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ON A NEW CLASS OF HARDY-TYPE INEQUALITIES WITH FRACTIONAL INTEGRALS AND FRACTIONAL DERIVATIVES

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Abstract. This paper is devoted to a new class of general weighted Hardy-type inequalities for arbitrary convex functions with some applications to different type of fractional integrals and fractional derivatives.

1. Introduction

In [5], A. Čižmešija et al. recently introduced a new class of general Hardy-type inequalities.

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures and A_k be an integral operator defined by

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,y) f(y) d\mu_2(y), \tag{1.1}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and nonnegative kernel, f is measurable function on Ω_2, and

$$K(x) := \int_{\Omega_2} k(x,y) d\mu_2(y), \qquad x \in \Omega_1. \tag{1.2}$$

Throughout the paper, we consider that $K(x) > 0$ a.e. on Ω_1.

Theorem 1.1. *Let $1 < p \leq q < \infty$. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ-finite measures, u be a weight function on Ω_1, v be a measurable μ_2- a.e. positive function on Ω_2, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$, and K be defined on Ω_1 by (1.2). Let $K(x) > 0$ for all*

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$x \in \Omega_1$ and let the function $x \mapsto u(x)\left(\frac{k(x,y)}{K(x)}\right)^q$ be integrable on Ω_1 for each fixed $y \in \Omega_2$. Suppose that $\Phi : I \to [0,\infty)$ is a bijective convex function on an interval $I \subseteq \mathbb{R}$. If there exist a real parameter $s \in (1,p)$ and a positive measurable function $V : \Omega_2 \to \mathbb{R}$ such that

$$A(s,V) \;=\; F(V,v) \sup_{y\in\Omega_2} V^{\frac{s-1}{p}}(y) \left(\int_{\Omega_1} u(x) \left(\frac{k(x,y)}{K(x)}\right)^q d\mu_1(x) \right)^{\frac{1}{q}} \;<\; \infty,$$

where

$$F(V,v) = \left(\int_{\Omega_2} V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) d\mu_2(y) \right)^{\frac{1}{p'}},$$

then there is a positive real constant C, such that the inequality

$$\left(\int_{\Omega_1} u(x)\Phi^q\left(A_k f(x)\right) d\mu_1(x) \right)^{\frac{1}{q}} \leq C \left(\int_{\Omega_2} v(y)\Phi^p\left(f(y)\right) d\mu_2(y) \right)^{\frac{1}{p}} \tag{1.3}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ with values in I and $A_k f$ be defined on Ω_1 by (1.1)*. Moreover, if C is the smallest constant for* (1.3) *to hold, then*

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Here, our aim is to construct new results related to this new class of Hardy-type inequalities for fractional integrals of a function with respect to another increasing function, Riemann-Liouville fractional integrals, Hadamard-type fractional integrals, Canavati-type fractional derivative, Caputo fractional derivative, Erdelyi-Kóber fractional integrals. Many authors gave improvements and generalizations of Hardy-type inequalities for convex functions as well as for superquadratic functions, (see [6], [7], [10], [11] [12], [13], [14], [16], [17]).

We also recall important result of G. H. Hardy. Let $[a,b], -\infty < a < b < \infty$ be a finite interval on real axis $\mathbb{R}$, and $1 \leq p \leq \infty$, then

$$\|I^\alpha_{a^+} f\|_p \leq K\|f\|_p, \qquad \|I^\alpha_{b^-} f\|_p \leq K\|f\|_p \tag{1.4}$$

holds, where

$$K = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)},$$

$I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha > 0$ denote the Riemann-Liouville fractional integrals defined by

$$I^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1}f(y)dy, \quad (x > a)$$

and

$$I^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (y-x)^{\alpha-1}f(y)dy, \quad (x < b),$$

where Γ is the Gamma function, i.e. $\Gamma(\alpha) = \int_0^\infty e^t\, t^{\alpha-1}dt$.

