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GENERALIZATIONS AND IMPROVEMENTS OF AN INEQUALITY OF HARDY-LITTLEWOOD-PÓLYA

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Abstract. Some generalizations of an inequality of Hardy-Littlewood-Pólya are presented. We discuss the n-exponential convexity and log-convexity of the functions associated with the linear functional defined by the generalized inequality and also prove the monotonicity property of the generalized Cauchy means obtained via this functional. Finally, we give several examples of the families of functions for which the results can be applied.

1. Introduction and preliminaries

The following theorem is given in the famous Hardy-Littlewood-Pólya book [2, Theorem 134].

Theorem 1.1. *If f is a convex and continuous function defined on* $[0, \infty)$ *and* $(a_k, k \in \mathbb{N})$ *are non-negative and non-increasing, then*

$$f\left(\sum_{k=1}^{n} a_k\right) \geq f(0) + \sum_{k=1}^{n} \left[f(ka_k) - f((k-1)a_k)\right]. \tag{1.1}$$

If f is concave, then the inequality in (1.1) *reverses. If f' is a strictly increasing function, there is equality only when a_k are equal up to a certain point and then zero.*

An example of the above theorem is given below (see [2, p.100]).

Corollary 1.2. *Let $a_k \in [0, \infty)$ and the sequence $(a_k,\, k \in \mathbb{N})$ is non-increasing. If $s > 1$, then we have*

$$\left(\sum_{k=1}^{n} a_k\right)^{s} \geq \sum_{k=1}^{n} {a_k}^{s} \left[k^{s} - (k-1)^{s}\right]. \tag{1.2}$$

If $0 < s < 1$, then the inequality in (1.2) *reverses.*

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Inequality (1.1) is of great interest and has been generalized in many different ways by various mathematicians.

In order to obtain our main results we need some definitions.

DEFINITION 1.3. *A sequence* $(a_k,\ k \in \mathbb{N}) \subset \mathbb{R}$ *is non-increasing in mean if*

$$\frac{1}{n}\sum_{k=1}^{n} a_k \geq \frac{1}{n+1}\sum_{k=1}^{n+1} a_k, \quad n \in \mathbb{N}. \tag{1.3}$$

A sequence $(a_k,\ k \in \mathbb{N}) \subset \mathbb{R}$ is non-decreasing in mean, if opposite inequality holds in (1.3).
In a similar way we can define when a finite sequence $(a_k,\ k = 1, \ldots, n) \subset \mathbb{R}$ is non-increasing in mean or non-decreasing in mean (see [4]).

REMARK 1.4. If we denote $S_k = \sum_{i=1}^{k} a_i,\ k \in \mathbb{N}$, then it is easy to see that the sequence $(a_k,\ k \in \mathbb{N})$ is non-increasing in mean (non-decreasing in mean) if and only if $S_{k-1} \geq (k-1)a_k$ $(S_{k-1} \leq (k-1)a_k)$ for $k = 2, 3, \ldots$.

In 1995, inequality (1.2) was improved by J. Pečarić and L. E. Persson in [5] and this improvement is given below:

THEOREM 1.5. *If the sequence* $(a_k > 0, k \in \mathbb{N})$ *is non-increasing in mean and if* s *is a real number such that* $s > 1$, *then*

$$\left(\sum_{k=1}^{\infty} a_k\right)^s \geq \sum_{k=1}^{\infty} {a_k}^s \left[k^s - (k-1)^s\right] \tag{1.4}$$

holds. If $0 < s < 1$, *then the inequality in* (1.4) *reverses.*

It is well known and easy to see that if a sequence $(a_k,\ k \in \mathbb{N})$ is non-increasing (non-decreasing), then it is also non-increasing in mean (non-decreasing in mean) but the reverse implications do not hold in general. This means that Theorem 1.5 is a genuine generalization of Theorem 1.2. The following property of a convex function will be useful further (see [7, p.2]).

DEFINITION 1.6. *A function* $f : I \to \mathbb{R}$ *is convex on* I *if*

$$(x_3 - x_2)\, f(x_1) + (x_1 - x_3)\, f(x_2) + (x_2 - x_1)\, f(x_3) \geq 0 \tag{1.5}$$

holds for all $x_1, x_2, x_3 \in I$ *such that* $x_1 < x_2 < x_3$.

Another characterization of a convex function will be needed later (see [7, p.2]).

PROPOSITION 1.7. *If* f *is a convex function defined on an interval* I *and if* $x_1 \leq y_1,\ x_2 \leq y_2,\ x_1 \neq x_2,\ y_1 \neq y_2$, *then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq \frac{f(y_2) - f(y_1)}{y_2 - y_1}. \tag{1.6}$$

If the function f *is concave, then the inequality reverses.*

By letting $x_1 = x$, $x_2 = x + h$, $y_1 = y$ and $y_2 = y + h$, $(x \leq y, h \geq 0)$ in (1.6), we have

$$f(x+h) - f(x) \leq f(y+h) - f(y). \tag{1.7}$$

The following definition of Wright-convex function is given in [7, p.7].

Definition 1.8. *A function $f : [a,b] \to \mathbb{R}$ is said to be Wright-convex if for all x, $y + h \in [a,b]$ such that $x \leq y$, $h \geq 0$,* (1.7) *holds. The function f is said to be Wright-concave if opposite inequality holds in* (1.7).

Remark 1.9. If $K([a,b])$ and $W([a,b])$ denote the set of all convex functions and the set of all Wright-convex functions defined on $[a,b]$, then $K([a,b]) \subsetneqq W([a,b])$. That is, a convex function is also a Wright-convex function but not conversely (*see*[7, p.7]).

Wright-convex functions have interesting and important generalization for functions of several variables (see [1]). Let $\mathbb{R}^m$ denote the m-dimensional vector lattice of points $\mathbf{x} = (x_1, \dots, x_m)$, $x_i \in \mathbb{R}$ for $i = 1, \dots, m$ with the partial ordering

$$\mathbf{x} = (x_1, \dots, x_m) \leq \mathbf{y} = (y_1, \dots, y_m)$$

if and only if $x_i \leq y_i$ for $i = 1, \dots, m$ (see [7, p.13]).

