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POSITIVE EXPONENTIAL SUMS AND ODD POLYNOMIALS

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Abstract. Given an odd integer polynomial $f(x)$ of a degree $k \geq 3$, we construct a non-negative valued, normed trigonometric polynomial with non-vanishing coefficients only at values of $f(x)$ not greater than n, and a small free coefficient $a_0 = O((\log n)^{-1/k})$. This gives an alternative proof of the bound for the maximal possible cardinality of a set of integers A, so that $A - A$ does not contain an integer value of $f(x)$. We also discuss other interpretations and an ergodic characterization of that bound.

1. Introduction

We consider polynomials $f(x) = \alpha_k x^k + ... + \alpha_1 x$ with integer coefficients, satisfying:

(1.1) For all j even, $\alpha_j = 0$; and the leading coefficient is $\alpha_k > 0$.

The main result of the paper is the following:

Theorem 1.1. *Given an integer polynomial f of a degree $k \geq 3$ satisfying (1.1), there exist cosine polynomials*

$$T(x) = b_0 + \sum_{0<f(j)\leq N} b_{f(j)} \cos(2\pi f(j)x), \tag{1.2}$$

such that for all x, $T(x) \geq 0$, and such that all coefficients b_j are non-negative, normed (i.e. $\sum b_j = 1$), and that $b_0 = O((\log N)^{-1/k})$.

We now discuss the background and implications of that result. Let D be a set of positive integers. Denote by $\mathcal{T}(D)$ the set of all cosine polynomials T such that its coefficient $b_j \neq 0$ only if $j \in D \cup \{0\}$, such that $T(x) \geq 0$ for all $x \in \mathbb{R}$, and $T(0) = 1$. Let $\mathcal{T}^+(D)$ be the subset of $\mathcal{T}(D)$ with non-negative coefficients. Kamae and Mendès France in [5] introduced the notion of van der Corput sets (VdC sets; or correlative sets), if $\inf_{T\in\mathcal{T}(D)} b_0 = 0$ (b_0

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is the free coefficient). One can also define VdC$^+$ sets as those for which $\inf_{T\in\mathcal{T}^+(D)} b_0 = 0$. Let $\gamma(n)$, $\gamma^+(n)$ be the arithmetic functions which measure how quickly a set is becoming a van der Corput set:

$$\gamma(n) = \inf_{T\in\mathcal{T}(D\cap[1,n])} b_0$$

and $\gamma^+(n)$ analogously for $\mathcal{T}^+(D)$ (and then $\gamma(n) \leq \gamma^+(n)$). Theorem 1.1 can now be rephrased as follows: for the sets of values of an odd integer polynomial f,

$$\gamma^+(n) = O((\log n)^{-1/k}). \tag{1.3}$$

Kamae and Mendès France, Ruzsa and Montgomery [8, 13, 14] described various characterizations of van der Corput sets and the function $\gamma(n)$, mainly related to uniform distribution properties of the set D. In particular, in [5] it was shown that van der Corput sets are intersective sets, and Ruzsa showed in [14] that an upper bound for the function γ is also an upper bound for the intersective property. A corollary of Theorem 1.1 is thus the following:

Corollary 1.2. *Let f be an integer polynomial satisfying (1.1). Suppose that N is a positive integer and that $A \subset \{1, ..., N\}$ is such that the difference between any two elements of A is never an integer value of f. Then $|A| = O(N(\log N)^{-1/k})$.*

This gives another proof of the upper bound for the difference property of odd polynomials, where the best current result (by Lucier [6], valid for all polynomials) is $N(\log N)^{-1/(k-1+o(1))}$.

Montgomery set a problem in [8] of finding any upper bounds for the van der Corput property for any "interesting" sets, such as the set of squares and more generally the set of values of an integer polynomial. Ruzsa in [13] announced the $\gamma(n) = O((\log n)^{-1/2})$ bound for the set of squares, but the proof was never published. One of the authors in [16, 17] proved bounds $\gamma(n) \leq \gamma^+(n) = O((\log n)^{-1/3})$ for the set of perfect squares, and $\gamma(n) \leq \gamma^+(n) = O((\log n)^{-1+o(1)})$ for the set of shifted primes. We also note that Theorem 1.1 can be extended to all integer polynomials of degree $k \geq 3$, but for now with the bound only $\gamma^+(n) = O((\log\log n)^{-1/k^2})$ [11].

Ruzsa showed that, by using only non-negative coefficients in the case of squares, one can not do better than $O((\log n)^{-1})$. We extend the same argument to show that Theorem 1.1 is close to optimal if only non-negative coefficients are used:

Theorem 1.3. *Let $f(x) = \beta x^k$, $\beta > 0$ an integer, and $k \geq 3$ an odd integer. Then*

$$\gamma^+(n) \geq (1/\varphi(k) + o(1))(\log n)^{-1}. \tag{1.4}$$

It is hoped that one can improve the van der Corput and intersective sets bounds by constructing cosine polynomials also with negative coefficients. Matolcsi and Ruzsa have recently announced progress in this direction in the case of perfect squares; and also discussed this in a more general setting of commutative groups [7].

The Theorems 1.1, 1.3 have an ergodic-theoretical interpretation, as was noted in [12]:

Corollary 1.4. *Let f be an integer polynomial of a degree $k \geq 3$ satisfying (1.1), H an arbitrary real Hilbert space, U an unitary operator on H, and P the projection to the kernel of $U - I$. If $x \in H$ is such that $Px \neq 0$, then there exists a positive integer $f(j)$ such that $(U^{f(j)}x, x) > 0$.*

Furthermore, if $(Px, x) > \gamma^+(n)(x, x)$, then there exists such $f(j) \leq n$, where $\gamma^+(n)$ is the best such bound valid universally for all H, U, with bounds (1.3), and (1.4) in the case $f(x) = \alpha_k x^k$.

The structure of the paper. We first introduce some notation related to the polynomial f. The degree of f will be always denoted by k, and let $l \geq 1$ be the smallest index such that $\alpha_l > 0$. Let $c(f) = \gcd(\alpha_k, ..., \alpha_l)$ be the content of the polynomial. Without loss of generality we always assume that for $x \geq 1$, $f(x) \geq 1$, and that $f(j)$, $j \geq 1$ is an increasing sequence (if not, we find the smallest j_0 such that it holds for $j \geq j_0$, and modify all the estimates by skipping the first j_0 values of f, this impacting only the implicit constants in the estimates).

