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CLASSIFICATION OF CONIC SECTIONS IN $PE_2(\mathbb{R})$

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Abstract. This paper gives a complete classification of conics in $PE_2(\mathbb{R})$. The classification has been made earlier (Reveruk [5]), but it showed to be incomplete and not possible to cite and use in further studies of properties of conics, pencil of conics, and of quadratic forms in pseudo-Euclidean spaces. This paper provides that. A pseudo-orthogonal matrix, pseudo-Euclidean values of a matrix, diagonalization of a matrix in a pseudo-Euclidean way are introduced. Conics are divided in families and by types, giving both of them geometrical meaning. The invariants of a conic with respect to the group of motions in $PE_2(\mathbb{R})$ are determined, making it possible to determine a conic without reducing its equation to canonical form. An overview table is given.

1. Pseudo-Euclidean plane

The pseudo-Euclidean plane is a real affine plane where the metric is introduced by the *absolute figure* $(\omega, \Omega_1, \Omega_2)$ consisting of the line ω at infinity and the points $\Omega_1, \Omega_2 \in \omega$. Any line passing through Ω_1 or Ω_2 is called an *isotropic* line and any point incident with ω is called an *isotropic* point.
Let $T = (x_0 : x_1 : x_2)$ denote any point in the plane presented in homogeneous coordinates. In the affine model, where

$$x = \frac{x_1}{x_0}, \qquad y = \frac{y_1}{y_0}$$

the absolute figure is determined by w: $x_0 = 0$; $\Omega_1 = (0 : 1 : 1)$ and $\Omega_2 = (0 : 1 : -1)$.
In the pseudo-Euclidean plane the scalar product for two vectors, e.g. $\mathbf{v_1} = (x_1, y_1)$ and $\mathbf{v_2} = (x_2, y_2)$, $x_i, y_i \in \mathbb{R}, i = 1, 2$ is defined as

$$\mathbf{v_1} \cdot \mathbf{v_2} = (x_1, y_1) \cdot (x_2, y_2) = x_1 x_2 - y_1 y_2. \tag{1.1}$$

Hence, the norm of the vector $\mathbf{v} = (x, y)$ is of the form

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{(x, y) \cdot (x, y)} = \sqrt{x^2 - y^2}. \tag{1.2}$$

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Since (1.2) may not always be real, one can distinguish three types of vectors in the pseudo-Euclidean plane:

$$\begin{array}{ll} & 1. \ \textit{spacelike vectors} \text{ if } \quad \mathbf{v}\cdot\mathbf{v} > 0; \\ (1.3) & 2. \ \textit{timelike vectors} \text{ if } \quad \mathbf{v}\cdot\mathbf{v} < 0; \\ & 3. \ \textit{lightlike vectors} \text{ (isotropic vectors) if } \quad \mathbf{v}\cdot\mathbf{v} = 0. \end{array}$$

As a consequence there are 3 types of straight lines: *spacelike lines*, *timelike lines*, *lightlike lines*.
Apparently, for two points $T_1 = (x_1, y_1)$ and $T_2 = (x_2, y_2)$

$$d(T_1, T_2) := \sqrt{(x_1 - x_2)^2 - (y_1 - y_2)^2} \tag{1.4}$$

defines the distance between them. Comparing (1.4) and (1.2), for $\mathbf{v} = \overrightarrow{T_1T_2}$ we have $|\mathbf{v}| = d(T_1, T_2)$. We will use the following notation: $d(T_1, T_2) = |T_1T_2|$.
If $0 = (0, 0)$ is the origin, the vectors $\overrightarrow{OT_1}$ and $\overrightarrow{OT_2}$, being both spacelike or both timelike, form an angle defined by

$$\cosh\alpha := \frac{x_1x_2 - y_1y_2}{\sqrt{{x_1}^2 - {y_1}^2}\sqrt{{x_2}^2 - {y_2}^2}}. \tag{1.5}$$

The transformations that keep the absolute figure invariant and preserve the above given metric quantities of a scalar product, distance, angle, are of the form

$$\begin{aligned} \overline{x} &= x\cosh\varphi + y\sinh\varphi + a \\ \overline{y} &= x\sinh\varphi + y\cosh\varphi + b. \end{aligned} \tag{1.6}$$

The transformations (1.6) form a group B_3, called the *motion group*. Hence, the group of pseudo-Euclidean motions consists of *translations* and *pseudo-Euclidean rotations*, that is

$$\begin{aligned} \overline{x} &= x + a \\ \overline{y} &= y + b \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overline{x} &= x\cosh\varphi + y\sinh\varphi \\ \overline{y} &= x\sinh\varphi + y\cosh\varphi. \end{aligned}$$

With the geometry of the pseudo-Euclidean plane (also known as *Minkowski plane* and *Lorentzian plane*) one can get acquainted through, for example, [4] and [3].