Definition 1.10. *A sequence $(\mathbf{a}_k, k \in \mathbb{N}) \subset \mathbb{R}^m$ is non-increasing in mean if*

$$\frac{1}{n}\sum_{k=1}^{n} \mathbf{a}_k \geq \frac{1}{n+1}\sum_{k=1}^{n+1} \mathbf{a}_k, \quad n \in \mathbb{N}, \tag{1.8}$$

where $\mathbf{a}_k = (a_k^1, \dots, a_k^m)$. A sequence $(\mathbf{a}_k, k \in \mathbb{N}) \subset \mathbb{R}^m$ is non-decreasing in mean, if opposite inequality holds in (1.8).

In [1], H. D. Brunk presented an interesting class of multivariate real-valued functions defined as follows:

Definition 1.11. *A real-valued function f on an m-dimensional rectangle $\mathbf{I} \subset \mathbb{R}^m$ is said to have non-decreasing increments if*

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) \leq f(\mathbf{y} + \mathbf{h}) - f(\mathbf{y}), \tag{1.9}$$

whenever $\mathbf{x}$, $\mathbf{y} + \mathbf{h} \in \mathbf{I}$, $\mathbf{0} \leq \mathbf{h} \in \mathbb{R}^m$, $\mathbf{x} \leq \mathbf{y}$. The function f is said to have non-increasing increments if opposite inequality holds in (1.9).

Remark 1.12. It is easy to see that if a function f is defined on $[a,b] \subset \mathbb{R}$ and has non-decreasing increment, then it is a Wright-convex function.

In this paper, we present some generalizations of the Hardy-Littlewood-Pólya inequality (1.1). First generalization is obtained by using a Wright-convex function and a sequence $(a_k, k = 1, \dots, n) \subset \mathbb{R}$ which is non-increasing in mean. Since the class of all Wright-convex functions contains the class of all convex functions, our first main result is the generalization of the inequality (1.1) in the sense that we can obtain (1.1) as a special case of our first main result. Second generalization is obtained by using a real-valued function defined on an m-dimensional rectangle $\mathbf{I} \subset \mathbb{R}^m$, having non-decreasing increments and a sequence $(\mathbf{a}_k, k = 1, \dots, n) \subset \mathbb{R}^m$, which is non-increasing in mean. We also discuss the case when the sequence is non-decreasing in mean. Further, we give the n-exponential convexity and log-convexity of the functions associated with the linear functional defined by the generalized inequality and we also deduce Lyapunov-type inequalities for this functional. We prove monotonicity property of the generalized Cauchy means obtained via this functional. Finally, we give several examples of the families of functions for which the results can be applied.

2. Main results

Inequality (1.1) was already proved in [2], but we give a new proof in a more general setting. Our first main result states:

Theorem 2.1. *Let a_k $(k = 1, \dots, n)$ be real numbers such that $a_k \geq 0$, $S_k = \sum_{i=1}^{k} a_i$, ka_k, $(k-1)a_k \in [a, b]$ for all $k = 2, \dots, n$ and $f : [a, b] \to \mathbb{R}$ be a Wright-convex function.*

(i) *If the sequence $(a_k,\, k = 1, \dots, n)$ is non-increasing in mean, then we have*

$$f\left(\sum_{k=1}^{n} a_k\right) \geq f(a_1) + \sum_{k=2}^{n} \left[f(ka_k) - f((k-1)a_k)\right]. \tag{2.1}$$

(ii) *If the sequence $(a_k,\, k = 1, \dots, n)$ is non-decreasing in mean, then we have*

$$f\left(\sum_{k=1}^{n} a_k\right) \leq f(a_1) + \sum_{k=2}^{n} \left[f(ka_k) - f((k-1)a_k)\right]. \tag{2.2}$$

If the function f is Wright-concave, then opposite inequalities hold in (2.1) *and* (2.2).

Proof. (i) Since the sequence $(a_k,\, k = 1, \dots, n) \subset \mathbb{R}$ is non-increasing in mean, by definition we have, $S_{k-1} \geq (k-1)\, a_k$ for $k = 2, \dots, n$. As f is a Wright-convex function, by setting $x = (k-1)\, a_k$, $y = S_{k-1}$ and $h = a_k$ $(k = 2, \dots, n)$ in (1.7), and also using the fact that $S_{k-1} + a_k = S_k$ $(k = 2, \dots, n)$, we have

$$f(S_k) - f(S_{k-1}) \geq f(ka_k) - f((k-1)a_k).$$

Summing over k from 2 to n, we have

$$\sum_{k=2}^{n} \left[f(S_k) - f(S_{k-1})\right] \geq \sum_{k=2}^{n} [f(ka_k) - f((k-1)a_k)],$$

which is equivalent to

$$f(S_n) - f(S_1) \geq \sum_{k=2}^{n} [f(ka_k) - f((k-1)a_k)],$$

and so (2.1) holds.

(ii) Since the sequence $(a_k,\ k = 1, \dots, n) \subset \mathbb{R}$ is non-decreasing in mean, by definition we have, $S_{k-1} \leq (k-1)a_k$ for $k = 2, \dots, n$. As f is a Wright-convex function, by setting $x = S_{k-1}$, $y = (k-1)a_k$ and $h = a_k$ $(k = 2, \dots, n)$ in (1.7), we have

$$f(S_k) - f(S_{k-1}) \leq f(ka_k) - f((k-1)a_k).$$

Now summing over k from 2 to n and after simplification, we have (2.2).

If f is a Wright-concave function, then opposite inequality holds in (1.7) and so opposite inequalities hold in (2.1) and (2.2). ∎

Since the class of Wright-convex (Wright-concave) functions properly contains the class of convex (concave) functions (see for example [7, p.7]), the following result is valid.