If $0 < \alpha < 1$ and $1 < p < \frac{1}{\alpha}$, then the operators $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ are bounded from $L_p(a,b)$ into $L_q(a,b)$, where $q = \frac{p}{1-\alpha p}$. This is known as Hardy-Littlewood theorem. Later on, the inequality (1.4) is discussed and proved by S. G. Samko et al. in [18, Theorem 2.6]. For details we refer [15, Remark 2.1](also see [18]).

G. H. Hardy proved the inequality (1.4) involving left-sided fractional integral in one of his initial paper, see [9]. The calculation for the constant K is hidden inside the proof.

Throughout this paper, all measures are assumed to be positive, all functions are assumed to be positive and measurable and expressions of the form $0 \cdot \infty$, $\frac{\infty}{\infty}$ and $\frac{0}{0}$ are taken to be equal to zero. Moreover, by a weight $u = u(x)$ we mean a non-negative measurable function on the actual interval or more general set. For a real parameter $0 \neq p \neq 1$, by p' we denote its conjugate exponent $p' = \frac{p}{p-1}$, that is $\frac{1}{p} + \frac{1}{p'} = 1$.

The paper is organized in the following way: After introduction, in Section 2, we construct and discuss a new class of generalized inequalities of Hardy-type using different kinds of fractional integrals and fractional derivatives.

2. The Main Results

Let us recall some facts about fractional derivatives needed in the sequel, for more details see e.g. [1], [8].

Let $0 < a < b \leq \infty$. By $C^m([a,b])$ we denote the space of all functions on $[a,b]$ which have continuous derivatives up to order m, and $AC([a,b])$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^m([a,b])$ we denote the space of all functions $g \in C^{m-1}([a,b])$ with $g^{(m-1)} \in AC([a,b])$. For any $\alpha \in \mathbb{R}$ we denote by $[\alpha]$ the integral part of α (the integer k satisfying $k \leq \alpha < k+1$) and $\lceil \alpha \rceil$ is the ceiling of α ($\min\{n \in \mathbb{N}, n \geq \alpha\}$). By $L_1(a,b)$

we denote the space of all functions integrable on the interval (a,b), and by $L_\infty(a,b)$ the set of all functions measurable and essentially bounded on (a,b). Clearly, $L_\infty(a,b) \subset L_1(a,b)$.

We start with definitions and some properties of the *fractional integrals of a function f with respect to given function g*. For details see e.g. [15, p. 99].

Let (a,b), $-\infty \le a < b \le \infty$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let g be an increasing function on (a,b) and g' be a continuous function on (a,b). The left- and right-sided fractional integrals of a function f with respect to another function g in $[a,b]$ are given by

$$(I^\alpha_{a+;g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{g'(t)f(t)dt}{[g(x)-g(t)]^{1-\alpha}}, \quad x > a$$

and

$$(I^\alpha_{b-;g}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{g'(t)f(t)dt}{[g(t)-g(x)]^{1-\alpha}}, \quad x < b,$$

respectively.

Our first result deals with fractional integral of f with respect to another increasing function g is given.

Theorem 2.1. *Let $1 < p \le q < \infty$, $\alpha > 0$, u be a weight function on (a,b), v be a.e. positive function on (a,b), g be increasing function on $(a,b]$ such that g' be continuous on (a,b), $I^\alpha_{a+;g}f$ denotes the left sided fractional integral of f with respect to another increasing function g. Suppose that $\Phi : I \to [0,\infty)$ is a bijective convex function on an interval $I \subseteq \mathbb{R}$. If there exist a real parameter $s \in (1,p)$ and $V : (a,b) \to \mathbb{R}$ is a positive measurable function such that*

$$\begin{aligned} (2.1) \quad A(s,V) = {} & \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y)dy \right)^{\frac{1}{p'}} \\ & \times \sup_{y\in(a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{\alpha\, g'(y)(g(x)-g(y))^{\alpha-1}}{(g(x)-g(a))^\alpha} \right)^q dx \right)^{\frac{1}{q}} < \infty, \end{aligned}$$

then there exists a positive constant C, such that the inequality

(2.2)
$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha}I^\alpha_{a+;g}f(x)\right)\right]^q dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b v(y)\Phi^p(f(y))dy\right)^{\frac{1}{p}}.$$

holds. Moreover, if C is the smallest constant for (2.2) *to hold, then*

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Proof. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{g'(y)}{\Gamma(\alpha)(g(x)-g(y))^{1-\alpha}}, & a < y \leq x\,; \\ 0, & x < y \leq b, \end{cases}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}(g(x)-g(a))^\alpha$, $A_k f(x) = \frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha}I^\alpha_{a+;g}f(x)$, and the inequality given in (1.3) reduces to (2.2). This complete the proof. ∎

