RAD HAZU. MATEMATIČKE ZNANOSTI
Vol. 18 = 519 (2014): 107-123

ON POTENTIAL INEQUALITY FOR THE ABSOLUTE VALUE OF FUNCTIONS

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Abstract. Potential inequality was introduced in [4] and later extended to the case of general convex and concave functions in [1]. The main goal of this paper is to derive the potential inequality for the case where the function at which the potential is evaluated is replaced by its absolute value. The results obtained, together with methods from [2], are used to construct new families of exponentially convex functions.

1. Introduction

Rao and Šikić [4] introduced the potential inequality for kernels and functions that satisfy the maximum principle (see below), but they proved it only for a special class of convex and concave functions. Elezović, Pečarić and Praljak [1] generalized the potential inequality to the class of naturally defined convex functions on $(0,+\infty)$, which enabled them to generate certain exponentially convex functions that were used to refine some of the known inequalities and to derive new ones.

In this article, we will look at the same class of kernels and functions and extend the results to the case where the function is replaced by its absolute value. Furthermore, we will generate more families of exponentially convex functions by applying methods from [2] that make use of the divided differences.

We will start off by introducing notation and the setup. We say that $N(x,dy)$ is a *positive kernel* on X if $N : X \times \mathcal{B}(X) \to [0,+\infty]$ is a mapping such that, for every $x \in X$, $A \mapsto N(x,A)$ is a σ-finite measure, and, for every $A \in \mathcal{B}(X)$, $x \mapsto N(x,A)$ is a measurable function. For a measurable function f, the *potential of f with respect to N at a point $x \in X$* is

$$(Nf)(x) = \int_X f(y)N(x,dy),$$

2010 *Mathematics Subject Classification.* 26D15.
Key words and phrases. Potential inequality, exponential convexity.

whenever the integral exists. The class of functions that have the potential at every point is denoted by $\mathcal{POT}(N)$.

For a measure μ on $(X, \mathcal{B}(X))$ and a measurable set $C \in \mathcal{B}(X)$ we will denote by $\hat{N}_C\mu$ the measure defined by

$$(\hat{N}_C\mu)(dy) = \int_C N(x, dy)\mu(dx).$$

If $C = X$ we will omit the subscript, i. e. $\hat{N}\mu$ will denote the measure $\hat{N}_X\mu$.

Definition 1.1. *Let N be a positive kernel on X and $\mathcal{R} \subset \mathcal{POT}(N)$. We say that N* satisfies the strong maximum principle on $\mathcal{R}$ *(with constant $M \geq 1$) if*

$$(Nf)(x) \leq Mu + N[f^+\mathbf{1}_{\{(Nf)\geq u\}}](x) \tag{1.1}$$

holds for every $x \in X$, $f \in \mathcal{R}$ and $u \geq 0$.

□

The main result from [1] is the following theorem

Theorem 1.2 (The potential inequality for convex functions). *Let $\Phi : (0, +\infty) \to \mathbb{R}$ be a convex function and $N(x, dy)$ a positive kernel on X which satisfies the strong maximum principle on $\mathcal{R}$ with constant M. Let $f \in \mathcal{R}$, $x \in X$ and $z > 0$ be such that $z \leq (Nf)(x)/M$ and denote by B_z the set*

$$B_z = \big\{y \in X : (Nf)(y) \geq z\big\}.$$

Then

$$\begin{aligned}\Phi\Big(\frac{1}{M}(Nf)(x)\Big) - \Phi(z) \quad &\leq \quad \frac{1}{M}N[f^+\varphi(Nf)\mathbf{1}_{B_z}](x) \\ &\qquad + \frac{1}{M}\varphi(z)N[f - f^+\mathbf{1}_{B_z}](x) - z\varphi(z).\end{aligned}$$

2. Potential Inequality for Absolute Values

Let N be a kernel that satisfies the strong maximum principle. If both f and $-f$ belong to $\mathcal{R}$, then a property similar to (1.1) holds for $|f|$ (see Lemma 2.1 bellow).

First, notice that condition (1.1) is equivalent to

$$(Nf)^+(x) \leq Mu + N[f^+\mathbf{1}_{\{(Nf)^+\geq u\}}](x). \tag{2.1}$$

Indeed, the two conditions are equivalent for $u > 0$ since the right hand sides of (1.1) and (2.1) are equal because the sets $\{Nf > u\}$ and $\{(Nf)^+ > u\}$ are equal. Furthermore, the left hand sides are also equal on the set $\{Nf > 0\} = \{(Nf)^+ > 0\}$, while the inequalities (1.1) and (2.1) are trivially satisfied on the set $\{Nf \leq 0\} = \{(Nf)^+ = 0\}$ since the right hand sides are nonnegative. Finally, for $u = 0$ the condition (2.1) holds trivially since $\{(Nf)^+ \geq 0\} = X$ and $(Nf)^+ \leq N(f^+)$.

On the other hand, when (2.1) holds for every $u \geq 0$, letting $u \searrow 0$ we get

$$(Nf)(x) \leq (Nf)^{+}(x) \leq M \cdot 0 + N[f^{+}\mathbf{1}_{\{(Nf)^{+}>0\}}] \leq N[f^{+}\mathbf{1}_{\{(Nf)\geq 0\}}].$$

The following lemma was proven in [4].

LEMMA 2.1. *Let N be a positive kernel that satisfies the strong maximum principle on $\mathcal{R}$. If f and $-f$ belong to $\mathcal{R}$, then*

$$|Nf| \leq 2Mu + N\big[|f| \cdot \mathbf{1}_{\{|Nf|\geq u\}}\big] \tag{2.2}$$

holds for every $u \geq 0$.