Corollary 2.2. *Let a_k $(k = 1, \dots, n)$ be real numbers such that $a_k \geq 0$, $S_k = \sum_{i=1}^{k} a_i$, ka_k, $(k-1)a_k \in [a, b]$ for all $k = 2, \dots, n$ and $f : [a, b] \to \mathbb{R}$ be a convex function.*

(i) *If the sequence $(a_k,\ k = 1, \dots, n)$ is non-increasing in mean, then* (2.1) *holds.*

(ii) *If the sequence $(a_k,\ k = 1, \dots, n)$ is non-decreasing in mean, then* (2.2) *holds.*

If the function f is concave, then opposite inequalities hold in (2.1) *and* (2.2).

The following corollary is an application of Corollary 2.2.

Corollary 2.3. *Let $f(x) = x^s$, where $x \in (0, \infty)$ and $s \in \mathbb{R}$.*

(i) *If the sequence $(a_k > 0,\ k = 1, \dots, n)$ is non-increasing in mean and $s \in \mathbb{R}$ such that $s < 0$ or $s > 1$, then*

$$\left(\sum_{k=1}^{n} a_k \right)^s \geq a_1^s + \sum_{k=2}^{n} {a_k}^s \left[k^s - (k-1)^s \right] \tag{2.3}$$

holds. If $0 < s < 1$, then the inequality in (2.3) *reverses.*

(ii) *If the sequence $(a_k > 0,\ k = 1, \dots, n)$ is non-decreasing in mean and $s \in \mathbb{R}$ such that $s < 0$ or $s > 1$, then*

$$\left(\sum_{k=1}^{n} a_k \right)^s \leq a_1^s + \sum_{k=2}^{n} {a_k}^s \left[k^s - (k-1)^s \right] \tag{2.4}$$

holds. If $0 < s < 1$, then the inequality in (2.4) *reverses.*

The multidimensional generalization is stated as follows:

THEOREM 2.4. *Let* $\mathbf{a}_k \in \mathbb{R}^m$ *be such that* $\mathbf{a}_k \geq 0$ *for all* $k = 1, \ldots, n$, $\mathbf{S}_k = \sum_{i=1}^{k} \mathbf{a}_i$, $k\mathbf{a}_k$, $(k-1)\mathbf{a}_k \in \mathbf{I} \subseteq \mathbb{R}^m$ *for all* $k = 2, \ldots, n$ *and* $f : \mathbf{I} \subseteq \mathbb{R}^m \to \mathbb{R}$ *be a real valued function having non-decreasing increments.*

(i) *If the sequence* $(\mathbf{a}_k,\, k = 1, \ldots, n)$ *is non-increasing in mean, then we have*

$$f\left(\sum_{k=1}^{n} \mathbf{a}_k\right) \geq f(\mathbf{a}_1) + \sum_{k=2}^{n} \left[f\left(k\mathbf{a}_k\right) - f\left((k-1)\,\mathbf{a}_k\right)\right]. \tag{2.5}$$

(ii) *If the sequence* $(\mathbf{a}_k,\, k = 1, \ldots, n)$ *is non-decreasing in mean, then we have*

$$f\left(\sum_{k=1}^{n} \mathbf{a}_k\right) \leq f\left(\mathbf{a}_1\right) + \sum_{k=2}^{n} \left[f\left(k\mathbf{a}_k\right) - f((k-1)\mathbf{a}_k)\right]. \tag{2.6}$$

If the function has non-increasing increments, then opposite inequalities hold in (2.5) *and* (2.6).

PROOF. The idea of the proof is the same as in Theorem 2.1.

(i) Since the sequence $(\mathbf{a}_k,\, k = 1, \ldots, n) \subset \mathbb{R}^m$ is non-increasing in mean, by definition we have, $\mathbf{S}_{k-1} \geq (k-1)\mathbf{a}_k$ for $k = 2, \ldots, n$. By setting $\mathbf{x} = (k-1)\mathbf{a}_k$, $\mathbf{y} = \mathbf{S}_{k-1}$ and $\mathbf{h} = \mathbf{a}_k$ $(k = 2, \ldots, n)$ in (1.9), where f has non-decreasing increments and also using the fact that $\mathbf{S}_{k-1} + \mathbf{a}_k = \mathbf{S}_k$ $(k = 2, \ldots, n)$, we have

$$f\left(\mathbf{S}_k\right) - f\left(\mathbf{S}_{k-1}\right) \geq f\left(k\mathbf{a}_k\right) - f\left((k-1)\,\mathbf{a}_k\right).$$

Summing over k from 2 to n, we have

$$\sum_{k=2}^{n} \left[f\left(\mathbf{S}_k\right) - f\left(\mathbf{S}_{k-1}\right)\right] \geq \sum_{k=2}^{n} [f(k\mathbf{a}_k) - f((k-1)\mathbf{a}_k)],$$

which is equivalent to

$$f(\mathbf{S}_n) - f(\mathbf{S}_1) \geq \sum_{k=2}^{n} [f(k\mathbf{a}_k) - f((k-1)\mathbf{a}_k)],$$

and so (2.5) holds.

(ii) Since the sequence $(\mathbf{a}_k,\, k = 1, \ldots, n) \subset \mathbb{R}^m$ is non-decreasing in mean, by definition we have, $\mathbf{S}_{k-1} \leq (k-1)\mathbf{a}_k$ for $k = 2, \ldots, n$. By setting $\mathbf{x} = \mathbf{S}_{k-1}$, $\mathbf{y} = (k-1)\mathbf{a}_k$ and $\mathbf{h} = \mathbf{a}_k$ $(k = 2, \ldots, n)$ in (1.9), where f has non-decreasing increments, we have

$$f\left(S_k\right) - f\left(\mathbf{S}_{k-1}\right) \leq f\left(k\mathbf{a}_k\right) - f\left((k-1)\,\mathbf{a}_k\right).$$

Now summing over k from 2 to n and after simplification, we have (2.6).