If we choose the function $\Phi : \mathbb{R}^+ \to \mathbb{R}$ defined by $\Phi(x) = x^t, t \geq 1$, then the following result is obtained.

Corollary 2.2. *Let $1 < p \leq q < \infty$, $t \geq 1$, $\alpha > 0$, u be a weight function on (a,b), v be a.e. positive function on (a,b), g be increasing function on $(a,b]$ such that g' be continuous on (a,b), $I^\alpha_{a+;g}f$ denotes the left sided fractional integral of f with respect to another increasing function g. If there exist a real parameter $s \in (1,p)$ and $V : (a,b) \to \mathbb{R}$ is a positive measurable function such that*

$$A(s,V) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y)v^{1-p'}(y)dy\right)^{\frac{1}{p'}}$$
$$\times \sup_{y\in(a,b)} V^{\frac{s-1}{p}}(y)\left(\int_y^b u(x)\left(\frac{\alpha\, g'(y)(g(x)-g(y))^{\alpha-1}}{(g(x)-g(a))^\alpha}\right)^q dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

(2.3)
$$\left(\int_a^b u(x)\left(\frac{\Gamma(\alpha+1)}{(g(x)-g(a))^\alpha}I^\alpha_{a+;g}f(x)\right)^{tq} dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b v(y)(f(y))^{tp}dy\right)^{\frac{1}{p}}$$

holds. Moreover, if C *is the smallest constant for* (2.3) *to hold, then*

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Here, we give a special case for the Riemman-Liouville fractional integral. If $g(x)=x$, then $I^{\alpha}_{a+;x}f(x)$ reduces to $I^{\alpha}_{a_+}f(x)$ left-sided Riemann–Liouville fractional integral, so the following result follows.

COROLLARY 2.3. *Let* $1<p\leq q<\infty$, $\alpha>0$, $t\geq 1$, u *be a weight function on* (a,b), v *be a.e. positive function on* (a,b), *and* $I^{\alpha}_{a+}f$ *denotes the left-sided Riemann-Liouville fractional integral of* f. *If there exist a real parameter* $s\in(1,p)$ *and* $V:(a,b)\to\mathbb{R}$ *is a positive measurable function such that*

$$A(s,V)=\left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y)v^{1-p'}(y)dy\right)^{\frac{1}{p'}} \times \sup_{y\in(a,b)} V^{\frac{s-1}{p}}(y)\left(\int_y^b u(x)\left(\frac{\alpha\,(x-y)^{\alpha-1}}{(x-a)^{\alpha}}\right)^{q}dx\right)^{\frac{1}{q}}<\infty, \tag{2.4}$$

then there exists a positive constant C, *such that the inequality*

$$\left(\int_a^b u(x)\left(\frac{\Gamma(\alpha+1)}{(x-a)^{\alpha}}I^{\alpha}_{a+}f(x)\right)^{tq}dx\right)^{\frac{1}{q}}\leq C\left(\int_a^b v(y)f^{tp}(y)dy\right)^{\frac{1}{p}} \tag{2.5}$$

holds. Moreover, if C *is the smallest constant for* (2.5) *to hold, then*

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

REMARK 2.4. If we take $g(x)=\log x$, then $I^{\alpha}_{a+;g}f(x)$ reduces to $J^{\alpha}_{a_+}f(x)$ left-sided Hadamard-type fractional integral that is defined for $\alpha>0$ by

$$(J^{\alpha}_{a_+}f)(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\log\frac{x}{y}\right)^{\alpha-1}\frac{f(y)dy}{y},\quad x>a.$$