PROOF. Since $(Nf)^{-} = [N(-f)]^{+}$, applying (2.1) for both f and $-f$ we get

$$|(Nf)(x)| = (Nf)^{+}(x) + (Nf)^{-}(x) \leq \\ 2Mu + N[f^{+}\mathbf{1}_{\{(Nf)^{+}\geq u\}}] + N[f^{-}\mathbf{1}_{\{(Nf)^{-}\geq u\}}].$$

Finally, since $f^{+} \leq |f|$, $f^{-} \leq |f|$ and $\{(Nf)^{+} \geq u\} \cup \{(Nf)^{-} \geq u\} = \{|Nf| \geq u\}$, the claim of the lemma follows from the last inequality. ∎

THEOREM 2.2 (The Potential Inequality for the Absolute Value of Functions). *Let $\Phi : (0, +\infty) \to \mathbb{R}$ be a convex function and let $\varphi = \Phi'_{+}$ be the right-continuous version of its derivative. Let $N(x, dy)$ be a positive kernel on X which satisfies the strong maximum principle on $\mathcal{R}$ with constant M, let $f, -f \in \mathcal{R}$, $x \in X$ and $z > 0$ be such that $2Mz \leq |Nf(x)|$ and denote by B_z the set*

$$B_z = \big\{y \in X : |Nf(y)| \geq z\big\}.$$

Then the following inequality holds

$$\Phi\Big(\frac{1}{2M}|Nf(x)|\Big) - \Phi(z) \leq \frac{1}{2M}N[|f|\varphi(|Nf|)\mathbf{1}_{B_z}](x) \\ - \frac{1}{2M}\varphi(z)\Big(N[|f|\mathbf{1}_{B_z}](x) - |Nf(x)|\Big) - z\varphi(z).$$

PROOF. Let $\tau(x) = \frac{1}{2M}|Nf(x)|$. Integration by parts gives

$$\begin{aligned}
\Phi(\tau(x)) - \Phi(z) &= \int_z^{\tau(x)} \varphi(u)du = u\varphi(u)\Big|_z^{\tau(x)} - \int_z^{\tau(x)} ud\varphi(u) \\
&= \tau\varphi(\tau(x)) - z\varphi(z) - \int_z^{\tau(x)} (u \pm \tau(x))d\varphi(u) \\
&= \int_z^{\tau(x)} (\tau(x) - u)d\varphi(u) + \varphi(z)(\tau(x) - z).
\end{aligned}$$

Since $d\varphi(u)$ is a positive measure and $\tau(x) \geq z$, using (2.2) we get

$$\Phi(\tau(x)) - \Phi(z) \quad \leq \quad \int_z^{\tau(x)} \frac{1}{2M} N[|f|\mathbf{1}_{\{|Nf|\geq u\}}]d\varphi(u) + \varphi(z)(\tau(x) - z).$$

Applying Fubini's theorem and the fact that $|f|\mathbf{1}_{\{|Nf|\geq u\}}$ is a nonnegative function, we further get

$$\begin{aligned}
\int_z^{\tau(x)} & N[|f|\mathbf{1}_{\{|Nf|\geq u\}}]\, d\varphi(u) \\
&= \int_z^{\tau(x)} \int_X |f(y)|\mathbf{1}_{\{|Nf(y)|\geq u\}} N(x,dy)\, d\varphi(u) \\
&= \int_X \left[\int_z^{\tau(x)} \mathbf{1}_{\{|Nf(y)|\geq u\}} d\varphi(u) \right] |f(y)| N(x,dy) \\
&\leq \int_X \left[\int_z^{+\infty} \mathbf{1}_{\{|Nf(y)|\geq u\}} d\varphi(u) \right] |f(y)| N(x,dy) \\
&= \int_X |f(y)| \Big(\varphi(|Nf(y)|) - \varphi(z) \Big) \mathbf{1}_{B_z}(y) N(x,dy) \\
&= N[|f|\varphi(|Nf|)\mathbf{1}_{B_z}](x) - \varphi(z) N[|f|\mathbf{1}_{B_z}](x).
\end{aligned}$$

Now the inequality from the theorem follows from the two inequalities above and linearity of potential. ∎

Let us further denote the set

$$B = \lim_{z \searrow 0} B_z = \big\{ x \in X : |Nf(x)| \neq 0 \big\}.$$

By integrating the potential inequality from Theorem 2.2 with respect to the variable x we can get the following, integral version of the inequality.

COROLLARY 2.3. *Let the assumptions of Theorem 2.2 hold for a function* $z : B \to (0, +\infty)$*, i. e.* $2Mz(x) \leq |Nf(x)|$ *for* $x \in B$*. Then, for* $C \subset B$ *and a measure* μ *on* $(X, \mathcal{B}(X))$*, the following inequality holds*

$$\begin{aligned}
\int_C & \Big(\Phi\big(\frac{1}{2M}|Nf(x)|\big) - \Phi(z(x)) \Big) \mu(dx) \\
&\leq \frac{1}{2M} \int_C \int_{B_{z(x)}} |f(y)|\varphi(|Nf(y)|) N(x,dy) \mu(dx) \\
- \frac{1}{2M} \int_C & \varphi(z(x)) \Big(N[|f|\mathbf{1}_{B_z}](x) - |Nf(x)| \Big) \mu(dx) - \int_C z(x)\varphi(z(x)) \mu(dx).
\end{aligned}$$

In particular, for $C = B_z$ and a constant function $z(x) \equiv z$, if the measure μ is σ-finite, we get

$$\begin{aligned}\int_{B_z} \Phi\big(\frac{1}{2M}|Nf(x)|\big)\mu(dx) - \Phi(z)\mu(B_z) \\ \leq \frac{1}{2M}\int_{B_z}|f(x)|\varphi(|Nf(x)|)(\hat{N}_{B_z}\mu)(dx) \\ + \frac{1}{2M}\varphi(z)\int_{B_z}\Big(N[|f|\mathbf{1}_{B_z}](x) - |Nf(x)|\Big)\mu(dx) - z\varphi(z)\mu(B_z).\end{aligned}$$

Proof. Integrating the potential inequality with respect to the measure μ we get

$$\begin{aligned}\int_C\Big(\Phi\big(\frac{1}{2M}|Nf(x)|\big) - \Phi(z(x))\Big)\mu(dx) \leq \frac{1}{2M}\int_C N[|f|\varphi(|Nf|)\mathbf{1}_{B_{z(x)}}]\mu(dx) \\ - \frac{1}{2M}\int_C \varphi(z(x))\Big(N[|f|\mathbf{1}_{B_z}](x) - |Nf(x)|\Big)\mu(dx) - \int_C z(x)\varphi(z(x))\mu(dx),\end{aligned}$$

which is the first inequality.