Let $F_n(x)$ be the normed Fejér's kernel

$$F_n(x) = \frac{1}{n} + 2\sum_{j=1}^{n}\left(\frac{1}{n} - \frac{j}{n^2}\right)\cos(2\pi jx).$$

Then $F_n(x) \geq 0$ and $F_n(0) = 1$. The key tool in our construction will be, following the idea of I. Ruzsa, construction of a polynomial of the type (1.2) which approximates $F_n(x)$. We may further restrict "allowable" indices j to those with an integer d as a factor, and define

$$G_{n,d}(x) = \frac{2}{K}\sum_{\alpha_k(dj)^k \leq n} \alpha_k k d^k j^{k-1}\left(\frac{1}{n} - \frac{\alpha_k(dj)^k}{n^2}\right)\cos(2\pi f(dj)x), \tag{1.5}$$

where K is chosen so that $G_{n,d}(0) = 1$ (K will be close to 1 for n large enough; and will be estimated in Section 2).

The structure of the proof is as follows: let $S(f, q)$ be the complete trigonometric sum

$$S(f, q) = \sum_{s=0}^{q-1} e(f(s)/q), \tag{1.6}$$

where $e(x) = \exp(2\pi i x)$. We will need the reduced complete trigonometric sum over the multiples of an integer d:

$$S_d(f,q) = \sum_{s=0}^{q-1} e(f(ds)/q).$$

For q small (the major arc estimates), we show that

$$G_{n,d}(a/q+\kappa) = \frac{1}{q} S_d(af,q) F_n(\kappa) + \text{error term}, \tag{1.7}$$

where the error term is small for small κ and large n as compared to q, d. For large q, we show by partial summation and by using the well-known Vinogradov's trigonometric sum estimates that $G_{n,d}(x)$ is small. The key step is the averaging step: we choose the constants $d_0, ..., d_s$ and normalized weights $w_0, \ldots, w_s$ such that for any q, $\sum_j w_j S_{d_j}(af,q) \geq -\delta$, $\delta = O((\log N)^{-1/k})$. Here N is the size of the largest non-zero coefficient in the family of polynomials G_{n,d_j}. The estimate $b_0 = O((\log N)^{-1/k})$ follows from this and the size of the error term.

Unfortunately, for polynomials which are not odd, this approach seems to fail as (1.7) does not hold. Namely, there is another factor difficult to control if one can not a-priori claim that the imaginary part of $S(f,q)$ is 0, as is the case for odd polynomials.

We prove Theorem 1.1 in Sections 2-5, and Theorem 1.3 in Section 6.

2. The major arcs

We will use the notation $O_k(.)$, $\ll_k$, $O_f(.)$, $\ll_f$ when the implicit constant depends implicitly on the degree k or the coefficients of the polynomial f (including the degree) respectively. We will often use the following relations: If $x, y \geq 1$ are integers such that $f(x), f(y) \ll_f n$, then

$$\begin{aligned}
f(x) &= O_f(x^k) = && (2.1)\\
&= \alpha_k x^k + O_f(n^{1-1/k}), && (2.2)\\
|f(x)-f(y)| &= |x-y| O_f(n^{1-1/k}) = && (2.3)\\
&= \alpha_k k |x-y| x^{k-1} + |x-y|^2 O_f(n^{1-2/k}). && (2.4)
\end{aligned}$$

The relations above can be computed easily by using $\beta = 2(|\alpha_{k-1}| + \cdots + |\alpha_1|)/\alpha_k$ and the relation $\alpha_k x^k \leq 2f(x)$ if $x \geq 1$ and $x \geq \beta$.

The following result by Chen [3] and Nechaev [10] gives a bound for the complete trigonometric sums.

Lemma 2.1. *Let f be an integer polynomial of a degree $k \geq 2$. Then for any positive integer q,*

$$\frac{1}{q}|S(f,q)| \leq c_0 \frac{\gcd(c(f),q)^{1/k}}{q^{1/k}}, \tag{2.5}$$

where c_0 is a constant depending only on k.

We now adapt (2.5) to bound the sum $S_d(af,q)$ over multiples of an integer d.

LEMMA 2.2. *Let f be an integer polynomial of a degree $k \geq 1$ with the last non-zero coefficient α_l. Then for any positive integer d and relatively prime integers a,q ($q>0$),*

$$\frac{1}{q}S_d(af,q) \geq -c_0\frac{|\alpha_l|}{r^{1/k}},$$

where $r = q/\gcd(q,d^l)$ and c_0 as in Lemma 2.1.

PROOF. Denote $g(x) = af(dx)$, and then $S_d(af,q) = S(g,q)$. The content of g divides any of its coefficients including $ad^l\alpha_l$, thus (as a,q are relatively prime)

$$\begin{aligned} \gcd(c(g),q) &\leq \gcd(ad^l\alpha_l,q) = \gcd(d^l\alpha_l,q) \leq |\alpha_l|\gcd(d^l,q) \\ &\leq |\alpha_l|^k\gcd(d^l,q). \end{aligned} \tag{2.6}$$

It suffices to insert (2.6) into (2.5). □

We now state the major arcs estimate.

PROPOSITION 2.3. *Let $G_{n,d}(x)$ be a trigonometric polynomial as in (1.5) for some integer polynomial f of a degree $k \geq 3$ satisfying (1.1), and n,d positive integers. Let $x = a/q+\kappa$. Then*

$$G_{n,d}(x) = \frac{1}{q}S_d(af,q)F_n(\kappa) + O_f(dqn^{-1/k}(1+|\kappa|n)).$$

PROOF. Without loss of generality we assume that $dqn^{-1/k} \leq \alpha_k^{-1/k}$ (otherwise the error term is of the order 1 and the claim is trivial). By writing $\operatorname{Re} e(y)$ instead of $\cos(2\pi y)$ and appropriate grouping, we get

$$G_{n,d}(x) =$$
$$\operatorname{Re}\sum_{s=0}^{q-1}\frac{1}{q}e(f(ds)a/q)\frac{2}{K}\sum_{\substack{\alpha_k(dj)^k\leq n\\ j\equiv s(\operatorname{mod} q)}}\alpha_k qkd^kj^{k-1}\left(\frac{1}{n}-\frac{\alpha_k(dj)^k}{n^2}\right)e(f(dj)\kappa).$$

Let $B/2$ be the inner sum in the expression above, $A = B/K$, and let

$$\begin{aligned} C &= 2 \sum_{\substack{\alpha_k (dj)^k \leq n \\ j \equiv s (\bmod q)}} e(f(dj)\kappa) \sum_{t=f(dj)}^{f(d(j+q))-1} \left(\frac{1}{n} - \frac{t}{n^2} \right), \\ D &= 2 \sum_{\substack{f(dj) \leq n \\ j \equiv s (\bmod q)}} e(f(dj)\kappa) \sum_{t=f(dj)}^{f(d(j+q))-1} \left(\frac{1}{n} - \frac{t}{n^2} \right), \\ F_n^*(\kappa) &= \frac{1}{n} + 2 \sum_{t=1}^{n} \left(\frac{1}{n} - \frac{t}{n^2} \right) e(t\kappa). \end{aligned}$$

Note that $F_n(x) = \operatorname{Re} F_n^*(x)$, but F_n^* has in general a non-zero imaginary part.