If there exist a real parameter $s \in (1,p)$ and a positive measurable function $V : (a,b) \to \mathbb{R}$ with $1 < p \le q < \infty,\ t \ge 1$, such that

$$A(s,V) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}}$$

$$\times \sup_{y \in (a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{\alpha \left(\log x - \log y\right)^{\alpha-1}}{y (\log x - \log a)^{\alpha}} \right)^q dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

(2.6)
$$\left(\int_a^b u(x) \left(\frac{\Gamma(\alpha+1)}{(\log x - \log a)^{\alpha}} J^{\alpha}_{a_+} f(x) \right)^{tq} dx \right)^{\frac{1}{q}} \le C \left(\int_a^b v(y) f^{tp}(y) dy \right)^{\frac{1}{p}}$$

holds. Moreover, if C is the smallest constant for (2.6) to hold, then

$$C \le \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Next we give result with respect to the generalized Riemann-Liouville fractional derivative.

Let us recall the definition, for details see [2].

Let $\alpha > 0$ and $n = [\alpha] + 1$ where $[\cdot]$ is the integral part and we define the *generalized Riemann-Liouville fractional derivative of* f of order α by

$$(D^{\alpha}_a f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx} \right)^n \int_a^x (x-y)^{n-\alpha-1} f(y)\, dy\, .$$

In addition, we stipulate

$$D^0_a f := f =: I^0_a f, \quad I^{-\alpha}_a f := D^{\alpha}_a f \text{ if } \alpha > 0.$$

If $\alpha \in \mathbb{N}$ then $D^{\alpha}_a f = \frac{d^{\alpha} f}{dx^{\alpha}}$, the ordinary α-order derivative.

The space $I^{\alpha}_a(L(a,b))$ is defined as the set of all functions f on $[a,b]$ of the form $f = I^{\alpha}_a \varphi$ for some $\varphi \in L(a,b)$, [18, Chapter 1, Definition 2.3]. According to Theorem 2.3 in [18, p. 43], the latter characterization is equivalent to the condition

$$I^{n-\alpha}_a f \in AC^n[a,b]\,, \tag{2.7}$$

$$\frac{d^j}{dx^j} I_a^{n-\alpha} f(a) = 0\,, \quad j = 0, 1, \dots, n-1\,.$$

A function $f \in L(a,b)$ satisfying (2.7) is said to have an *integrable fractional derivative* $D_a^\alpha f$, [18, Chapter 1, Definition 2.4].

The following lemma help us to prove the next result. For details see [2].

LEMMA 2.5. *Let $\beta > \alpha \geq 0$, $n = [\beta] + 1$, $m = [\alpha] + 1$. Identity*

$$D_a^\alpha f(x) = \frac{1}{\Gamma(\beta - \alpha)} \int_a^x (x-y)^{\beta - \alpha - 1} D_a^\beta f(y)\, dy\,, \quad x \in [a,b]\,.$$

is valid if one of the following conditions holds:

(i) $f \in I_a^\beta (L(a,b))$.
(ii) $I_a^{n-\beta} f \in AC^n[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \dots n$.
(iii) $D_a^{\beta-k} f \in C[a,b]$ *for* $k = 1, \dots, n$, $D_a^{\beta-1} f \in AC[a,b]$ *and* $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \dots n$.
(iv) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta - \alpha \notin \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \dots, n$ *and* $D_a^{\alpha-k} f(a) = 0$ *for* $k = 1, \dots, m$.
(v) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta - \alpha = l \in \mathbb{N}$, $D_a^{\beta-k} f(a) = 0$ *for* $k = 1, \dots, l$.
(vi) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$ *and* $f(a) = f'(a) = \cdots = f^{(n-2)}(a) = 0$.
(vii) $f \in AC^n[a,b]$, $D_a^\beta f \in L(a,b)$, $D_a^\alpha f \in L(a,b)$, $\beta \notin \mathbb{N}$ *and* $D_a^{\beta-1} f$ *is bounded in a neighborhood of* $t = a$.