The second inequality follows by taking $C = B_z$ and $z(x) \equiv z$ and by applying Fubini's theorem on the first integral of the right-hand side. ∎

Let us denote by Φ_p the following class of functions

$$\Phi_p(\tau) = \begin{cases} \frac{\tau^p}{p(p-1)}, & p \neq 0,1 \\ -\log\tau, & p = 0 \\ \tau\log\tau, & p = 1 \end{cases} \tag{2.3}$$

Corollary 2.4. *Under the assumptions of Corollary 2.3, for $p \in \mathbb{R}\backslash\{0,1\}$ the following inequality holds*

$$\begin{aligned}\frac{1}{p(p-1)}\int_{B_z}|Nf(x)|^p\mu(dx) \leq \frac{(2M)^{p-1}}{(p-1)}\int_{B_z}|f(x)||Nf(x)|^{p-1}(\hat{N}_{B_z}\mu)(dx) \\ - \frac{(2Mz)^{p-1}}{(p-1)}\int_{B_z}\Big(N[|f|\mathbf{1}_{B_z}](x) - |Nf(x)|\Big)\mu(dx) - \frac{(2Mz)^p\mu(B_z)}{p}.\end{aligned}$$

Furthermore, for $q = p/(p-1)$ the following inequality holds

$$\begin{aligned}\frac{1}{p(p-1)}\int_{B_z}|Nf|^p d\mu \leq \frac{(2M)^{p-1}}{(p-1)}\left[\int_{B_z}|f|^p d(\hat{N}_{B_z}\mu)\right]^{\frac{1}{p}}\left[\int_{B_z}|Nf|^p d(\hat{N}_{B_z}\mu)\right]^{\frac{1}{q}} \\ - \frac{(2Mz)^{p-1}}{(p-1)}\int_{B_z}\Big(N[|f|\mathbf{1}_{B_z}](x) - |Nf(x)|\Big)\mu(dx) - \frac{(2Mz)^p\mu(B_z)}{p}.\end{aligned}$$

Proof. Applying the second inequality from Corollary 2.3 for convex functions Φ_p, $p \in \mathbb{R}\backslash\{0,1\}$, and rearranging we get the first inequality. The second inequality follows from the first by applying Hölder's inequality on the first integral of the right-hand side. ∎

Corollary 2.5. *Under the assumptions of Theorem 2.2, if* $B_z = X$ *and* $\varphi(z) \geq 0$*, then for every* $x \in X$ *the following inequality holds*

$$\Phi(\frac{1}{2M}|Nf(x)|) - \Phi(z) \leq \frac{1}{2M}N[|f|\varphi(|Nf|)](x) - z\varphi(z).$$

Furthermore, for a σ*-finite measure* μ *on* $(X, \mathcal{B}(X))$*, the following inequality holds*

$$\int_X \Phi(\frac{1}{2M}|Nf(x)|)\mu(dx) - \Phi(z)\mu(X) \leq \frac{1}{2M}\int_X |f(x)|\varphi(|Nf(x)|)(\hat{N}\mu)(dx) - z\varphi(z)\mu(X).$$

Proof. Under the assumptions of the corollary, the second term on the right hand side of the potential inequality from Theorem 2.2 is nonpositive, so the first inequality follows.

The second inequality follows by integrating the first with respect to the measure μ over the set X and applying Fubini's theorem on the right hand side integral. ∎

Corollary 2.6. *Under the assumptions of Corollary 2.5, the following inequality holds for* $p > 1$

$$\int_X |Nf(x)|^p\mu(dx) \leq p(2M)^{p-1}\int_X |f(x)|Nf(x)|^{p-1}(\hat{N}\mu)(dx) - (p-1)(2Mz)^p\mu(X).$$

Furthermore, for $q = p/(p-1)$ *the following inequality holds*

$$\int_X |Nf(x)|^p\mu(dx) \leq p(2M)^{p-1}\left[\int_X |f|^p d(\hat{N}\mu)\right]^{\frac{1}{p}}\left[\int_X |Nf|^p d(\hat{N}\mu)\right]^{\frac{1}{q}} - (p-1)(2Mz)^p\mu(X).$$

Proof. Applying Corollary 2.5 for convex functions Φ_p, $p > 1$, and rearranging we get the first inequality. The second inequality follows from the first by applying Hölder's inequality on the right-hand side integral. ∎

If Theorem 2.2 or Corollary 2.5 hold for $z > 0$, then they hold for every z', $0 < z' \leq z$. Letting $z' \to 0$ we can get further inequalities.

In the following corollaries we will assume that either φ is nonnegative, or that for every $x \in B$ there exists a function $g_x \in L^1(N(x,\cdot))$ such that

$|f\varphi(|Nf|)| \le g_x$. In either case, by the monotone convergence theorem in the former and by the dominated convergence theorem in the latter, we have

$$\lim_{z\searrow 0} N[|f|\varphi(|Nf|)\mathbf{1}_{B_z}] = N[|f|\varphi(|Nf|)\mathbf{1}_B]$$

since $|f|\varphi(|Nf|)\mathbf{1}_{B_z} \to_{z\to 0} |f|\varphi(|Nf|)\mathbf{1}_B$ pointwise.