If m is chosen so that $\alpha_k (dm)^k \leq n < \alpha_k (d(m+1))^k$, then one easily gets

$$|\alpha_k d^k m^k - n| \leq \alpha_k d^k ((m+1)^k - m^k) \leq \alpha_k k 2^{k-1} d(dm)^{k-1}.$$

We now have

$$\alpha_k d^k m^k = n + O_f(dn^{1-1/k}). \tag{2.7}$$

Using (2.7) and $\sum_{1 \leq j \leq m} j^{k-1} = (1/k)m^k + O_k(m^{k-1})$, we estimate K:

$$\begin{aligned} K &= 2 \sum_{\alpha_k (dj)^k \leq n} \alpha_k k d^k j^{k-1} \left(\frac{1}{n} - \frac{\alpha_k (dj)^k}{n^2} \right) = \\ &= \frac{2}{n} \alpha_k d^k (m^k + O_k(m^{k-1})) - \frac{1}{n^2} \alpha_k^2 d^{2k} (m^{2k} + O_k(m^{2k-1})) = \\ &= 1 + O_f(dn^{-1/k}). \end{aligned} \tag{2.8}$$

Similarly, by using the elementary fact that

$$\sum_{\substack{1 \leq j \leq m \\ j \equiv s (\bmod q)}} j^{k-1} = O_k \left(\frac{1}{q} m^k + m^{k-1} \right), \tag{2.9}$$

one gets that $|B| \leq 1 + O_f(dqn^{-1/k})$. The assumption $dqn^{-1/k} \ll_f 1$ implies $B = O_f(1)$, thus

$$A - B = B(1/K - 1) = O_f(dn^{-1/k}). \tag{2.10}$$

If $1 \leq j \leq m$ and $f(dj) \leq t \leq f(d(j+q)) - 1$, then (2.2) and (2.3) imply that

$$t = \alpha_k (dj)^k + O_f(dqn^{1-1/k}). \tag{2.11}$$

Using (2.11) and (2.4), we get

$$\begin{aligned}
(2.12)\quad & \sum_{t=f(dj)}^{f(d(j+q))-1} \left(\frac{1}{n}-\frac{t}{n^2}\right) \\
&= (f(d(j+q))-f(dj))\left(\frac{1}{n}-\frac{\alpha_k(dj)^k}{n^2}+\frac{O_f(dqn^{1-1/k})}{n^2}\right) \\
&= \left(\alpha_k qkd^k j^{k-1}+O_f(d^2q^2n^{1-2/k})\right)\left(\frac{1}{n}-\frac{\alpha_k(dj)^k}{n^2}+O_f(dqn^{-1-1/k})\right) \\
&= \alpha_k qkd^k j^{k-1}\left(\frac{1}{n}-\frac{\alpha_k(dj)^k}{n^2}\right)+\alpha_k qkd^k j^{k-1}O_f(dqn^{-1-1/k}) \\
&\quad + O_f(d^2q^2n^{-2/k}).
\end{aligned}$$

By summing (2.12) over all the summands "j" in the definition of C, we get

$$|B-C| = O_f(dqn^{-1-1/k}) \sum_{\substack{\alpha_k(dj)^k\le n\\ j\equiv s \pmod q}} \alpha_k qkd^k j^{k-1} + O_f(d^2q^2n^{-2/k}) \sum_{\substack{\alpha_k(dj)^k\le n\\ j\equiv s \pmod q}} 1.$$

Now (2.9) implies

$$B-C=O_f(dqn^{-1/k}). \tag{2.13}$$

From (2.7), as $\alpha_k d^k(m+q)^k-\alpha_k d^k m^k \le \alpha_k d^k qk(m+q)^{k-1}$ and $d(m+q) \ll_f n^{1/k}$, we get $\alpha_k d^k(m+q)^k = n+O_f(dqn^{1-1/k})$. Therefore (2.2) implies

$$f(d(m+q)) = n+O_f(dqn^{1-1/k}). \tag{2.14}$$

Choose m_* so that $f(dm_*)\le n< f(d(m_*+1))$. Assume that $m_*\le m$ (the second case is proved analogously). If $f(d(m_*+1))\le t\le f(d((m+q))-1$, then (2.14) implies

$$t=n+O_f(dqn^{1-1/k}). \tag{2.15}$$

Similarly as before, one shows that

$$f(d(m_*+1)) = n+O_f(dn^{1-1/k}) \tag{2.16}$$

and

$$f(d(m_*+q)) = n+O_f(dqn^{1-1/k}). \tag{2.17}$$

Now, C and D only differ in the number of summands $(1/n-t/n^2)$, thus by (2.15), one gets

$$|C-D| \le 2\sum_{t=f(d(m_*+1))}^{f(d((m+q))-1} \left|\frac{1}{n}-\frac{t}{n^2}\right| = O_f(dqn^{-1-1/k})(f(d(m+q))-f(d(m_*+1))).$$

Using (2.14) and (2.16), we deduce that

$$C-D=O_f(dqn^{-1/k}). \tag{2.18}$$

We now estimate $D - F_n^*(\kappa)$. If $1 \le j \le m_*$ and $f(dj) \le t \le f(d(j+q)) - 1$, the relations $|e(x) - e(y)| \le 2\pi|x-y|$ and (2.3) imply

$$|e(f(dj)\kappa) - e(t\kappa)| \le 2\pi|\kappa|(f(d(j+q)) - f(dj)) = O_f(|\kappa| dqn^{1-1/k}). \tag{2.19}$$