If f has non-increasing increments, then opposite inequality holds in (1.9) and so opposite inequalities hold in (2.5) and (2.6). ∎

Consider the inequality (2.1) and define a functional

$$\Phi(f) = f\left(\sum_{k=1}^{n} a_k\right) - f(a_1) - \sum_{k=2}^{n}\left[f(ka_k) - f((k-1)a_k)\right], \tag{2.7}$$

where a_k $(k = 1, \ldots, n)$ are real numbers such that $a_k \geq 0$, $S_k = \sum_{i=1}^{k} a_i$, ka_k and $(k-1)a_k \in [a,b]$ for all $k = 2, \ldots, n$. If the function f is convex on $[a,b]$ and the sequence $(a_k,\, k = 1, \ldots, n) \subset \mathbb{R}$ is non-increasing in mean, then Corollary 2.2 (i) implies that $\Phi(f) \geq 0$.

Now, we give mean value theorems for the functional Φ. These theorems enable us to define various classes of means that can be expressed in terms of a linear functional.

Theorem 2.5. *Let a_k $(k = 1, \ldots, n)$ be real numbers such that $a_k \geq 0$, $S_k = \sum_{i=1}^{k} a_i$, ka_k, $(k-1)a_k \in [a,b]$ for all $k = 2, \ldots, n$ and the sequence $(a_k,\, k = 1, \ldots, n)$ is non-increasing in mean. Suppose that Φ is a linear functional defined as in* (2.7) *and $f \in C^2([a,b])$. Then there exists $\xi \in [a,b]$ such that*

$$\Phi(f) = \frac{f''(\xi)}{2}\,\Phi(f_0),$$

where $f_0(x) = x^2$.

Proof. Analogous to the proof of Theorem 2.2 in [6]. ∎

We can prove the following Cauchy type mean value theorem by following the proof of Theorem 2.4 in [6].

Theorem 2.6. *Let a_k $(k = 1, \ldots, n)$ be real numbers such that $a_k \geq 0$, $S_k = \sum_{i=1}^{k} a_i$, ka_k, $(k-1)a_k \in [a,b]$ for all $k = 2, \ldots, n$ and the sequence $(a_k,\, k = 1, \ldots, n)$ is non-increasing in mean. Suppose that Φ is a linear functional defined as in* (2.7) *and $f, g \in C^2([a,b])$. Then there exists $\xi \in [a,b]$ such that*

$$\frac{\Phi(f)}{\Phi(g)} = \frac{f''(\xi)}{g''(\xi)}, \tag{2.8}$$

provided that the denominators are non-zero.

Remark 2.7. (i) By taking $f(x) = x^s$ and $g(x) = x^q$ in (2.8), where $s, q \in \mathbb{R} \setminus \{0,1\}$ are such that $s \neq q$, we have

$$\xi^{s-q} = \frac{q(q-1)\,\Phi(x^s)}{s(s-1)\,\Phi(x^q)}.$$

(ii) If the inverse of the function f''/g'' exists, then (2.8) gives

$$\xi = \left(\frac{f''}{g''}\right)^{-1}\left(\frac{\Phi(f)}{\Phi(g)}\right).$$

3. n-exponential convexity and log-convexity of the functions associated with the difference of the generalized inequality

We begin this section by recollecting definitions and properties which are going to be explored here and also some useful characterizations of these properties. In the sequel, let I be an open interval in $\mathbb{R}$.

Definition 3.1. *A function $h : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j h\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for every $\alpha_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \ldots, n$ (see [6]).

Definition 3.2. *A function $h : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.*

Remark 3.3. From the above definition, it is clear that 1-exponentially convex functions in the Jensen sense are non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex functions in the Jensen sense for all $k \in \mathbb{N}$, $k \leq n$.

By the definition of positive semi-definite matrices and some basic linear algebra, we have the following proposition.

Proposition 3.4. *If $h : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense, then the matrix $\left[h\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is a positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leq n$. Particularly,*

$$\det\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0 \quad \text{for every } k \in \mathbb{N},\ k \leq n,\ \ x_i \in I,\ \ i = 1, \ldots, n.$$

Definition 3.5. *A function $h : I \to \mathbb{R}$ is exponentially convex in the Jensen sense if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.*

Definition 3.6. *A function $h : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.*

Lemma 3.7. *A function $h : I \to (0, \infty)$ is* log*-convex in the Jensen sense, that is, for every $x, y \in I$,*

$$h^2\left(\frac{x + y}{2}\right) \leq h(x)\, h(y)$$

holds if and only if the relation

$$\alpha^2 h(x) + 2\alpha\beta\, h\left(\frac{x+y}{2}\right) + \beta^2 h(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ *and* $x, y \in I$.

REMARK 3.8. It follows that a function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. Also, by using basic convexity theory, a function is log-convex if and only if it is 2-exponentially convex.

The following definition of divided difference is given in [7, p.14].

DEFINITION 3.9. *The second-order divided difference of a function* $f : [a, b] \to \mathbb{R}$ *at mutually distinct points* $y_0, y_1, y_2 \in [a, b]$ *is defined recursively by*

$$[y_i; f] = f(y_i), \quad i = 0, 1, 2,$$

$$[y_i, y_{i+1}; f] = \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i}, \quad i = 0, 1,$$

$$[y_0, y_1, y_2; f] = \frac{[y_1, y_2; f] - [y_0, y_1; f]}{y_2 - y_0}. \tag{3.1}$$

REMARK 3.10. The value $[y_0, y_1, y_2; f]$ is independent of the order of the points y_0, y_1 and y_2. This definition may be extended to include the case in which some or all the points coincide (see [7, p.16]). Namely, taking the limit $y_1 \to y_0$ in (3.1), we get

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_2; f] = \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, \; y_2 \neq y_0,$$

provided that f' exists; and furthermore, taking the limits $y_i \to y_0$, $i = 1, 2$, in (3.1), we get

$$\lim_{y_2 \to y_0} \lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_0; f] = \frac{f''(y_0)}{2},$$

provided that f'' exists.