COROLLARY 2.6. *Let $1 < p \leq q < \infty$, $t \geq 1$, $\beta > \alpha \geq 0$, u be a weight function on (a,b), v be a.e. positive function on (a,b), $D_a^\alpha f$ denotes the generalized Riemann-Liouville fractional derivative of f and let the assumptions of the Lemma 2.5 be satisfied. If there exist a real parameter $s \in (1,p)$ and a positive measurable function $V : (a,b) \to \mathbb{R}$ such that*

$$A(s,V) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}}$$
$$\times \sup_{y \in (a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(\beta-\alpha)(x-y)^{\beta-\alpha-1}}{(x-a)^{\beta-\alpha}} \right)^q dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

(2.8)
$$\left(\int_a^b u(x) \left(\frac{\Gamma(\beta - \alpha + 1)}{(x-a)^{\beta-\alpha}} D_a^\alpha f(x) \right)^{tq} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b v(y) (D_a^\beta f(y))^{tp} dy \right)^{\frac{1}{p}}$$

holds. Moreover, if C *is the smallest constant for* (2.8) *to hold, then*

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

PROOF. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}, & a < y \leq x\,; \\ 0, & x < y \leq b, \end{cases}$$

we get that $K(x) = \frac{(x-a)^{\beta-\alpha}}{\Gamma(\beta-\alpha+1)}$ and $A_k f(x) = \frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^{\alpha} f(x)$. Replace f by $D_a^{\beta} f$. Then the inequality (1.3) becomes
(2.9)

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(\beta-\alpha+1)}{(x-a)^{\beta-\alpha}} D_a^{\alpha} f(x) \right) \right]^q dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b v(y) \Phi^p \left(D_a^{\beta} f(y) \right) dy \right)^{\frac{1}{p}}.$$

For $t \geq 1$, the function $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be defined by $\Phi(x) = x^t$, then (2.9) becomes (2.8). ∎

Now we define Canavati-type fractional derivative ($\nu-$fractional derivative of f), for details see [1] and [3]. We consider

$$C_a^{\nu}([a,b]) = \{ f \in C^n([a,b]) : I^{1-\bar{\nu}} f^{(n)} \in C^1([a,b]) \},$$

$\nu > 0$, $n = [\nu], [.]$ is the integral part, and $\bar{\nu} = \nu - n, 0 \leq \bar{\nu} < 1$.
For $f \in C_a^{\nu}([a,b])$, the Canavati-ν fractional derivative of f is defined by

$$D_a^{\nu} f = D I_a^{1-\bar{\nu}} f^{(n)},$$

where $D = d/dx$.

LEMMA 2.7. *Let* $\nu > \gamma > 0$, $n = [\nu]$, $m = [\gamma]$. *Let* $f \in C_a^{\nu}([a,b])$, *be such that* $f^{(i)}(a) = 0$, $i = m, m+1, ..., n-1$. *Then*

(*i*) $f \in C_a^{\gamma}([a,b])$

(*ii*) $(D_a^{\gamma} f)(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-t)^{\nu-\gamma-1} (D_a^{\nu} f)(t) dt,$

for every $x \in [a,b]$.

In the following Corollary, we construct new inequality for the Canavati-type fractional derivative.

COROLLARY 2.8. *Let* $1 < p \leq q < \infty$, $t \geq 1$, u *be a weight function on* (a,b), v *be a.e. positive function on* (a,b), *and let the assumptions in Lemma*

2.7 be satisfied. If there exist a real parameter $s \in (1,p)$ and $V : (a,b) \to \mathbb{R}$ is a positive measurable function such that

$$A(s,V) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}}$$

$$\times \sup_{y \in (a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(\nu-\gamma)(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}} \right)^q dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

$$\left(\int_a^b u(x) \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x) \right)^{tq} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b v(y) (D_a^\nu f(y))^{tp} dy \right)^{\frac{1}{p}} \tag{2.10}$$

holds. Moreover, if C is the smallest constant for (2.10) *to hold, then*

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Proof. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)}, & a < y \leq x\,; \\ 0, & x < y \leq b, \end{cases}$$

we get that $K(x) = \frac{(x-a)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}$ and $A_k f(x) = \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x)$. Replace f by $D_a^\nu f$. Then the inequality given in (1.3) become

$$\left(\int_a^b u(x) \left[\Phi \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D_a^\gamma f(x) \right) \right]^q dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b v(y) \Phi^p \left(D_a^\nu f(y) \right) dy \right)^{\frac{1}{p}}. \tag{2.11}$$