Comparing D and $F_n^*(\kappa)$, we see that

$$\begin{aligned} |D - F_n^*(\kappa)| \quad \le \quad & \frac{1}{n} + 2 \sum_{t=1}^{f(ds)-1} \left| \frac{1}{n} - \frac{t}{n^2} \right| |e(t\kappa)| \\ & +2 \sup_{\substack{1 \le j \le m_*,\, j \equiv s \,(\mathrm{mod}\, q) \\ f(dj) \le t \le f(d(j+q))-1}} |e(f(dj)\kappa) - e(t\kappa)| \sum_{t=f(ds)}^{n} \left| \frac{1}{n} - \frac{t}{n^2} \right| \\ & +2 \sum_{t=n+1}^{f(d(m_*+q))-1} \left| \frac{1}{n} - \frac{t}{n^2} \right| |e(f(dj)\kappa)|. \end{aligned} \tag{2.20}$$

Combining (2.15), (2.17), (2.19) and (2.20) we conclude that

$$D - F_n^*(\kappa) = O_f(dqn^{-1/k}) + O_f(dq|\kappa|n^{1-1/k}). \tag{2.21}$$

Now note that $S_d(af, q)$ is real, as f is an odd polynomial. The claim now follows by combining (2.10), (2.13), (2.18) and (2.21). ∎

Let $\mathfrak{M}(Q, R)$ denote the major arcs, namely the set of all $x \in \mathbb{R}$ which can be approximated by a rational a/q, $\gcd(a,q) = 1$, where $q \le Q$, so that $|x - a/q| \le 1/qR$, and let $\mathfrak{m}(Q,R) = \mathbb{R} \backslash \mathfrak{M}(Q,R)$ be the minor arcs. We also define a function $\tau(d,q)$ which will describe the behavior of the principal part of the major arcs estimate:

$$\begin{aligned} \tau^*(d,q) &= \begin{cases} 1 & q|d^l, \\ -c_0|\alpha_l| r^{-1/k} & \text{otherwise}, \end{cases} \\ \tau(d,q) &= \max\{\tau^*(d,q), -1\}, \end{aligned}$$

where $r = q/\gcd(q, d^l)$ and c_0 is the constant from Lemmas 2.1, 2.2. We now combine all the results of this section.

Corollary 2.4. ***The major arcs estimate.*** *Let $G_{n,d}(x)$ be a trigonometric polynomial as in (1.5) for some integer polynomial f of a degree $k \ge 3$ satisfying (1.1). Assume $1 \le Q < R$ are given. Let $x \in \mathfrak{M}(Q,R)$ written as $x = a/q + \kappa$, where $\gcd(a,q) = 1$, $q \le Q$ and $|\kappa| \le 1/(qR)$. Then*

$$G_{n,d}(x) \ge \tau(d,q) F_n(\kappa) + O_f(dn^{-1/k}(Q + n/R)).$$

Proof. Recall that $F_n(\kappa)$ is non-negative. In the case $q|d^l$ this follows from $S_d(f,q) = q$ and Proposition 2.3, otherwise from Lemma 2.2 and Proposition 2.3. ∎

3. The minor arcs

We derive the following Lemma from the well-known estimates of Vinogradov.

Lemma 3.1. *Let $f(x) = \alpha_k x^k + ... + \alpha_1 x$ be an integer polynomial. If $m, d, 1 \leq Q < R$ are constants so that $Q \geq \alpha_k d^k m^{1/k}$ and $x \in \mathfrak{m}(Q,R)$, then*

$$\sup_{1 \leq m_* \leq m} \left| \sum_{j=1}^{m_*} e(f(dj)x) \right| \ll_f \left(d^{-k} QR\right)^{1/(k-1/k)} + m^{1-\rho},$$

where $\rho = 1/(8k^2(\log k + 1.5 \log\log k + 4.2))$.

Proof. We write $g(j)$ instead of $f(dj)x$, or more precisely: let $g(y) = \beta_k y^k + ... + \beta_1 y$ where $\beta_j = \alpha_j d^j x$. Let $1 \leq m_* \leq m$. Applying Vinogradov exponential sum bounds, it is easy to see that if $Q \geq \alpha_k d^k m_*^{1/k}$, $m_* \geq (\alpha_k^{-1} d^{-k} QR)^{1/(k-1/k)}$ and $x \in \mathfrak{m}(Q,R)$, then $\beta = (\beta_k, ..., \beta_1)$ is of the second class (see [18], Section 11 for the definition of the second class vectors and the bounds), thus

$$\left| \sum_{j=1}^{m_*} e(f(dj)x) \right| \ll_f m_*^{1-\rho} \leq m^{1-\rho}. \tag{3.1}$$

Trivially for any m_*,

$$\left| \sum_{j=1}^{m_*} e(f(dj)x) \right| \leq m_*. \tag{3.2}$$

The claim now follows by summing (3.1) for $m_* \geq \left(\alpha_k^{-1} d^{-k} QR\right)^{1/(k-1/k)}$ and (3.2) for $m_* \leq (\alpha_k^{-1} d^{-k} QR)^{1/(k-1/k)}$. ∎

Proposition 3.2. ***The minor arcs estimate.*** *Let $G_{n,d}(x)$ be the trigonometric polynomial as in (1.5) for some odd integer polynomial f and $1 \leq Q < R$ constants. Let $x \in \mathfrak{m}(Q,R)$. Also assume that $d \leq n^{1/k}$ and $\alpha_k d^k n^{1/k^2} \leq Q$. Then*

$$G_{n,d}(x) \ll_f n^{-1/k} \left(QR\right)^{1/(k-1/k)} + dn^{-\rho/k}.$$

Proof. Choose m so that $\alpha_k d^k m^k \leq n < \alpha_k d^k (m+1)^k$. Then

$$\frac{d^k m^{k-1}}{n} + \frac{d^{2k} m^{2k-1}}{n^2} \ll_f dn^{-1/k}. \tag{3.3}$$

We introduce the notation

$$\begin{aligned} g(j) &= \alpha_k k d^k j^{k-1} \left(\frac{1}{n} - \frac{\alpha_k (dj)^k}{n^2} \right), \\ h(j) &= \cos(2\pi f(dj)x) = \operatorname{Re} e(f(dj)x). \end{aligned}$$