REMARK 3.11. Convex functions can be characterized by second order divided difference (see [7, p.16]): a function $f : [a, b] \to \mathbb{R}$ is convex if and only if for all choices of three distinct points $y_0, y_1, y_2, \in [a, b]$, $[y_0, y_1, y_2; f] \geq 0$.

Next, we study the n-exponential convexity and log-convexity of the functions associated with the linear functional Φ defined in (2.7).

THEOREM 3.12. *Let* $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ *be a family of functions defined on* $[a, b]$ *such that the function* $s \mapsto [y_0, y_1, y_2; f_s]$ *is n-exponentially convex in the Jensen sense on* I *for every three mutually distinct points* $y_0, y_1, y_2 \in [a, b]$. *Let* Φ *be a linear functional defined as in* (2.7). *Then the following statements hold:*

(i) *The function* $s \mapsto \Phi(f_s)$ *is n-exponentially convex in the Jensen sense on* I *and the matrix* $\left[\Phi(f_{\frac{s_j+s_k}{2}})\right]_{j,k=1}^{m}$ *is a positive semi-definite matrix for all* $m \in \mathbb{N}$*,* $m \leq n$ *and* $s_1, \ldots, s_m \in I$*. Particularly,*

$$\det\left[\Phi(f_{\frac{s_j+s_k}{2}})\right]_{j,k=1}^{m} \geq 0, \ \forall \ m \in \mathbb{N}, \ m \leq n.$$

(ii) *If the function* $s \mapsto \Phi(f_s)$ *is continuous on* I*, then it is n-exponentially convex on* I*.*

PROOF. The idea of the proof is the same as that of Theorem 3.1 in [6].

(i) Let $\alpha_j \in \mathbb{R}$ $(j = 1, \ldots, n)$ and consider the function

$$\varphi(y) = \sum_{j,k=1}^{n} \alpha_j \alpha_k f_{\frac{s_j+s_k}{2}}(y),$$

where $s_j \in I$ and $f_{\frac{s_j+s_k}{2}} \in \Omega$. Then

$$[y_0, y_1, y_2; \varphi] = \sum_{j,k=1}^{n} \alpha_j \alpha_k \left[y_0, y_1, y_2; f_{\frac{s_j+s_k}{2}}\right]$$

and since $\left[y_0, y_1, y_2; f_{\frac{s_j+s_k}{2}}\right]$ is n-exponentially convex in the Jensen sense on I by assumption, it follows that

$$[y_0, y_1, y_2; \varphi] = \sum_{j,k=1}^{n} \alpha_j \alpha_k \left[y_0, y_1, y_2; f_{\frac{s_j+s_k}{2}}\right] \geq 0.$$

And so, by using Remark 3.11, we conclude that φ is a convex function. Hence,

$$\Phi\left(\varphi\right) \geq 0,$$

which is equivalent to

$$\sum_{j,k=1}^{n} \alpha_j \alpha_k \Phi\left(f_{\frac{s_j+s_k}{2}}\right) \geq 0$$

and so we conclude that the function $s \mapsto \Phi(f_s)$ is n-exponentially convex in the Jensen sense on I.

The remaining part follows from Proposition 3.4.

(ii) If the function $s \mapsto \Phi\left(f_s\right)$ is continuous on I, then from (i) and by Definition 3.2, it follows that it is n-exponentially convex on I.

∎

The following corollary is an immediate consequence of the above theorem.

COROLLARY 3.13. *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex in the Jensen sense on I for every three mutually distinct points $y_0, y_1, y_2 \in [a, b]$. Let Φ be a linear functional defined as in (2.7). Then the following statements hold:*

(i) *The function $s \mapsto \Phi(f_s)$ is exponentially convex in the Jensen sense on I and the matrix $\left[\Phi(f_{\frac{s_j+s_k}{2}})\right]_{j,k=1}^{n}$ is a positive semi-definite matrix for all $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in I$. Particularly,*

$$\det\left[\Phi(f_{\frac{s_j+s_k}{2}})\right]_{j,k=1}^{n} \geq 0, \;\; \forall \;\, n \in \mathbb{N}.$$

(ii) *If the function $s \mapsto \Phi(f_s)$ is continuous on I, then it is exponentially convex on I.*

COROLLARY 3.14. *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is 2-exponentially convex in the Jensen sense on I for every three mutually distinct points $y_0, y_1, y_2 \in [a, b]$. Let Φ be a linear functional defined as in (2.7) and also assume that $\Phi(f_s)$ is strictly positive for $f_s \in \Omega$. Then the following statements hold:*

(i) *If the function $s \mapsto \Phi(f_s)$ is continuous on I, then it is 2-exponentially convex on I and so it is log-convex on I and for $r, s, t \in I$ such that $r < s < t$, we have*

$$[\Phi(f_s)]^{t-r} \leq [\Phi(f_r)]^{t-s}\,[\Phi(f_t)]^{s-r}, \tag{3.2}$$

known as Lyapunov's inequality. If $r < t < s$ or $s < r < t$, then opposite inequality holds in (3.2).

(ii) *If the function $s \mapsto \Phi(f_s)$ is differentiable on I, then for every $s, q, u, v \in I$ such that $s \leq u$ and $q \leq v$, we have*

$$\mu_{s,q}(\Phi, \Omega) \leq \mu_{u,v}(\Phi, \Omega), \tag{3.3}$$

where

$$\mu_{s,q}(\Phi, \Omega) = \begin{cases} \left(\dfrac{\Phi(f_s)}{\Phi(f_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\dfrac{\frac{d}{ds}\Phi(f_s)}{\Phi(f_s)}\right), & s = q, \end{cases} \tag{3.4}$$

for $f_s, f_q \in \Omega$.

PROOF. The idea of the proof is the same as that of Corollary 3.2 in [6].

(i) The claim that the function $s \mapsto \Phi(f_s)$ is log-convex on I is an immediate consequence of Theorem 3.12 and Remark 3.8 and (3.2) can be obtained by replacing the convex function f with the convex function

$f(z) = \log \Phi(f_z)$ for $z = r, s, t$ in (1.5), where $r, s, t \in I$ such that $r < s < t$.