If we take $\Phi(x) = x^t, t \geq 1, x \in \mathbb{R}^+$, then (2.11) becomes (2.10). ∎

Next, we give the result for Caputo fractional derivative, for details see [1, p. 449]. The Caputo fractional derivative is defined as:

Let $\alpha \geq 0$, $n = \lceil \alpha \rceil$, $g \in AC^n([a,b])$. The Caputo fractional derivative is given by

$$D_{*a}^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{g^{(n)}(y)}{(x-y)^{\alpha-n+1}} dy,$$

for all $x \in [a,b]$. The above function exists almost everywhere for $x \in [a,b]$.

The following result cover the case of the Caputo fractional derivative.

Corollary 2.9. *Let $1 < p \leq q < \infty$, $t \geq 1$, u be a weight function on (a,b), v be a.e. positive function on (a,b), and $D^{\alpha}_{*a}f$ denotes the Caputo fractional derivative of f, $f \in AC^n([a,b])$. If there exist a real parameter $s \in (1,p)$ and $V : (a,b) \to \mathbb{R}$ is a positive measurable function such that*

$$A(s,V) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) dy\right)^{\frac{1}{p'}} \tag{2.12}$$

$$\times \sup_{y\in(a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(n-\alpha)(x-y)^{n-\alpha-1}}{(x-a)^{n-\alpha}}\right)^q dx\right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality

$$\left(\int_a^b u(x) \left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}} D^{\alpha}_{*a} f(x)\right)^{tq} dx\right)^{\frac{1}{q}} \leq C \left(\int_a^b v(y) (f^{(n)}(y))^{tp} dy\right)^{\frac{1}{p}} \tag{2.13}$$

holds. Moreover, if C is the smallest constant for (2.13) *to hold, then*

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

Proof. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{n-\alpha-1}}{\Gamma(n-\alpha)}, & a < y \leq x\,; \\ 0, & x < y \leq b \end{cases}$$

we get that $K(x) = \frac{(x-a)^{n-\alpha}}{\Gamma(n-\alpha+1)}$ and $A_k f(x) = \frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}} D^{\alpha}_{*a} f(x)$. Replace f by $f^{(n)}$. Then the inequality given in (1.3) takes the form

$$\left(\int_a^b u(x) \left[\Phi\left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{n-\alpha}} D^{\alpha}_{*a} f(x)\right)\right]^q dx\right)^{\frac{1}{q}} \leq C \left(\int_a^b v(y) \Phi^p\left(f^{(n)}(y)\right) dy\right)^{\frac{1}{p}}. \tag{2.14}$$

If we choose $\Phi(x) = x^t, t \geq 1, x \in \mathbb{R}^+$, then (2.14) becomes (2.13). $\square$

We continue with the following lemma that is given [4].

LEMMA 2.10. *Let* $\nu > \gamma \geq 0$, $n = [\nu]+1$, $m = [\gamma]+1$ *and* $f \in AC^n([a,b])$. *Suppose that one of the following conditions hold:*

(a) $\nu, \gamma \notin \mathbb{N}_0$ *and* $f^i(0) = 0$ *for* $i = m, ..., n-1$.
(b) $\nu \in \mathbb{N}_0, \gamma \notin \mathbb{N}_0$ *and* $f^i(0) = 0$ *for* $i = m, ..., n-2$.
(c) $\nu \notin \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^i(0) = 0$ *for* $i = m-1, ..., n-1$.
(d) $\nu \in \mathbb{N}_0, \gamma \in \mathbb{N}_0$ *and* $f^i(0) = 0$ *for* $i = m-1, ..., n-2$.
Then

$$D^{\gamma}_{a^+} f(x) = \frac{1}{\Gamma(\nu-\gamma)} \int_a^x (x-y)^{\nu-\gamma-1} D^{\nu}_{a^+} f(y) dy$$

for all $a \leq x \leq b$.