By partial summation, using the notation $\Delta g(j) = g(j+1) - g(j)$, $H(j) = \sum_{i=1}^{j} h(i)$, we get

$$\begin{aligned} G_{n,d}(x) &= \frac{2}{K}\sum_{j=1}^{m} g(j)h(j) = \frac{2}{K}\left(g(m)H(m) - \sum_{j=1}^{m-1}\Delta g(j)H(j)\right) \\ &\ll_f \frac{1}{K}\left(\frac{d^k m^{k-1}}{n} + \frac{d^{2k}m^{2k-1}}{n^2}\right)\sup_{1\le m_*\le m}|H(m_*)|. \end{aligned}$$

By (2.8) and as $d \le n^{1/k}$, $1/K = O_f(1)$. As $Q \ge \alpha_k d^k n^{1/k^2}$ and as by choice of m, $n^{1/k} \ge m$, we get $Q \ge \alpha_k d^k m^{1/k}$. We now combine Lemma 3.1 and (3.3):

$$\begin{aligned} G_{n,d}(x) &\ll {}_f d n^{-1/k}\left(\left(d^{-k}QR\right)^{1/(k-1/k)} + n^{1/k-\rho/k}\right) \\ &\le n^{-1/k}\left(QR\right)^{1/(k-1/k)} + dn^{-\rho/k}. \end{aligned}$$

∎

4. Cancelling out the leading term

Recall the definition of the functions $\tau^*(d,q)$, $\tau(d,q)$ in Section 2, estimating the principal part of the major arcs estimate. For clarity of presentation, denote by $\alpha = c_0|\alpha_l|$, $\beta = 1/k$, and then

$$\tau(d,q) = \begin{cases} 1 & \text{if } q|d^l, \\ \max\{-\alpha r^{-\beta}, -1\} & \text{otherwise,} \end{cases}$$

where $r = q/(q,d^l)$. We use in this section only the facts that $\alpha > 0$, $0 < \beta < 1$.

THEOREM 4.1. ***"Averaging"***. *Assume $\delta > 0$ is given. Then there exist integer constants $s > 0$, $1 = d_0 < d_1 < \dots < d_s$, $d_s = O(\exp(c_1\delta^{-1/\beta})$, c_1 depending only on α, β, and a real constant $\lambda > 0$ such that for any integer q,*

$$\frac{1}{\Lambda}\sum_{j=0}^{s}\lambda^j\tau(d_j,q) \ge -\delta, \tag{4.1}$$

where $\Lambda = 1 + \lambda + \dots + \lambda^s$.

We first discuss the case when q is a prime power $q = p^k$, which encodes the key idea of this section. If p is a prime, then

$$\tau^*(p^j,p^k) = \begin{cases} 1 & k \le jl, \\ -\alpha p^{-\beta(k-jl)} & \text{otherwise.} \end{cases}$$

LEMMA 4.2. *Let p be a prime, and μ any real constant satisfying*

$$1 > \mu \ge \frac{\alpha+1}{\alpha+p^\beta}. \tag{4.2}$$

Then for any positive integer constants s, k,

$$\sum_{j=0}^{s} \mu^j \tau^*(p^j, p^k) \geq -\mu^{s+1}/(1-\mu). \tag{4.3}$$

PROOF. It can be easily deduced from (4.2) that

$$\mu p^{\beta l} \ > \ 1, \tag{4.4}$$

$$-\frac{\alpha p^{\beta(l-1)}}{\mu p^{\beta l}-1} \ \geq \ -\frac{1}{1-\mu}. \tag{4.5}$$

Assume m is the largest integer so that $m \leq s$, $ml < k$. Then for all $j \leq m$, $\tau^*(p^j, p^k) = -\alpha p^{\beta(jl-k)}$. Denote the left side of (4.3) by A_s. We first apply $k \geq ml+1$, then (4.4) and finally (4.5):

$$\begin{aligned} A_m \ &= \ -\alpha \sum_{j=0}^{m} \mu^j p^{\beta(jl-k)} \geq -\alpha \sum_{j=0}^{m} \mu^j p^{\beta l(j-m-1)} = -\alpha \mu^m p^{-\beta} \sum_{j=0}^{m} (\mu p^{\beta l})^{-j} \\ &\geq \ -\alpha \mu^m p^{-\beta} \sum_{j=0}^{\infty} (\mu p^{\beta l})^{-j} = -\frac{\alpha p^{\beta(l-1)} \mu^{m+1}}{\mu p^{\beta l}-1} \geq -\frac{\mu^{m+1}}{1-\mu}. \end{aligned}$$

The case $m = s$ is now proved. If $m < s$, then for $m < j \leq s$, $\tau^*(p^j, p^k) = 1$, thus

$$A_s = A_m + \sum_{j=m+1}^{s} \mu^j \geq -\frac{\mu^{m+1}}{1-\mu} + \sum_{j=m+1}^{s} \mu^j = -\mu^{s+1}/(1-\mu).$$

□

We now improve Lemma 4.2 and (4.2), so that also for small p, μ can be close to 1/2.

LEMMA 4.3. *Let p be a prime, $1 > \mu > 1/2$ and $a \geq 1$ an integer satisfying*

$$p^{\beta a l} \geq \frac{\alpha p^{-\beta}(1-\mu) + 2\mu - 1}{\mu(2\mu-1)}. \tag{4.6}$$

Then for any positive integers s, k,

$$\sum_{j=0}^{s} \mu^j \tau(p^{aj}, p^k) \geq -\mu^{s+1}/(1-\mu). \tag{4.7}$$

PROOF. We follow the steps of the proof of Lemma 4.2, and first note that (4.6) implies

$$\mu p^{\beta a l} \ > \ 1, \tag{4.8}$$

$$-\frac{\alpha p^{-\beta} + \mu p^{\beta a l} - 1}{\mu p^{\beta a l}-1} \ \geq \ -\frac{\mu}{1-\mu}. \tag{4.9}$$

Denote the left side of (4.7) by B_s. Let m be the largest integer so that $m \le s$, $aml < k$. Then for all $j \le m$, $\tau(p^{aj}, p^k) \ge -\alpha p^{\beta(ajl-k)}$. In the calculation below we apply that and the following facts respectively: $\tau \ge -1$ for $j = m$; $k \ge aml + 1$; then (4.8) and finally (4.9). We thus have

$$\begin{aligned} B_m &\ge -\alpha \sum_{j=0}^{m-1} \mu^j p^{\beta(ajl-k)} - \mu^m \ge -\alpha p^{-\beta} \sum_{j=0}^{m-1} \mu^j p^{\beta al(j-m)} - \mu^m \\ &\ge -\alpha \mu^{m-1} p^{-\beta(al+1)} \sum_{j=0}^{m-1} (\mu p^{\beta al})^{-j} - \mu^m \\ &\ge -\mu^m \frac{\alpha p^{-\beta} + \mu p^{\beta al} - 1}{\mu p^{\beta al} - 1} \ge -\frac{\mu^{m+1}}{1-\mu}. \end{aligned}$$

The rest of the proof is analogous to the proof of Lemma 4.2. ∎

We now set $\lambda = 1/2^\beta$, and combine Lemmas 4.2 and 4.3 to find the prime power components of d_j in Theorem 4.1.

