3})$-*quadruples.*

Since there exist a $D(\frac{2m+1}{2}+\frac{2n+1}{2}\sqrt{-3})$-quadruple in $\mathbb{Z}[(1+\sqrt{-3})/2]$, by multiplying by 2 the elements of this quadruple we obtain a $D(4m+2+(4n+2)\sqrt{-3})$-quadruple in $\mathbb{Z}[\sqrt{-3}]$, up to $z=2-2\sqrt{-3}$, $2+2\sqrt{-3}$.

3.3. $D(4m + 4n\sqrt{-3})$-*quadruples.*

We have shown in § 2.3. that $D(8m+(8n+4)\sqrt{-3})$ and $D(8m+4+8n\sqrt{-3})$-quadruples in $\mathbb{Z}[\sqrt{-3}]$ are obtained by multiplying by 2 the elements of $D(2m+(2n+1)\sqrt{-3})$ and $D(2m+1+2n\sqrt{-3})$-quadruples in $\mathbb{Z}[(1+\sqrt{-3})/2]$ up to the the $D(-4)$-quadruple whose elements are not in $\mathbb{Z}[\sqrt{-3}]$.

The set

$$\{1, 9k^2 - 8k, 9k^2 - 2k + 1, 36k^2 - 20k + 1\}$$

is $D(8k)$-quadruple ([7]) if $k \neq 0, 1$, so there exists a $D(8m + 8n\sqrt{-3})$-quadruple in $\mathbb{Z}[\sqrt{-3}]$.

z	k	m	u	D_4 in	exceptions of z
$8A+4+(8B+4)\sqrt{-3}$	$\frac{3A-2B+4}{2} + \frac{A+2}{2}\sqrt{-3}$	$\frac{\sqrt{-3}}{6}$	$2\sqrt{-3}$	$\mathbb{Z}[\sqrt{-3}]$	$-12-4\sqrt{-3}$ $-12+4\sqrt{-3}$

Table 6

It is easy to check that for those exceptions of z in Table 6, the polynomial formula D_4 gives the set with two equal elements. Therefore for certain z, we found the following $D(z)$-quadruples in $\mathbb{Z}[\sqrt{-3}]$:

- $\{2 + \sqrt{-3}, 2 - 2\sqrt{-3}, 2 - 3\sqrt{-3}, 6 - 11\sqrt{-3}\}$ is the $D(-12 + 4\sqrt{-3})$-quadruple,
- $\{2 - \sqrt{-3}, 2 + 2\sqrt{-3}, 2 + 3\sqrt{-3}, 6 + 11\sqrt{-3}\}$ is the $D(-12 - 4\sqrt{-3})$-quadruple,
- $\{2+\sqrt{-3}, -2+2\sqrt{-3}, -2+\sqrt{-3}, -10+5\sqrt{-3}\}$ is the $D(8)$-quadruple.

Remark 3.1. Concerning the list of possible exceptions given in Theorem 1.1 and Theorem 1.2, we can easily observe that $3 = -1\cdot(\sqrt{-3})^2$, $-4 = -1\cdot 2^2$, $\frac{1}{2} \pm \frac{1}{2}\sqrt{-3} = -1 \cdot (\frac{1}{2} \mp \frac{1}{2}\sqrt{-3})^2$ and $2 \pm 2\sqrt{-3} = -1 \cdot (1 \mp \sqrt{-3})^2$. So, we are not surprised that the key point lies in an investigation on the existence of $D(-1)$-quadruples in rings $\mathbb{Z}[(1 + \sqrt{-3})/2]$ and $\mathbb{Z}[\sqrt{-3}]$. In an analogy to $D(-1)$-quadruple conjecture in the ring of integers and the problem of existence of $D(-1)$-quadruples in $\mathbb{Z}[\sqrt{-t}], t > 0$ studied in [29] and [30], we might expect that for such z there does not exists a $D(z)$-quadruple.

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Diofantov problem za cijele brojeve kvadratnog polja $\mathbb{Q}(\sqrt{-3})$

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Sažetak. Rješavamo Diofantov problem za cijele brojeve kvadratnog polja $\mathbb{Q}(\sqrt{-3})$ konstruiranjem $D(z)$-četvorki u prstenu $\mathbb{Z}[\sqrt{-3}]$ za svaki z koji se može prikazati kao razlika dva kvadrata u $\mathbb{Q}(\sqrt{-3})$, do na konačno mnogo mogućih izuzetaka.

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