COROLLARY 2.11. *Let* $1 < p \leq q < \infty$, $t \geq 1$, u *be a weight function on* (a,b), v *be a.e. positive function on* (a,b), *and let the assumptions in Lemma 2.10 be satisfied. Let* $D^{\gamma}_{a^+} f$ *denotes the Caputo fractional derivative of* f, $f \in AC^n([a,b])$. *If there exist a real parameter* $s \in (1,p)$ *and* $V : (a,b) \to \mathbb{R}$ *is a positive measurable function such that*

$$A(s,V) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}}$$
$$\times \sup_{y \in (a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{(\nu-\gamma)(x-y)^{\nu-\gamma-1}}{(x-a)^{\nu-\gamma}} \right)^q dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, *such that the inequality*

$$\left(\int_a^b u(x) \left(\frac{\Gamma(n-\alpha+1)}{(x-a)^{\nu-\gamma}} D^{\gamma}_{a^+} f(x) \right)^{tq} dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b v(y) (D^{\nu}_{a^+} f(y))^{tp} dy \right)^{\frac{1}{p}} \tag{2.15}$$

holds. Moreover, if C *is the smallest constant for* (2.15) *to hold, then*

$$C \leq \inf_{\substack{1<s<p \\ V>0}} A(s,V).$$

PROOF. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{(x-y)^{\nu-\gamma-1}}{\Gamma(\nu-\gamma)}, & a < y \leq x\,; \\ 0, & x < y \leq b, \end{cases}$$

we get that $K(x) = \frac{(x-a)^{\nu-\gamma}}{\Gamma(\nu-\gamma+1)}$ and $A_k f(x) = \frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D^{\gamma}_{a+} f(x)$. Replace f by $D^{\nu}_{a+} f$. Then the inequality given in (1.3) takes the form

(2.16)
$$\left(\int_a^b u(x) \left[\Phi \left(\frac{\Gamma(\nu-\gamma+1)}{(x-a)^{\nu-\gamma}} D^{\gamma}_{a+} f(x) \right) \right]^q dx \right)^{\frac{1}{q}} \leq C \left(\int_a^b v(y) \Phi^p \left(D^{\nu}_{a+} f(y) \right) dy \right)^{\frac{1}{p}}.$$

For $t \geq 1$, the function $\Phi : \mathbb{R}^+ \to \mathbb{R}$ is defined by $\Phi(x) = x^t$, then (2.16) becomes (2.15). □

Now we present definitions and some properties of the *Erdélyi-Kober type fractional integrals*. Some of these definitions and results were presented in Samko et al. in [18].

Let (a, b) , $(0 \leq a < b \leq \infty)$ be a finite or infinite interval of the half-axis $\mathbb{R}^+$. Also let $\alpha > 0, \sigma > 0$, and $\eta \in \mathbb{R}$. We consider the left- and right-sided integrals of order $\alpha \in \mathbb{R}$ defined by

$$(I^{\alpha}_{a+;\sigma;\eta} f)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \frac{t^{\sigma\eta+\sigma-1} f(t) dt}{(x^{\sigma} - t^{\sigma})^{1-\alpha}}, \quad x > a \tag{2.17}$$

and

$$(I^{\alpha}_{b-;\sigma;\eta} f)(x) = \frac{\sigma x^{\sigma\eta}}{\Gamma(\alpha)} \int_x^b \frac{t^{\sigma(1-\eta-\alpha)-1} f(t) dt}{(t^{\sigma} - x^{\sigma})^{1-\alpha}}, \qquad x < b, \tag{2.18}$$

respectively. Integrals (2.17) and (2.18) are called the Erdélyi-Kober type fractional integrals.