COROLLARY 2.7. *Under the assumptions of Theorem 2.2, if* $\varphi(0+)$ *is finite, then for every* $x \in X$ *we have*

$$\Phi\big(\frac{1}{2M}|Nf(x)|\big) - \Phi(0+) \le \frac{1}{2M}N[|f|\varphi(|Nf|)\mathbf{1}_B](x) - \frac{1}{2M}\varphi(0+)\Big(N[|f|\mathbf{1}_B](x) - |Nf(x)|\Big). \tag{2.4}$$

Furthermore, if μ *is a* σ*-finite measure on* $(X, \mathcal{B}(X))$*, then the following inequality holds*

$$\int_B \Phi(\frac{1}{2M}(|Nf(x)|)\mu(dx) - \Phi(0+)\mu(B) \le \frac{1}{2M}\int_B |f(x)|\varphi(|Nf(x)|)(\hat{N}_B\mu)(dx) - \frac{1}{2M}\varphi(0+)\int_B \Big(N[|f|\mathbf{1}_B](x) - |Nf(x)|\Big)\mu(dx).$$

PROOF. Since $\lim_{z\to 0} z\varphi(z) = 0{\cdot}\varphi(0+) = 0$, letting $z \to 0$ in the potential inequality from Theorem 2.2, the last term on the right hand side disappears and inequality (2.4) follows.

The second inequality of the corollary follows by integrating the first with respect to the measure μ and applying Fubini's theorem on the first integral of the right hand side. ∎

COROLLARY 2.8. *Under the assumptions of Corollary 2.7, for* $p > 1$ *the following inequality holds*

$$\int_B |Nf(x)|^p \mu(dx) \le p(2M)^{p-1}\int_B |f(x)||Nf(x)|^{p-1}(\hat{N}\mu)(dx).$$

Furthermore, for $q = p/(p-1)$ *the following inequality holds*

$$\int_B |Nf|^p d\mu \le p(2M)^{p-1}\left[\int_B |f|^p d(\hat{N}\mu)\right]^{\frac{1}{p}}\left[\int_B |Nf|^p d(\hat{N}\mu)\right]^{\frac{1}{q}}.$$

PROOF. The first inequality holds since convex functions Φ_p, $p > 1$, satisfy the assumptions of Corollary 2.7 with $\Phi_p(0+) = \varphi_p(0+) = 0$. The second inequality follows from the first by applying Hölder's inequality on the right-hand side integral. ∎

When one or both of the measures μ and $\hat{N}_C\mu$ is bounded by the other up to a multiplicative constant, then we can state further inequalities. In that regard, let K_1 and K_2 be positive constants, if they exist, such that

$$K_1\mu \le \hat{N}_C\mu \tag{2.5}$$

and

$$\hat{N}_C\mu \le K_2\mu. \tag{2.6}$$

COROLLARY 2.9. *Let the assumptions of Corollary 2.4 hold. If N and μ satisfy (2.6) with $C = B_z$, then for $p > 1$*

$$\int_{B_z} |Nf|^p d\mu \le pK_2(2M)^{p-1}\left[\int_{B_z} |f|^p d\mu\right]^{\frac{1}{p}}\left[\int_{B_z} |Nf|^p d\mu\right]^{\frac{1}{q}}$$
$$- p(2Mz)^{p-1}\int_{B_z}\Big(N[|f|\mathbf{1}_{B_z}](x) - |Nf(x)|\Big)\mu(dx) - (p-1)(2Mz)^p\mu(B_z).$$

When N and μ satisfy both (2.5) and (2.6) with $C = B_z$, then for $0 < p < 1$

$$\int_{B_z} |Nf|^p d\mu \ge pK_1^{1/p}K_2^{1/q}(2M)^{p-1}\left[\int_{B_z} |f|^p d\mu\right]^{\frac{1}{p}}\left[\int_{B_z} |Nf|^p d\mu\right]^{\frac{1}{q}}$$
$$- p(2Mz)^{p-1}\int_{B_z}\Big(N[|f|\mathbf{1}_{B_z}](x) - |Nf(x)|\Big)\mu(dx) - (p-1)(2Mz)^p\mu(B_z),$$

while for $p < 0$

$$\int_{B_z} |Nf|^p d\mu \le pK_1^{1/q}K_2^{1/p}(2M)^{p-1}\left[\int_{B_z} |f|^p d\mu\right]^{\frac{1}{p}}\left[\int_{B_z} |Nf|^p d\mu\right]^{\frac{1}{q}}$$
$$- p(2Mz)^{p-1}\int_{B_z}\Big(N[|f|\mathbf{1}_{B_z}](x) - |Nf(x)|\Big)\mu(dx) - (p-1)(2Mz)^p\mu(B_z).$$

PROOF. The inequalities follow directly from (2.5), (2.6) and Corollary 2.4. □

COROLLARY 2.10. *Let the assumptions of Corollary 2.8 hold. If N and μ satisfy (2.6) with $C = B$, then for $p > 1$*

$$\left[\int_B |Nf|^p d\mu\right]^{\frac{1}{p}} \le pK_2(2M)^{p-1}\left[\int_B |f|^p d\mu\right]^{\frac{1}{p}}$$

and

$$\left[\int_B |Nf|^p d\mu\right]^{\frac{1}{p}} \le pK_2^{1/q}(2M)^{p-1}\left[\int_B |f|^p d(\hat{N}\mu)\right]^{\frac{1}{p}}.$$

PROOF. The inequalities follow directly from (2.6) and Corollary 2.8. □

REMARK 2.11. **Concave case:** When Φ is concave, $d\varphi(u)$ is a negative measure and the inequalities in Theorem 2.2, Corollaries 2.3 and 2.7 are reversed. The reversed inequality in Corollary 2.5 holds if $\varphi(z) \leq 0$.

3. Exponential Convexity

Using the potential inequality from the previous section we will construct linear functionals that are nonnegative for convex functions. This will enable us to construct new families of exponentially convex functions by applying a method from [2].