(ii) Since by (i) the function $s \mapsto \Phi(f_s)$ is log-convex on I, that is, the function $s \mapsto \log \Phi(f_s)$ is convex on I. Applying Proposition 1.7 with setting $f(z) = \log \Phi(f_z)$, we get

$$\frac{\log \Phi(f_s) - \log \Phi(f_q)}{s - q} \leq \frac{\log \Phi(f_u) - \log \Phi(f_v)}{u - v} \tag{3.5}$$

for $s \leq u$, $q \leq v$, $s \neq q$, $u \neq v$; and therefore, we conclude that

$$\mu_{s,q}(\Phi, \Omega) \leq \mu_{u,v}(\Phi, \Omega).$$

If $s = q$, we consider the limit when $q \to s$ in (3.5) and conclude that

$$\mu_{s,s}(\Phi, \Omega) \leq \mu_{u,v}(\Phi, \Omega).$$

The case $u = v$ can be treated similarly.

□

Remark 3.15. Note that the results from Theorem 3.12, Corollary 3.13 and Corollary 3.14 still hold when two of the points $y_0, y_1, y_2 \in [a, b]$ coincide, say $y_1 = y_0$, for a family of differentiable functions f_s such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense on I); and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 3.10 and by using suitable characterizations of convexity.

4. Examples

In this section, we present several families of functions which fulfil the conditions of Theorem 3.12, Corollary 3.13 and Corollary 3.14 and Remark 3.15. This enables us to construct large families of functions which are exponentially convex.

Example 4.1. Consider the family of functions

$$\Omega_1 = \{g_s : \mathbb{R} \to [0,\ \infty) : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases} \frac{1}{s^2} e^{sx}, & s \neq 0, \\ \frac{1}{2} x^2, & s = 0. \end{cases}$$

We have $\frac{d^2}{dx^2} g_s(x) = e^{sx} > 0$, which shows that g_s is convex on $\mathbb{R}$ for every $s \in \mathbb{R}$ and $s \mapsto \frac{d^2}{dx^2} g_s(x)$ is exponentially convex by definition (*see also* [3]).

In order to prove that the function $s \mapsto [y_0, y_1, y_2; g_s]$ is exponentially convex, it is enough to show that
(4.1)
$$\sum_{j,k=1}^{n} \alpha_j \alpha_k \left[y_0, y_1, y_2; g_{\frac{s_j+s_k}{2}}\right] = \left[y_0, y_1, y_2; \sum_{j,k=1}^{n} \alpha_j \alpha_k g_{\frac{s_j+s_k}{2}}\right] \geq 0,$$

$\forall\ n \in \mathbb{N}$, $\alpha_j, s_j \in \mathbb{R}$, $j = 1 \ldots, n$. By Remark 3.11, (4.1) will hold if $\Upsilon(x) := \sum_{j,k=1}^{n} \alpha_j \alpha_k g_{\frac{s_j+s_k}{2}}$ is convex. Since $s \mapsto g_s''(x)$ is exponentially convex, i.e., $\sum_{j,k=1}^{n} \alpha_j \alpha_k g''_{\frac{s_j+s_k}{2}} \geq 0$, $\forall\ n \in \mathbb{N}$, $\alpha_j, s_j \in \mathbb{R}$, $j = 1, \ldots, n$, which shows the convexity of $\Upsilon(x)$ and so (4.1) holds. Now, as the function $s \mapsto [y_0, y_1, y_2; g_s]$ is exponentially convex, $s \mapsto [y_0, y_1, y_2; g_s]$ is exponentially convex in the Jensen sense and by using Corollary 3.13, we have $s \mapsto \Phi(g_s)$ is exponentially convex in the Jensen sense. Since this mapping is continuous (although the mapping $s \mapsto g_s$ is not continuous for $s = 0$), $s \mapsto \Phi(g_s)$ is exponentially convex.
For this family of functions, by taking $\Omega = \Omega_1$ in (3.4), $\mu_{s,q}(\Phi, \Omega_1)$ become

$$\mu_{s,q}(\Phi, \Omega_1) = \begin{cases} \left(\frac{\Phi(g_s)}{\Phi(g_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi(id \cdot g_s)}{\Phi(g_s)} - \frac{2}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{\Phi(id \cdot g_0)}{3\Phi(g_0)}\right), & s = q = 0. \end{cases}$$

By using Theorem 2.6, it can be seen that

$$M_{s,q}(\Phi, \Omega_1) = \log \mu_{s,q}(\Phi, \Omega_1)$$

satisfy $a \leq M_{s,q}(\Phi, \Omega_1) \leq b$, which shows that $M_{s,q}(\Phi, \Omega_1)$ is a mean.

Example 4.2. Consider the family of functions

$$\Omega_2 = \{f_s : (0, \infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0, 1, \\ -\ln x, & s = 0, \\ x \ln x, & s = 1. \end{cases}$$

Here, $\frac{d^2}{dx^2} f_s(x) = x^{s-2} = e^{(s-2)\ln x} > 0$, which shows that f_s is convex for $x > 0$ and $s \mapsto \frac{d^2}{dx^2} f_s(x)$ is exponentially convex by definition (*see also* [3]). It is easy to prove that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex. Arguing as in Example 4.1, we have $s \mapsto \Phi(f_s)$ is exponentially convex.
If $r, s, t \in \mathbb{R}$ are such that $r < s < t$, then from (3.2), we have

$$\Phi(f_s) \leq [\Phi(f_r)]^{\frac{t-s}{t-r}} [\Phi(f_t)]^{\frac{s-r}{t-r}}. \tag{4.2}$$