Now, we give the following result for Erdélyi-Kober type fractional integrals.

Corollary 2.12. *Let $1 < p \leq q < \infty$, $t \geq 1, \alpha > 0$, u be a weight function on (a, b), v be a.e. positive function on (a, b), $I^{\alpha}_{a+;\sigma;\eta} f$ denotes the Erdélyi-Kober type fractional integrals of f, and ${}_2F_1(a, b; c; z)$ denotes the hypergeometric function. If there exist a real parameter $s \in (1, p)$ and $V : (a, b) \to \mathbb{R}$ is a positive measurable function such that*

$$A(s, V) = \left(\int_a^b V^{\frac{-p'(s-1)}{p}}(y) v^{1-p'}(y) dy \right)^{\frac{1}{p'}}$$

$$\times \sup_{y \in (a,b)} V^{\frac{s-1}{p}}(y) \left(\int_y^b u(x) \left(\frac{\alpha \, \sigma \, x^{-\sigma\eta} y^{\sigma\eta+\sigma-1} (x^{\sigma} - y^{\sigma})^{\alpha-1}}{(x^{\sigma} - a^{\sigma})^{\alpha} \; {}_2F_1(x)} \right)^q dx \right)^{\frac{1}{q}} < \infty,$$

then there exists a positive constant C, such that the inequality
(2.19)

$$\left(\int_a^b u(x)\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)} I^\alpha_{a+;\sigma;\eta} f(x)\right)^{tq} dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b v(y) f^{tp}(y) dy\right)^{\frac{1}{p}}.$$

holds. Moreover, if C is the smallest constant for (2.19) *to hold, then*

$$C \leq \inf_{\substack{1<s<p\\ V>0}} A(s,V).$$

where

$${}_2F_1(x) = {}_2F_1\left(-\eta, \alpha; \alpha+1; 1-\left(\frac{a}{x}\right)^\sigma\right)$$

and

$${}_2F_1(y) = {}_2F_1\left(\eta, \alpha; \alpha+1; 1-\left(\frac{b}{y}\right)^\sigma\right).$$

Proof. Applying Theorem 1.1 with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx, d\mu_2(y) = dy$,

$$k(x,y) = \begin{cases} \frac{1}{\Gamma(\alpha)} \frac{\sigma x^{-\sigma(\alpha+\eta)}}{(x^\sigma - y^\sigma)^{1-\alpha}} y^{\sigma\eta+\sigma-1}, & a<y\leq x\,; \\ 0, & x<y\leq b, \end{cases}$$

we get that $K(x) = \frac{1}{\Gamma(\alpha+1)}\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(-\eta,\alpha;\alpha+1;1-\left(\frac{a}{x}\right)^\sigma)$ and $A_k f(x) = \frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)} I^\alpha_{a+;\sigma;\eta} f(x)$, then inequality (1.3) becomes

(2.20)

$$\left(\int_a^b u(x)\left[\Phi\left(\frac{\Gamma(\alpha+1)}{\left(1-\left(\frac{a}{x}\right)^\sigma\right)^\alpha {}_2F_1(x)} I^\alpha_{a+;\sigma;\eta} f(x)\right)\right]^q dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b v(y)\Phi^p\left(f(y)\right) dy\right)^{\frac{1}{p}}.$$

If we choose $\Phi(x) = x^t, t \geq 1, x \in \mathbb{R}^+$, then (2.20) becomes (2.19). ∎

Remark 2.13. Similar result can be obtained for the right sided fractional integral of f with respect to another increasing function g, right sided Riemann-Liouville fractional integral, right sided Hadamard-type fractional integrals, right sided Erdélyi-Kober type fractional integrals, but here we omit the details.

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O novoj klasi nejednakosti Hardyjeva tipa s razlomljenim integralima i razlomljenim derivacijama

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SAŽETAK. Ovaj rad posvećen je novoj klasi općenitih težinskih nejednakosti Hardyjeva tipa za proizvoljnu konveksnu funkciju s primjenama na razne vrste razlomljenih integrala i razlomljenih derivacija.

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