Let us define linear functionals $A_1 = A_{1;f,N,z,x}$ and $A_2 = A_{2;f,N,z,\mu}$ with

$$\begin{aligned} A_1(\Phi) = & \frac{1}{2M} N[|f|\varphi(|Nf|)\mathbf{1}_{B_z}](x) - \frac{1}{2M}\varphi(z)\Big(N[|f|\mathbf{1}_{B_z}](x) - |Nf(x)|\Big) \\ & - \Phi(\frac{1}{2M}|Nf(x)|) + \Phi(z) - z\varphi(z) \end{aligned}$$

$$\begin{aligned} A_2(\Phi) = & \frac{1}{2M}\int_{B_z} |f(x)|\varphi(|Nf(x)|)(\hat{N}_{B_z}\mu)(dx) \\ & - \frac{1}{2M}\varphi(z)\int_{B_z}\Big(N[|f|\mathbf{1}_{B_z}](x) - |Nf(x)|\Big)\mu(dx) \\ & - \int_{B_z}\Phi(\frac{1}{2M}|Nf(x)|)\mu(dx) + \Phi(z)\mu(B_z) - z\varphi(z)\mu(B_z). \end{aligned}$$

Linear functionals A_k, $k = 1, 2$, depend on function f, kernel N, measure μ and points x and z, but if these choices are clear from context, we will omit them from the notation.

Similarly, we define linear functionals $A_3 = A_{3;f,N,x}$ and $A_4 = A_{4;f,N,\mu}$ with

$$\begin{aligned} A_3(\Phi) = & \frac{1}{2M} N[|f|\varphi(|Nf|)\mathbf{1}_{B}](x) - \frac{1}{2M}\varphi(0+)\Big(N[|f|\mathbf{1}_{B}](x) - |Nf(x)|\Big) \\ & - \Phi(\frac{1}{2M}|Nf(x)|) + \Phi(0+), \\ A_4(\Phi) = & \frac{1}{2M}\int_{B} |f(x)|\varphi(|Nf(x)|)(\hat{N}_{B}\mu)(dx) \\ & - \frac{1}{2M}\varphi(0+)\int_{B}\Big(N[|f|_B](x) - |Nf(x)|\Big)\mu(dx) \\ & - \int_{B}\Phi(\frac{1}{2M}|Nf(x)|)\mu(dx) + \Phi(0+)\mu(B). \end{aligned}$$

We will first give mean value theorems for the linear functionals A_k, $k = 1, ..., 4$.

THEOREM 3.1. *Let* $k \in \{1, ..., 4\}$ *and let there exist a constant* b *such that* $|(Nf)(x)| \le b$ *for every* $x \in X$. *Furthermore, define the constant* a *as:*

(*i*) $a = z$ *for* $k = 1$ *and* $k = 2$, *where* $z > 0$ *is the number from the definition of the linear functionals* A_1 *and* A_2, *respectively*

(*ii*) $a = 0$ *for* $k = 3$ *and* $k = 4$.

If $\Psi \in C^2[a, b]$, *then there exists* $\xi_k \in [a, b]$ *such that*

$$A_k(\Psi) = \Psi''(\xi_k)A_k(\Phi_2),$$

where Φ_2 *is given by (2.3).*

PROOF. Let

$$m = \min_{\tau\in[a,b]} \Psi''(\tau) \quad \text{and} \quad M = \max_{\tau\in[a,b]} \Psi''(\tau).$$

The function $M\Phi_2 - \Psi$ is convex since

$$\frac{d^2}{d\tau^2}\Big(M\frac{\tau^2}{2} - \Psi(\tau)\Big) = M - \Psi''(\tau) \ge 0.$$

From the statement and proofs of the inequalities from Theorem 2.2 and Corollaries 2.3 and 2.7 it is clear that they are meaningful for convex functions defined on the interval $[a, b]$. Since, by the assumptions of the theorem, the convex function $M\Phi_2 - \Psi$ satisfies the assumptions of Theorem 2.2 (for $k = 1$), Corollaries 2.3 (for $k = 2$) and 2.7 (for $k = 3$ or 4) we have

$$0 \le A_k\Big(M\Phi_2 - \Psi\Big), \qquad k = 1, ..., 4,$$

i. e.

$$A_k(\Psi) \le MA_k(\Phi_2), \qquad k = 1, ..., 4. \tag{3.1}$$

Similarly, the inequality

$$mA_k(\Phi_2) \le A_k(\Psi), \qquad k = 1, ..., 4 \tag{3.2}$$

holds since $\Psi - m\Phi_2$ is convex.

Since Φ_2 is a convex function we have $A_k(\Phi_2) \ge 0$. If $A_k(\Phi_2) = 0$, then $A_k(\Psi) = 0$ and for ξ_k we can take any point in $[a, b]$. If $A_k(\Phi_2) > 0$, then from from (3.1) and (3.2) we conclude

$$m \le \frac{A_k(\Psi)}{A_k(\Phi_2)} \le M,$$

so the existence of ξ_k, $k = 1, ..., 4$, follows from continuity of Ψ''. ∎

THEOREM 3.2. *Let $k \in \{1, ...4\}$ and let N, f, a and b be as in Theorem 3.1. If $\Psi, \widetilde{\Psi} \in C^2[a,b]$ and $A_k(\Phi_2) \neq 0$, then there exists $\xi_k \in [a,b]$ such that*

$$\frac{\Psi''(\xi_k)}{\widetilde{\Psi}''(\xi_k)} = \frac{A_k(\Psi)}{A_k(\widetilde{\Psi})}. \tag{3.3}$$

PROOF. Let us define the function ϕ by

$$\phi(\tau) = \Psi(\tau)A_k(\widetilde{\Psi}) - \widetilde{\Psi}(\tau)A_k(\Psi).$$

The function ϕ satisfies the assumptions of Theorem 3.1 and, hence, there exists $\xi_k \in [a,b]$ such that $A_k(\phi) = \phi''(\xi_k)A_k(\Phi_2)$. Since $A_k(\phi) = 0$ and $A_k(\Phi_2) \neq 0$, it follows that $0 = \phi''(\xi_k) = \Psi''(\xi_k)A_k(\widetilde{\Psi}) - \widetilde{\Psi}''(\xi_k)A_k(\Psi)$, so equality (3.3) follows. ◻

We will continue this section with few basic notions and results on exponential convexity that will be used here.

DEFINITION 3.3. *A function $\psi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} \xi_i \xi_j \psi\Big(\frac{x_i + x_j}{2}\Big) \geq 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, ..., n$.