If $r < t < s$ or $s < r < t$, then opposite inequality holds in (4.2).
Particularly, for $r, s, t \in \mathbb{R} \setminus \{0, 1\}$ such that $r < s < t$, we have

$$\begin{aligned}&\frac{\left(\sum_{k=1}^{n} a_k\right)^s - a_1^s - \sum_{k=2}^{n} a_k^s (k^s - (k-1)^s)}{s(s-1)} \geq \\ &\left[\frac{\left(\sum_{k=1}^{n} a_k\right)^r - a_1^r - \sum_{k=2}^{n} a_k^r (k^r - (k-1)^r)}{r(r-1)}\right]^{\frac{t-s}{t-r}} \\ &\times \left[\frac{\left(\sum_{k=1}^{n} a_k\right)^t - a_1^t - \sum_{k=2}^{n} a_k^t \left(k^t - (k-1)^t\right)}{t(t-1)}\right]^{\frac{s-r}{t-r}},\end{aligned} \tag{4.3}$$

where $a_k > 0$ $(k = 1, \ldots, n)$ are real numbers, $S_k = \sum_{i=1}^{k} a_i$, ka_k and $(k-1)a_k \in [a, b]$ for all $k = 2, \ldots, n$. In fact, for $s < 0$ or $s > 1$, (4.3) is the improvement of the inequality (2.3) and for $0 < s < 1$, the inequality in (4.3) reverses.
By taking $\Omega = \Omega_2$ in (3.4), $\Xi_{s,q} := \mu_{s,q}(\Phi, \Omega_2)$ are of the form

$$\Xi_{s,q} = \left(\frac{q(q-1)}{s(s-1)} . \frac{\left(\sum_{k=1}^{n} a_k\right)^s - a_1^s - \sum_{k=2}^{n} a_k^s \ \left(k^s - (k-1)^s\right)}{\left(\sum_{k=1}^{n} a_k\right)^q - a_1^q - \sum_{k=2}^{n} a_k^q \left(k^q - (k-1)^q\right)}\right)^{\frac{1}{s-q}}, \ s \neq q \neq 0, 1,$$

$$\Xi_{s,0} = \left(\frac{1}{s(s-1)} . \frac{\left(\sum_{k=1}^{n} a_k\right)^s - a_1^s - \sum_{k=2}^{n} a_k^s \left(k^s - (k-1)^s\right)}{\ln(na_1) - \ln\left(\sum_{k=1}^{n} a_k\right)}\right)^{\frac{1}{s}}, \ s \neq 0, 1,$$

$$\Xi_{s,1} = \left(\frac{\frac{1}{s(s-1)}\left(\left(\sum_{k=1}^{n} a_k\right)^s - a_1^s - \sum_{k=2}^{n} a_k^s \ \left(k^s - (k-1)^s\right)\right)}{\sum_{k=1}^{n} a_k \left(\ln\left(\sum_{k=1}^{n} a_k\right) - k\ln(ka_k)\right) + \sum_{k=2}^{n} a_k (k-1)\ln\left((k-1)\, a_k\right)}\right)^{\frac{1}{s-1}},$$
$s \neq 0, 1$,

$$\Xi_{0,1} = \frac{\sum_{k=1}^{n} a_k \left(\ln\left(\sum_{k=1}^{n} a_k\right) - k\ln(ka_k)\right) + \sum_{k=2}^{n} a_k (k-1)\ln\left((k-1)\, a_k\right)}{\ln(na_1) - \ln\left(\sum_{k=1}^{n} a_k\right)},$$

$$\begin{aligned}\Xi_{s,s} &= \exp\left(\frac{1-2s}{s(s-1)}\right) \times \\ &\exp\left(\frac{\left(\sum_{k=1}^{n} a_k\right)^s \ln\left(\sum_{k=1}^{n} a_k\right) - \sum_{k=1}^{n} a_k^s k^s \ln(ka_k) + \sum_{k=2}^{n} a_k^s (k-1)^s \ln\left((k-1)\, a_k\right)}{\left(\sum_{k=1}^{n} a_k\right)^s - a_1^s - \sum_{k=2}^{n} a_k^s \left(k^s - (k-1)^s\right)}\right),\end{aligned}$$
$s \neq 0, 1$,

$$\begin{aligned}&\Xi_{0,0} = \\ &\exp\left(\frac{\left(\ln\left(\sum_{k=1}^{n} a_k\right) - \ln a_1\right)\ln\left(e^2 a_1 \sum_{k=1}^{n} a_k\right) - \sum_{k=2}^{n} \left(\ln k - \ln(k-1)\right)\ln\left(e^2 k(k-1) a_k^2\right)}{2\left(\ln\left(\sum_{k=1}^{n} a_k\right) - \ln(na_1)\right)}\right),\end{aligned}$$

$$\Xi_{1,1}^1 = \exp\left(\frac{\sum_{k=1}^n a_k\left(\ln\left(\sum_{k=1}^n a_k\right)\ln\left(e^{-2}\sum_{k=1}^n a_k\right) - k\ln(ka_k)\ln\left(e^{-2}ka_k\right)\right)}{2\left(\sum_{k=1}^n a_k\left(\ln\left(\sum_{k=1}^n a_k\right) - k\ln(ka_k)\right) + \sum_{k=2}^n a_k(k-1)\ln((k-1)a_k)\right)}\right)$$

$$\times\exp\left(\frac{\sum_{k=2}^n a_k(k-1)\ln((k-1)a_k)\ln\left(e^{-2}(k-1)a_k\right)}{2\left(\sum_{k=1}^n a_k\left(\ln\left(\sum_{k=1}^n a_k\right) - k\ln(ka_k)\right) + \sum_{k=2}^n a_k(k-1)\ln((k-1)a_k)\right)}\right).$$

If Φ is positive, then Theorem 2.6 applied for $f = f_s \in \Omega_2$ and $g = f_q \in \Omega_2$ yields that there exists $\xi \in [a, b]$ such that

$$\xi^{s-q} = \frac{\Phi(f_s)}{\Phi(f_q)}.$$

Since the function $\xi \mapsto \xi^{s-q}$ is invertible for $s \neq q$, we have

$$a \leq \left(\frac{\Phi(f_s)}{\Phi(f_q)}\right)^{\frac{1}{s-q}} \leq b \tag{4.4}$$

which, together with the fact that $\mu_{s,q}(\Phi, \Omega_2)$ is continuous, symmetric and monotonous (by (3.3)) shows that $\mu_{s,q}(\Phi, \Omega_2)$ is a mean.