LEMMA 4.4. *There exist a constant c_2 depending only on α, β, so that the following holds: for any positive integer s and prime number $p \le 2^s$, there exist integers $0 = a_0 \le a_1 \le ... \le a_s$ such that for any positive k,*

$$\sum_{j=0}^{s} \lambda^j \tau(p^{a_j}, p^k) \ge -\frac{1+\alpha}{1-\lambda} \lambda^{s+1}, \tag{4.10}$$

$$p^{a_s} < 2^{c_2 s}. \tag{4.11}$$

PROOF. We will distinguish small and large primes, and will apply below Lemma 4.2 for large, and Lemma 4.3 for small primes. Let $p_* = p_*(\alpha, \beta)$ be the smallest prime such that (4.2) holds for $p = p_*$ and $\mu = \lambda = 1/2^\beta$ (and then it holds for all $p \ge p_*$). Let $a_* = a_*(\alpha, \beta)$ be the smallest integer so that (4.6) holds for $p = 2$, $a = a_*$ and $\mu = \lambda = 1/2^\beta$ (and then it holds for all primes p and the same $a = a_*$). We distinguish two cases:

(i) Assume p is small, i.e. $p < p_*$. Then we set $a_j = a_* j$. Because of definition of a_*, we can apply Lemma 4.3 and get

$$\sum_{j=0}^{s} \lambda^j \tau(p^{a_j}, p^k) \ge -\frac{1}{1-\lambda} \lambda^{s+1}. \tag{4.12}$$

We also see that

$$p^{a_s} < p_*^{a_* s}. \tag{4.13}$$

(ii) Let p be large, that means $p^* \le p \le 2^s$. We find an integer q so that $p_*^q \le p < p_*^{q+1}$, and let b, r be the quotient and the remainder of dividing s by q, thus $s = bq + r$. Let $a_j = \lfloor j/q \rfloor$, where $\lfloor x \rfloor$ is the largest integer not greater than x. First note that the function $f(x) = (\alpha + x^q)/(\alpha + x)^q$ is increasing for $x \geq 1$ (e.g. by differentiating). Now applying this, the definition of p_* and $p \geq p_*^q$, we get

$$\lambda^q \geq \left(\frac{\alpha+1}{\alpha+p_*^\beta}\right)^q \geq \frac{\alpha+1}{\alpha+p_*^{\beta q}} \geq \frac{\alpha+1}{\alpha+p^\beta}. \tag{4.14}$$

Denote the right side of (4.10) with $C_s(k)$ and let $C_s^*(k)$ be the same sum with τ^* instead of τ. We can now apply Lemma 4.2 with $\mu = \lambda^q = 1/2^{\beta q}$, and get

$$\begin{aligned} C^*_{bq-1}(k) &= \sum_{j=0}^{b-1}(1+\lambda+...+\lambda^{q-1})\mu^j\tau^*(p^j,p^k) \geq \\ &\geq -\frac{1+\lambda+...+\lambda^{q-1}}{1-\mu}\mu^b = -\sum_{j=bq}^{\infty}\lambda^j. \end{aligned} \tag{4.15}$$

We analyze two cases. Suppose $k \leq bl$. Then $\tau^*(p^f,p^k) = 1$. We use (4.15) and get

$$C_s^*(k) = C^*_{bq-1}(k) + \sum_{j=bq}^{s}\lambda^j \geq -\sum_{j=s+1}^{\infty}\lambda^j = -\frac{1}{1-\lambda}\lambda^{s+1}.$$

Now assume $k > bl$. Then $\tau^*(p^b,p^k) \leq 0$ and also for all $j \leq b-1$, $\tau^*(p^j,p^k) = p^{-\beta}\tau^*(p^j,p^{k-1})$. We now get from (4.15) that

$$C^*_{bq-1}(k) = p^{-\beta}C^*_{bq-1}(k-1) \geq -p^{-\beta}\sum_{j=bq}^{\infty}\lambda^j. \tag{4.16}$$

It is easy to deduce from (4.14) that

$$-p^{-\beta} \geq -\lambda^q. \tag{4.17}$$

As $\tau^*(p^b,p^k) \geq -\alpha p^{-\beta}$, because of (4.16), (4.17) and finally $bq+q \geq s+1$, we get

$$\begin{aligned} C_s^*(k) &= C^*_{bq-1}(k) + \sum_{j=bq}^{s}\lambda^j\tau^*(p^b,p^k) \geq -p^{-\beta}\sum_{j=bq}^{\infty}\lambda^j - \sum_{j=bq}^{s}\lambda^j\alpha p^{-\beta} \geq \\ &\geq -(1+\alpha)\sum_{j=bq+q}^{\infty}\lambda^j \geq -\frac{1+\alpha}{1-\lambda}\lambda^{s+1}. \end{aligned}$$

As $C_s(k) \geq C_s^*(k)$, we see that (4.10) holds in both cases. Finally,

$$p^{a_s} = p^b < p_*^{b(q+1)} \leq p_*^{2s}. \tag{4.18}$$

We get (4.11) from (4.13) and (4.18), with $c_2 = \max\{2, a_*\}\log_2 p_*$. □

We now show why the left side of (4.1) can be reduced to analysis of a prime factor.

LEMMA 4.5. *Let $d_0, d_1, ..., d_s$ be a sequence of integers such that $d_j | d_{j+1}$. Then for each integer q, there exists a prime p such that for all j,*

$$\tau(d_j, q) \geq \tau(p^{a_j}, p^k), \tag{4.19}$$

where p^{a_j}, p^k are the factors in the prime decomposition of d_j, q respectively.

PROOF. If $q = 1$, then $k = 0$, so both sides of (4.19) are equal to 1. Assume now that $q > 1$. Let $m + 1$ be the smallest index such that $q | d_{m+1}^l$ (if there is no such m, we set $m = s$). If $m = 0$, then $q | d_j^l$ for all j, so both sides of (4.19) are equal to 1. In that case, we choose any prime p in the decomposition of q.