A function $\psi : I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

DEFINITION 3.4. *A function $\psi : I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for every $n \in \mathbb{N}$.*

A function $\psi : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous on I.

Definition of positive semi-definite matrices and some basic algebra gives us the following proposition

PROPOSITION 3.5. *If ψ is an n-exponentially convex in the Jensen sense on I, then for every choice of $x_i \in I$, $i = 1, ..., n$, the matrix $\Big[\psi\Big(\frac{x_i+x_j}{2}\Big)\Big]_{i,j=1}^{k}$ is a positive semi-definite matrix for all $k \in \mathbb{N}$, $k \leq n$. In particular, for all $k \in \mathbb{N}$, $\det\Big[\psi\Big(\frac{x_i+x_j}{2}\Big)\Big]_{i,j=1}^{k} \geq 0$ for all $k \leq n$.*

REMARK 3.6. It is known that $\psi : I \to \mathbb{R}$ is log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\Big(\frac{x+y}{2}\Big) + \beta^2\psi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. Moreover, a function is log-convex if and only if it is 2-exponentially convex.

We will also make use of the divided differences.

Definition 3.7. *The second order divided differences of a function* $\Psi : I \to \mathbb{R}$*, where* I *is an interval in* $\mathbb{R}$*, at mutually different points* $\tau_0, \tau_1, \tau_2 \in I$ *is defined recursively by*

$$\begin{aligned}[][\tau_i, ; \Psi] &= \Psi(\tau_i), \quad i = 0, 1, 2, \\ [\tau_i, \tau_{i+1}; \Psi] &= \frac{\Psi(\tau_{i+1}) - \Psi(\tau_i)}{\tau_{i+1} - \tau_i}, \quad i = 0, 1, \\ [\tau_0, \tau_1, \tau_2; \Psi] &= \frac{[\tau_1, \tau_2; \Psi] - [\tau_0, \tau_1; \Psi]}{\tau_2 - \tau_0}. \qquad (3.4)\end{aligned}$$

Remark 3.8. The value $[\tau_0, \tau_1, \tau_2; \Psi]$ is independent of the order of the points τ_0, τ_1, τ_2. This definition may be extended to include the case in which some or all of the points coincide by taking limits. If Ψ' exists, then by taking the limit $\tau_1 \to \tau_0$ in (3.4) we get

$$\lim_{\tau_1 \to \tau_0} [\tau_0, \tau_1, \tau_2; \Psi] = [\tau_0, \tau_0, \tau_2; \Psi] = \frac{\Psi(\tau_2) - \Psi(\tau_0) - \Psi'(\tau_0)(\tau_2 - \tau_0)}{(\tau_2 - \tau_0)^2}, \tau_2 \neq \tau_0.$$

Furthermore, if Ψ'' exists, then by taking the limits $\tau_i \to \tau_0$, $i = 1, 2$ in (3.4) we get

$$\lim_{\tau_2 \to \tau_0} \lim_{\tau_1 \to \tau_0} [\tau_0, \tau_1, \tau_2; \Psi] = [\tau_0, \tau_0, \tau_0; \Psi] = \frac{\Psi''(\tau_0)}{2}.$$

Notice that $\Psi \mapsto [\tau_0, \tau_1, \tau_2; \Psi]$ is a linear functional that is nonnegative for a convex function Ψ.

The following theorem will enable us to construct families of n-exponentially and exponentially convex functions by applying the linear functionals A_k on a family of functions with the same property.

Theorem 3.9. *Let* $\Omega = \{\Psi_p : p \in J\}$*, where* J *is an interval in* $\mathbb{R}$*, be a family of functions* $\Psi_p : (0, +\infty) \to \mathbb{R}$ *such that the function* $p \mapsto [\tau_0, \tau_1, \tau_2; \Psi_p]$ *is* n*-exponentially convex in the Jensen sense on* J *for every three mutually different points* $\tau_0, \tau_1, \tau_2 \in (0, +\infty)$*. Then:*

- (i) *for* $k = 1$ *or* 2*, the mapping* $p \mapsto A_k(\Psi_p)$ *is an* n*-exponentially convex function in the Jensen sense on* J*. If the function* $p \mapsto A_k(\Psi_p)$ *is continuous on* J*, then it is* n*-exponentially convex on* J*.*
- (ii) *if* $\Psi_p(0+)$ *is finite for every* $p \in J$*, then the same conclusions as in* (i) *hold for* $k = 3$ *and* 4*.*

Proof. For $\xi_i \in \mathbb{R}$ and $p_i \in J$, $i = 1, ..., n$, we define the function

$$\Psi(\tau) = \sum_{i,j=1}^{n} \xi_i \xi_j \Psi_{\frac{p_i + p_j}{2}}(\tau).$$

Due to the linearity of the divided differences and the assumption that the function $p \mapsto [\tau_0, \tau_1, \tau_2; \Psi_p]$ is n-exponentially convex in the Jensen sense we have

$$[\tau_0, \tau_1, \tau_2; \Psi] = \sum_{i,j=1}^{n} \xi_i \xi_j [\tau_0, \tau_1, \tau_2; \Psi_{\frac{p_i+p_j}{2}}] \geq 0.$$

This implies that Ψ is a convex functions and, due to the assumptions, it satisfies the assumptions of Theorem 2.2 (for $k = 1$ or 2) and Corollary 2.7 (for $k = 3$ or 4). Hence, by the potential inequality,

$$0 \leq A_k(\Psi) = \sum_{i,j=1}^{n} \xi_i \xi_j A_k\big(\Psi_{\frac{p_i+p_j}{2}}\big), \qquad k = 1, ..., 4$$

so the function $p \mapsto A_k(\Psi_p)$ is n-exponentially convex in the Jensen sense on J. If it is also continuous on J, then it is n-exponentially convex by definition. ∎