If $a = 0$ and we consider functions defined on $[0, \infty)$, then we can obtain inequalities and means of the same form, but for parameters s and q restricted to $(0, \infty)$. More precisely, we consider the family of functions

$$\tilde{\Omega}_2 = \{\tilde{f}_s : [0, \infty) \to \mathbb{R} : s \in (0, \infty)\}$$

defined by

$$\tilde{f}_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 1, \\ x \ln x, & s = 1, \end{cases}$$

with the convention that $0 \ln 0 = 0$.
If $r, s, t \in (0, \infty) \setminus \{1\}$ are such that $r < s < t$, then from (4.2), we have

$$\frac{\left(\sum_{k=1}^n a_k\right)^s - \sum_{k=1}^n a_k^s(k^s-(k-1)^s)}{s(s-1)} \geq$$
$$\left[\frac{\left(\sum_{k=1}^n a_k\right)^r - \sum_{k=1}^n a_k^r(k^r-(k-1)^r)}{r(r-1)}\right]^{\frac{t-s}{t-r}}$$
$$\times\left[\frac{\left(\sum_{k=1}^n a_k\right)^t - \sum_{k=1}^n a_k^t\left(k^t-(k-1)^t\right)}{t(t-1)}\right]^{\frac{s-r}{t-r}},$$

which is in fact the improvement of inequality (1.2) for $s > 1$. For $s > 0$ and $q > 0$, by taking $\Omega = \tilde{\Omega}_2$ in (3.4), $\tilde{\Xi}_{s,q} =: \mu_{s,q}(\Phi, \tilde{\Omega}_2)$ are of the same form as $\Xi_{s,q}$.

Example 4.3. Consider the family of functions

$$\Omega_3 = \{h_s : (0, \infty) \to (0, \infty) : s \in (0, \infty)\}$$

defined by

$$h_s(x) = \begin{cases} \frac{s^{-x}}{\ln^2 s}, & s \neq 1, \\ \frac{x^2}{2}, & s = 1. \end{cases}$$

We have $\frac{d^2}{dx^2}h_s(x) = s^{-x} > 0$, which shows that h_s is convex for all $s > 0$. Since $s \mapsto \frac{d^2}{dx^2}h_s(x) = s^{-x}$ is the Laplace transform of a non-negative function (*see* [3, 8]), it is exponentially convex. It is easy to see that the function $s \mapsto [y_0, y_1, y_2; h_s]$ is also exponentially convex. Arguing as in Example 4.1, we have $s \mapsto \Phi(h_s)$ is exponentially convex.
In this case, by taking $\Omega = \Omega_3$ in (3.4), $\mu_{s,q}(\Phi, \Omega_3)$ are of the form

$$\mu_{s,q}(\Phi, \Omega_3) = \begin{cases} \left(\frac{\Phi(h_s)}{\Phi(h_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi(id \cdot h_s)}{s\Phi(h_s)} - \frac{2}{s \ln s}\right), & s = q \neq 1, \\ \exp\left(-\frac{\Phi(id \cdot h_1)}{3\Phi(h_1)}\right), & s = q = 1. \end{cases}$$

By using Theorem 2.6, it follows that

$$M_{s,q}\left(\Phi, \Omega_3\right) = -L(s,q) \log \mu_{s,q}\left(\Phi, \Omega_3\right)$$

satisfy $a \leq M_{s,q}\left(\Phi, \Omega_3\right) \leq b$ and so $M_{s,q}\left(\Phi, \Omega_3\right)$ is a mean, where $L(s,q)$ is a logarithmic mean defined by $L(s,q) = \frac{s-q}{\log s - \log q}$, $s \neq q$, $L(s,s) = s$.

Example 4.4. Consider the family of functions

$$\Omega_4 = \{k_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s}.$$

Here, $\frac{d^2}{dx^2}k_s(x) = e^{-x\sqrt{s}} > 0$, which shows that k_s is convex for all $s > 0$. Since $s \mapsto \frac{d^2}{dx^2}k_s(x) = e^{-x\sqrt{s}}$ is the Laplace transform of a non-negative function (*see* [3, 8]), it is exponentially convex. It is easy to prove that the function $s \mapsto [y_0, y_1, y_2; k_s]$ is also exponentially convex. Arguing as in Example 4.1, we have $s \mapsto \Phi(k_s)$ is exponentially convex.
In this case, by taking $\Omega = \Omega_4$ in (3.4), $\mu_{s,q}(\Phi, \Omega_4)$ are of the form

$$\mu_{s,q}(\Phi, \Omega_4) = \begin{cases} \left(\frac{\Phi(k_s)}{\Phi(k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi(id \cdot k_s)}{2\sqrt{s}\Phi(k_s)} - \frac{1}{s}\right), & s = q. \end{cases}$$

By using Theorem 2.6, it is easy to see that

$$M_{s,q}\left(\Phi, \Omega_4\right) = -(\sqrt{s} + \sqrt{q}) \log \mu_{s,q}\left(\Phi, \Omega_4\right)$$

satisfy $a \leq M_{s,q}\left(\Phi, \Omega_4\right) \leq b$, showing that $M_{s,q}\left(\Phi, \Omega_4\right)$ is a mean.

Remark 4.5. From (3.4), it is clear that $\mu_{s,q}(\Phi, \Omega)$ for $\Omega = \Omega_1, \Omega_3$ and Ω_4 are monotonous functions in parameters s and q.

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Poopćenja i poboljšanja nejednakosti Hardy-Littlewood-Pólya

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Sažetak. U radu su dana neka poopćenja i poboljšanja nejednakosti Hardy-Littlewood-Pólya. Diskutirana je n-eksponencijalna konveksnost i logaritamska konveksnost funkcionala definiranog poopćenom nejednakošću kao i monotonost odgovarajućih poopćenih Cauchyjevih sredina.

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