Assume now that $1 \leq m \leq s$, and let $r = q/\gcd(q, d_m^l)$ and let p be any prime in the prime decomposition of r. For $j \geq m + 1$, both sides of (4.19) are equal to 1. For $j \leq m$, it is straightforward to check (4.19). $\square$

We now complete the proof of Theorem 4.1. Recall that $\lambda = 1/2^\beta$. Let c_3 be the largest of the constants $(1 + \alpha)/(1 - \lambda)$ and α/λ, and choose s so that

$$\lambda\delta/c_3 \leq \lambda^{s+1} \leq \delta/c_3. \tag{4.20}$$

Let $\Lambda = 1 + \lambda + \cdots + \lambda^s$, $m = 2^s$ and $2 = p_1 < p_2 < \ldots < p_t$ be all prime numbers between 1 and m, and let a_j^i be the exponents constructed in Lemma 4.4, associated to the prime p_i, $i = 1, \ldots, t$, $j = 0, \ldots, s$. We set

$$d_j = \prod_{i=1}^{t} p_i^{a_j^i}.$$

Let p be the smallest prime number constructed in Lemma 4.5. If $p \leq m$, then $p = p_i$, for some $i = 1, \ldots, t$. Now applying Lemma 4.5, Lemma 4.4, (4.20) and $\Lambda \geq 1$, we deduce that for any positive integer q,

$$\frac{1}{\Lambda}\sum_{j=0}^{s} \lambda^j \tau(d_j, q) \geq \frac{1}{\Lambda}\sum_{j=0}^{s} \lambda^j \tau(p_i^{a_j^i}, p_i^k) \geq -\frac{1 + \alpha}{(1 - \lambda)\Lambda}\lambda^{s+1} \geq -\delta.$$

Now assume that $p > m$. Then Lemma 4.5 and (4.20) imply that

$$\frac{1}{\Lambda}\sum_{j=0}^{s} \lambda^j \tau(d_j, q) \geq -\alpha p^{-\beta} \geq -\delta.$$

We deduce that (4.1) holds. Now we estimate d_s. By (4.20) and the definition of m, we get $m \leq (c_3/\delta)^{1/\beta}$ and thus

$$s \leq \frac{1}{\log 2}\log(c_3/\delta)^{1/\beta}. \tag{4.21}$$

The prime number theorem implies that $t \leq c_4(m/\log m)$, for some constant c_4, so

$$t \leq c_4 \frac{(c_3/\delta)^{1/\beta}}{\log(c_3/\delta)^{1/\beta}}. \tag{4.22}$$

Finally, by applying (4.11), (4.21) and (4.22), we get that $d_s \leq \exp(c_2 st) \leq \exp(c_1 \delta^{-1/\beta})$, where $c_1 = c_2 c_3^{1/\beta} c_4/\log 2$.

5. Proof of Theorem 1.1

We choose $\delta > 0$, and find first, by applying the Theorem 4.1, the constants $d_0, ..., d_s$ and $\lambda > 0$ such that for all integers q,

$$\frac{1}{\Lambda}\sum_{j=0}^{s} \lambda^j \tau(d_j, q) \ \geq\ -\delta/2, \tag{5.1}$$

$$d_s \ \leq\ \exp(c_5 \delta^{-k}), \tag{5.2}$$

where $\Lambda = 1 + \lambda + \cdots \lambda^s$ and c_5 depends on the degree and the coefficients of the polynomial f. Let c_6 and c_7 be the implicit constants from Corollary 2.4 and Proposition 3.2 respectively. To streamline the calculations below, we define $c_8 = 2(\alpha_k + 1)\max\{c_5, c_6, c_7\}/c_5$ and $d_* = c_8 \exp(c_5 \delta^{-k})$. Then it is easy to check that

$$\max\{c_6, c_7\}(\alpha_k + 1) d_*^{-1} \leq \frac{\delta}{2} \tag{5.3}$$

and that $d_j \leq d_*$ for all $j = 1, ..., s$. Compiling the constraints and the error terms from Corollary 2.4 and Proposition 3.2, we see that it is now enough to choose the constants n, Q, R so that:

$$\begin{aligned} c_6 d_* n^{-1/k}(Q + n/R) \ &\leq\ \delta/2, \\ d_* \ &\leq\ n^{1/k}, \\ \alpha_k d_*^k n^{1/k^2} \ &\leq\ Q, \\ c_7\left(n^{-1/k}\,(QR)^{1/(k-1/k)} + d_* n^{-\rho/k}\right) \ &\leq\ \delta/2, \end{aligned}$$

where $\rho = 1/(8k^2(\log k + 1.5\log\log k + 4.2)$. One can check using (5.3) that the choice $n = d_*^{k^8}$, $Q = \alpha_k d_*^{1.5k^6}$ and $R = d_*^{k^8 - k^7 + k^5 - 2.5k^4}$ satisfies all these relations. We now define the cosine polynomial

$$T(x) = \delta + \frac{(1-\delta)}{\Lambda}\sum_{j=0}^{s} \lambda^j G_{n,d_j}(x).$$

Clearly $T(0) = 1$. Now for $x \in \mathfrak{M}(Q, R)$, Corollary 2.4, (5.1) with the choice of constants above imply $T(x) \geq \delta + (1-\delta)(-\delta/2 - \delta/2) \geq 0$. Similarly for $x \in \mathfrak{m}(Q, R)$, $T(x) \geq 0$. Choose m_j such that $\alpha_k d_j^k m_j^k \leq n < \alpha_k d_j^k (m_j + 1)^k$, for $j = 0, \ldots, s$, and let $m = \max\{m_0, \ldots, m_s\}$. For given $\delta > 0$, the largest non-zero coefficient of the polynomial T is of the order at most $N = P(d_* m)$. From (2.1), we get $N = O_f(\exp(c_5(k+k^8)\delta^{-k}))$, thus $\delta = O_f((\log N)^{-1/k})$.

6. Proof of the lower bound

The proof of the lower bound mimics the construction of I. Ruzsa in the case of $f(x) = x^2$ [15].