Corollary 3.10. *Let $\Omega = \{\Psi_p : p \in J\}$, where J is an interval in $\mathbb{R}$, be a family of functions $\Psi_p : (0, +\infty) \to \mathbb{R}$ such that the function $p \mapsto [\tau_0, \tau_1, \tau_2; \Psi_p]$ is 2-exponentially convex in the Jensen sense on J for every three mutually different points $\tau_0, \tau_1, \tau_2 \in (0, +\infty)$. Then for $k = 1$ and $k = 2$ the following statements hold:*

(*i*) *If the function $p \mapsto A_k(\Psi_p)$ is continuous on J, then it is 2-exponentially convex and, thus, log-convex.*

(*ii*) *If the function $p \mapsto A_k(\Psi_p)$ is strictly positive and differentiable on J, then for every $p, q, r, s \in J$, such that $p \leq r$ and $q \leq s$, we have*

$$\mu^k_{p,q}(\Omega) \leq \mu^k_{r,s}(\Omega),$$

where

$$\mu^k_{p,q}(\Omega) = \begin{cases} \left(\frac{A_k(\Psi_p)}{A_k(\Psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(\frac{\frac{d}{dp}A_k(\Psi_p)}{A_k(\Psi_p)}\right), & p = q \end{cases} \tag{3.5}$$

for $\Psi_p, \Psi_q \in \Omega$.

If, additionally, $\Psi_p(0+)$ is finite for every $p \in J$, then statements (i) and (ii) hold for $k = 3$ and 4 as well.

Proof. (*i*) This is an immediate consequence of Theorem 3.9 and Remark 3.6

(*ii*) By (*i*), the function $p \mapsto A_k(\Psi_p)$ is log-convex on J, that is, the function $p \mapsto \log A_k(\Psi_p)$ is convex. Therefore

$$\frac{\log A_k(\Psi_p) - \log A_k(\Psi_q)}{p - q} \leq \frac{\log A_k(\Psi_r) - \log A_k(\Psi_s)}{r - s} \tag{3.6}$$

for $p \leq r$, $q \leq s$, $p \neq r$, $q \neq s$, which implies that

$$\mu_{p,q}^k(\Omega) \leq \mu_{r,s}^k(\Omega), \qquad k = 1, ..., 4.$$

The cases $p = r$ and $q = s$ follow from (3.6) by taking limits $p \to r$ or $q \to s$. ∎

REMARK 3.11. The results from Theorem 3.9 (Corollary 3.10) still hold when two of the points $\tau_0, \tau_1, \tau_2 \in (0, +\infty)$ coincide, say $\tau_0 = \tau_1$, for a family of differentiable functions Ψ_p such that the function $p \mapsto [\tau_0, \tau_0, \tau_2; \Psi_p]$ is n-exponentially convex in the Jensen sense (2-exponentially convex in the Jensen sense) and, furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property.

We will end this sections with several examples of families of functions that satisfy the assumptions of Theorem 3.9 and Corollary 3.10 (and Remark 3.11), which, as a consequence, gives us large families of exponentially convex functions.

EXAMPLE 3.12. Consider a family of functions $\Omega_1^k = \{\Phi_p : p \in J_k\}$, where Φ_p are the power functions defined by (2.3) and $J_1 = J_2 = \mathbb{R}$ and $J_3 = J_4 = (1, +\infty)$.

We have $\frac{d^2}{d\tau^2}\Phi_p(\tau) = \tau^{p-2} > 0$ which shows that Φ_p are convex on $(0, +\infty)$. Similarly as in the proof of Theorem 3.9, let us, for $\xi_i \in \mathbb{R}$ and $p_i \in J_k$, $i = 1, ..., n$, define the function

$$\Phi(\tau) = \sum_{i,j=1}^{n} \xi_i \xi_j \Phi_{\frac{p_i+p_j}{2}}(\tau).$$

Since the function $p \mapsto \frac{d^2}{d\tau^2}\Phi_p(\tau) = \tau^{p-2} = e^{(p-2)\ln\tau}$ is exponentially convex (by definition), it follows that

$$\Phi''(\tau) = \sum_{i,j=1}^{n} \xi_i \xi_j \Phi''_{\frac{p_i+p_j}{2}}(\tau) = \left(\sum_{i=1}^{n} \xi_i e^{(p_i-2)\ln\tau}\right)^2 \geq 0$$

is a convex function. Therefore

$$0 \leq [\tau_0, \tau_1, \tau_2; \Phi] = \sum_{i,j=1}^{n} \xi_i \xi_j [\tau_0, \tau_1, \tau_2; \Phi_{\frac{p_i+p_j}{2}}],$$

so $p \mapsto [\tau_0, \tau_1, \tau_2; \Psi_p]$ is n-exponentially convex in the Jensen sense. Now, by Theorem 3.9, it follows that the mappings $p \mapsto A_k(\Phi_p)$ are exponentially convex in the Jensen sense. It is straightforward to check that these mappings are continuous, so they are exponentially convex.

For these families of functions, $\mu^k_{p,q}(\Omega^k_1)$ from (3.5), for $k = 1$ and $k = 2$, are equal to

$$\mu^k_{p,q}(\Omega^k_1) = \begin{cases} \left(\frac{A_k(\Phi_p)}{A_k(\Phi_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(\frac{1-2p}{p(p-1)} - \frac{A_k(\Phi_0\Phi_p)}{A_k(\Phi_p)}\right), & p = q \neq 0, 1 \\ \exp\left(-1 - \frac{A_k(\Phi_0\Phi_1)}{2A_k(\Phi_1)}\right), & p = q = 1 \\ \exp\left(1 - \frac{A_k(\Phi_0^2)}{2A_k(\Phi_0)}\right), & p = q = 0 \end{cases}$$

while for $k = 3$ and $k = 4$ they have the same form, but are only defined for $p, q > 1$.

Furthermore, if a linear functional A_k and the functions $\Psi = \Phi_p$ and $\widetilde{\Psi} = \Phi_q$ are such that the assumptions of Theorem 3.2 are satisfied, we can define a two-parameter family of means. Indeed, since $(\Psi/\widetilde{\Psi})^{-1}(\tau) = \tau^{1/(p-q)}$, the number $E^k_{p,q}(\Omega^k_1) = \mu^k_{p,q}(\Omega^k_1)$ satisfies

$$0 \leq E^k_{p,q}(\Omega^k_1) \leq K.$$

Example 3.13. Let $\Omega_2 = \{\Psi_p : p \in \mathbb{R}\}$ be a family of functions defined by

$$\Psi_p(\tau) = \begin{cases} \frac{1}{p^2} e^{px}, & p \neq 0, \\ \frac{1}{2}\tau^2, & p = 0. \end{cases}$$

We have $\frac{d^2}{d\tau^2}\Psi_p(\tau) = e^{p\tau} > 0$ which shows that Ψ_p are convex and $p \mapsto \frac{d^2}{d\tau^2}\Psi_p(\tau)$ is exponentially convex (by definition). Arguing as in Example 3.12 we get that the mappings $p \mapsto A_k(\Psi_p)$, $k = 1, ..., 4$, are exponentially convex. In this case, the functions (3.5) are equal to

$$\mu^k_{p,q}(\Omega_2) = \begin{cases} \left(\frac{A_k(\Psi_p)}{A_k(\Psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(\frac{A_k(id \cdot \Psi_p)}{A_k(\Psi_p)} - \frac{2}{p}\right), & p = q \neq 0 \\ \exp\left(\frac{A_k(id \cdot \Psi_0)}{3A_k(\Psi_0)}\right), & p = q = 0, \end{cases}$$

where $id(\tau) = \tau$ is the identity function.

Again, if a linear functional A_k and the functions $\Psi = \Psi_p$ and $\widetilde{\Psi} = \Psi_q$ are such that the assumptions of Theorem 3.2 are satisfied, then, since $(\Psi/\widetilde{\Psi})^{-1}(\tau) = \ln(\tau)/(p - q)$, we have

$$0 \leq E^k_{p,q}(\Omega_2) = \ln \mu^k_{p,q}(\Omega_2) \leq K,$$

so $E^k_{p,q}(\Omega_2)$ are means.

Example 3.14. Consider a family of functions $\Omega_3 = \{\Psi_p : p \in (0, +\infty)\}$ given by

$$\Psi_p(\tau) = \begin{cases} \frac{p^{-\tau}}{\ln^2 \tau}, & p \neq 1, \\ \frac{1}{2}\tau^2, & p = 1. \end{cases}$$

Since $\frac{d^2}{d\tau^2}\Psi_p(\tau) = p^{-\tau}$ is the Laplace transform of a nonnegative function (see [5]), it is exponentially convex. Arguing as in Example 3.12 we get that $p \mapsto A_k(\Psi_p)$, $k = 1, ..., 4$, are exponentially convex functions.

For this family of functions, $\mu^k_{p,q}(\Omega_3)$ from (3.5) becomes

$$\mu^k_{p,q}(\Omega_3) = \begin{cases} \left(\frac{A_k(\Psi_p)}{A_k(\Psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(-\frac{A_k(id\cdot\Psi_p)}{pA_k(\Psi_p)} - \frac{2}{p\ln p}\right), & p = q \neq 1 \\ \exp\left(-\frac{2A_k(id\cdot\Psi_1)}{3A_k(\Psi_1)}\right), & p = q = 1. \end{cases}$$

If the assumptions of Theorem 3.2 are satisfied for the linear functional A_k and the functions $\Psi = \Psi_p$ and $\widetilde{\Psi} = \Psi_q$, then

$$E^k_{p,q}(\Omega_3) = -L(p,q)\ln \mu^k_{p,q}(\Omega_3)$$

satisfies $0 \leq E^k_{p,q}(\Omega_3) \leq K$, i. e. $E^k_{p,q}(\Omega_3)$ is a mean. Here, $L(p,q)$ is the logarithmic mean defined by $L(p,q) = (p-q)/(\ln p - \ln q)$, $p \neq q$, $L(p,p) = p$.

Example 3.15. Consider a family of functions $\Omega_4 = \{\Psi_p : p \in (0,+\infty)\}$ given by

$$\Psi_p(\tau) = \frac{e^{-\tau\sqrt{p}}}{p}.$$

Since $\frac{d^2}{d\tau^2}\Psi_p(\tau) = e^{-\tau\sqrt{p}}$ is the Laplace transform of a nonnegative function (see [5]), it is exponentially convex. Arguing as before, we get that $p \mapsto A_k(\Psi_p)$, $k = 1, ..., 4$, are exponentially convex functions.

For this family of functions, $\mu^k_{p,q}(\Omega_4)$ from (3.5) becomes

$$\mu^k_{p,q}(\Omega_3) = \begin{cases} \left(\frac{A_k(\Psi_p)}{A_k(\Psi_q)}\right)^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(-\frac{A_k(id\cdot\Psi_p)}{2\sqrt{p}A_k(\Psi_p)} - \frac{1}{p}\right), & p = q. \end{cases}$$

If the assumptions of Theorem 3.2 are satisfied for the linear functional A_k and the functions $\Psi = \Psi_p$ and $\widetilde{\Psi} = \Psi_q$, then

$$E^k_{p,q}(\Omega_4) = -\left(\sqrt{p} + \sqrt{q}\right)\ln \mu^k_{p,q}(\Omega_4)$$

satisfies $0 \leq E^k_{p,q}(\Omega_4) \leq K$, which shows that $E^k_{p,q}(\Omega_4)$ is a mean.

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O potencijalnoj nejednakosti za apsolutne vrijednosti funkcija

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SAŽETAK. Potencijalna nejednakost izvedena je u [4] i kasnije proširena na opću klasu konveksnih i konkavnih funkcija u [1]. Glavni cilj ovog članka je izvesti potencijalnu nejednakost u slučaju kada funkciju u kojoj se računa potencijal zamijenimo s apsolutnom vrijednošću te funkcije. Dobiveni rezultati, zajedno s metodama iz [2], koriste se pri konstrukciji novih klasa eksponencijalno konveksnih funkcija.

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Received: 15.7.2012.