LEMMA 6.1. *Let $k \geq 3$ be an odd integer, $\beta > 0$ an integer, and $p \equiv 1 \pmod{k}$ a prime, $p > \beta$. Then there exists a collection of integers $d_1, d_2, ..., d_s$, $s = (p-1)/k$ such that for any integer j, $(j,p) = 1$,*

$$\sum_{i=1}^{s} \cos(2\pi\beta j^k d_i/p) \leq -\sqrt{s/(k-2)}.$$

PROOF. Assume without loss of generality that $\beta = 1$ (we can do it as $(\beta, p) = 1$). As the congruence $x^k \equiv y^k (\operatorname{mod} p)$ has k solutions for a fixed y relatively prime with p, we can divide the set of $p-1$ reduced residue classes $\operatorname{mod} p$ into k equivalence classes $Q_1, ..., Q_k$ of size $s = (p-1)/k$, defined as: $a_1 \sim a_2$ if for some j, $(j,p) = 1$,

$$a_1 a_2^{-1} \equiv j^k \pmod{p}$$

(the a_2^{-1} is the multiplicative inverse of $a_2 \operatorname{mod} p$). As k is odd, $a \sim -a$. We conclude that the sum A_m defined below is real, on the left-hand side independent of j, $(j,p) = 1$ and on the right-hand side independent of $a \in Q_m$:

$$A_m = \sum_{a \in Q_m} e(j^k a/p) = \frac{1}{k} \sum_{j=1}^{p-1} e(j^k a/p). \tag{6.1}$$

By definition,

$$\sum_{m=1}^{k} A_m = \sum_{a,(a,p)=1} e(j^k a/p) = -1. \tag{6.2}$$

We now evaluate $\sum_{m=1}^{k} A_m^2$ by using the right-hand side of (6.1), and get

$$\begin{aligned} \sum_{m=1}^{k} sk^2 A_m^2 &= \sum_{a=1}^{p-1} \left| \sum_{x=1}^{p-1} e(x^k a/p) \right|^2 = \sum_{a=1}^{p-1} \sum_{x,y=1}^{p-1} e((x^k - y^k)a/p) = \\ &= \sum_{x,y=1}^{p-1} \sum_{a=1}^{p-1} e((x^k - y^k)a/p) = (kp - p + 1)(p-1), \end{aligned}$$

$$\sum_{m=1}^{k} A_m^2 = p - s, \tag{6.3}$$

where we used that $x^k \equiv y^k$ has k solutions $\bmod p$ for a fixed y relatively prime with p. Now suppose all $A_m \geq -c$ for some $c \geq 0$. If there are k^- numbers $A_m < 0$, $1 \leq k^- \leq k$, then $\sum_{m, A_m<0} A_m^2 \leq k^- c^2$, and by using (6.2), $\sum_{m, A_m \geq 0} A_m^2 \leq (k^- c - 1)^2$. Combining that with (6.3), we easily get $c \geq \sqrt{s/(k-2)}$. Now we can find $A_m \leq -\sqrt{s/(k-2)}$, and choose $d_1, ..., d_s$ to be the elements of Q_m. ∎

We now complete the proof of Theorem 1.3. Let $f(x) = \beta x^k$. Without loss of generality assume $\beta = 1$. Choose any cosine polynomial $T(x)$ as in (1.2) of degree n with coefficients $b_0, ..., b_n$ where b_j is non-zero only if j is an integer value of $f(x)$, $T(x) \geq 0$ for all $x \in \mathbb{R}$ and $T(0) = 1$. By calculating $\sum_{i=1}^{s} T(d_i/p)$, $p \equiv 1 \pmod{k}$, p a prime, applying Lemma 6.1 and noting that $1 + \sqrt{s(k-2)} \leq \sqrt{p}$, we get

$$\sum_{p|j,\, j=0,\ldots,n} b_j \geq \frac{1}{\sqrt{p}}. \tag{6.4}$$

Let $m \leq n$ be an integer to be chosen later. We multiply (6.4) by $\log p$ and sum over primes $p \leq m$, $p \equiv 1 \pmod{k}$. We get

$$\begin{aligned} \sum_{j=0}^{n} b_j \sum_{p|j, p\equiv 1 (\mathrm{mod}\, k), p\leq m} \log p &= \sum_{p\equiv 1 (\mathrm{mod}\, k), p\leq m} \left(\log p \sum_{p|j,\, j=0,\ldots,n} b_j \right) \\ &\geq \sum_{p\equiv 1 (\mathrm{mod}\, k), p\leq m} \frac{\log p}{\sqrt{p}}. \end{aligned} \tag{6.5}$$

By the theorem on primes in arithmetic sequences (e.g. [9], (11.32)),

$$\sum_{p\equiv 1 (\mathrm{mod}\, k), p\leq m} \log p = \frac{m}{\varphi(k)} + mO\left(\frac{1}{E(m)}\right) = m\left(\frac{1}{\varphi(k)} + o(1)\right),$$

where $E(x) = \exp(c_1\sqrt{\log x})$, c_1 an absolute constant and $\varphi(k)$ is the Euler's totient function. Now it is easy to check (e.g. by dividing the segment $0, ..., m$ into $E^{1/2}(m)$ equal segments and estimating the sum over them), that

$$\sum_{p\equiv 1 (\mathrm{mod}\, k), p\leq m} \frac{\log p}{\sqrt{p}} = \sqrt{m}\left(\frac{2}{\varphi(k)} + o(1)\right).$$

On the left-hand side of (6.5), the coefficient of b_j, $j > 0$ is $\leq \log j \leq \log n$ (where n is the largest non-zero coefficient of $T(x)$). Again by the theorem on primes in aritmetic sequences, the coefficient of b_0 is $m(1/\varphi(k) + o(1))$. As all $b_j \geq 0$ and $\sum_{j=0}^{n} b_j = T(0) = 1$, we thus get

$$mb_0\left(\frac{1}{\varphi(k)} + o(1)\right) + \log n \geq \sqrt{m}\left(\frac{2}{\varphi(k)} + o(1)\right).$$

We express b_0 as $\geq$ of a function of m, maximize over m and obtain $b_0 \geq (1/\varphi(k) + o(1)) / \log n$.

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Pozitivne eksponencijalne sume i neparni polinomi

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Sažetak. Ako je zadan neparan cjelobrojan polinom $f(x)$ stupnja $k \geq 3$, konstruiramo normirani trigonometrijski polinom s nenegativnim koeficijentima, koji nisu nula samo za vrijednosti $f(x)$ ne veće od n, te s malim slobodnim koeficijentom $a_0 = O((\log n)^{-1/k})$. To daje alternativni dokaz ograde za najveći mogući kardinalni broj skupa prirodnih brojeva A takvog da $A - A$ ne sadrži vrijednost od $f(x)$. Također razmatramo druge interpretacije, te ergodsku karakterizaciju te ograde.

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