2. Conic equation

General second-degree equation in two variables can be written in the form

$$F(x, y) \equiv a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{01}x + 2a_{02}y + a_{00} = 0 \tag{2.1}$$

where $a_{11}\ldots a_{00} \in \mathbb{R}$ and at least one of the numbers $a_{11}, a_{12}, a_{22} \neq 0$. All the solutions of the equation (2.1) represent the locus of points in a plane which

is called a *conic section* or simply, a *conic*.
Using the matrix notation, we have

$$
\begin{aligned}
F(x,y) &\equiv \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \\
&= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 2 \begin{bmatrix} a_{01} & a_{02} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + a_{00} = 0
\end{aligned} \tag{2.2}
$$

where

$$
A := \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \quad \text{and} \quad \sigma := \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \tag{2.3}
$$

are real, symmetric matrices. In the sequel we will use the following functions of the coefficients $a_{ij}, i, j = 0, 1, 2$

$$
\begin{aligned}
&I_1 := a_{11} - a_{22}, \qquad I_2 := det\sigma = \begin{vmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{vmatrix}, \\
&I_3 := detA = \begin{vmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{vmatrix}, \\
&I_4 := \begin{vmatrix} a_{00} & a_{01} \\ a_{01} & a_{11} \end{vmatrix} - \begin{vmatrix} a_{00} & a_{02} \\ a_{02} & a_{22} \end{vmatrix}, \qquad I_5 := a_{00}.
\end{aligned} \tag{2.4}
$$

The aim is to determine the invariants of conics with respect to the motion group B_3 in the pseudo-Euclidean plane. For that purpose, let's first apply on the conic equation (2.1) the "pseudo-Euclidean rotation" from (1.6) given by:

$$
\begin{aligned}
x &= \overline{x} \cosh\varphi + \overline{y} \sinh\varphi \\
y &= \overline{x} \sinh\varphi + \overline{y} \cosh\varphi.
\end{aligned} \tag{2.5}
$$

Using matrix notation, (2.5) can be represented as

$$
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix}, \qquad R := \begin{bmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{bmatrix}. \tag{2.6}
$$

Let's focus on the properties of the matrix R given in (2.6):

a) $\det R = \begin{vmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{vmatrix} = \cosh^2\varphi - \sinh^2\varphi = 1$

b) $R^{-1} = \begin{bmatrix} \cosh\varphi & -\sinh\varphi \\ -\sinh\varphi & \cosh\varphi \end{bmatrix}$

c) $R^T = R$

d) Denoting columns of R by $\mathbf{v_1} = (\cosh\varphi, \sinh\varphi)$, $\mathbf{v_2} = (\sinh\varphi, \cosh\varphi)$ we get

$$\mathbf{v_1} \cdot \mathbf{v_2} = (\cosh\varphi, \sinh\varphi) \cdot (\sinh\varphi, \cosh\varphi) = \cosh\varphi \sinh\varphi - \sinh\varphi \cosh\varphi = 0.$$

Computing norms of the vectors $\mathbf{v_1}, \mathbf{v_2}$, that is

$$\begin{aligned}&|\mathbf{v_1}| = \sqrt{\mathbf{v_1}\cdot\mathbf{v_1}} = \sqrt{(\cosh\varphi, \sinh\varphi)\cdot(\cosh\varphi, \sinh\varphi)} = \\ &= \sqrt{\cosh^2\varphi - \sinh^2\varphi} = \sqrt{1} = 1\end{aligned}$$

$$\begin{aligned}&|\mathbf{v_2}| = \sqrt{\mathbf{v_2}\cdot\mathbf{v_2}} = \sqrt{(\sinh\varphi, \cosh\varphi)(\sinh\varphi, \cosh\varphi)} = \\ &= \sqrt{\sinh^2\varphi - \cosh^2\varphi} = \sqrt{-1} = i,\end{aligned}$$

we conclude the columns of R are orthonormal in the pseudo-Euclidean sense.

Because of the aforementioned properties of the matrix R, we will say that R is a *pseudo-orthogonal* matrix.
Hence, applying (2.6) on the conic equation (2.1),

$$\begin{aligned}&\begin{bmatrix} \overline{x} & \overline{y} \end{bmatrix}\begin{bmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}\begin{bmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{bmatrix}\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} + \\ &+2\begin{bmatrix} a_{01} & a_{02} \end{bmatrix}\begin{bmatrix} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{bmatrix}\begin{bmatrix} \overline{x} \\ \overline{y} \end{bmatrix} + a_{00} = 0\end{aligned}$$

one gets

$$F(\overline{x}, \overline{y}) \equiv \begin{bmatrix} 1 & \overline{x} & \overline{y} \end{bmatrix}\begin{bmatrix} \overline{a_{00}} & \overline{a_{01}} & \overline{a_{02}} \\ \overline{a_{01}} & \overline{a_{11}} & \overline{a_{12}} \\ \overline{a_{02}} & \overline{a_{12}} & \overline{a_{22}} \end{bmatrix}\begin{bmatrix} 1 \\ \overline{x} \\ \overline{y} \end{bmatrix} = 0, \tag{2.7}$$

where

$$\begin{aligned}\overline{a_{11}} &= a_{11}\cosh^2\varphi + a_{22}\sinh^2\varphi + 2a_{12}\cosh\varphi\sinh\varphi \\ \overline{a_{12}} &= (a_{11} + a_{22})\cosh\varphi\sinh\varphi + a_{12}(\cosh^2\varphi + \sinh^2\varphi) \\ \overline{a_{22}} &= a_{11}\sinh^2\varphi + a_{22}\cosh^2\varphi + 2a_{12}\cosh\varphi\sinh\varphi \\ \overline{a_{01}} &= a_{01}\cosh\varphi + a_{02}\sinh\varphi \\ \overline{a_{02}} &= a_{01}\sinh\varphi + a_{02}\cosh\varphi \\ \overline{a_{00}} &= a_{00}.\end{aligned} \tag{2.8}$$

This yields I_1, I_2, I_3, I_4, I_5 are invariant with respect to the rotations (2.5). For example,

$$\begin{aligned}\overline{I_3} &= \begin{vmatrix} \overline{a_{00}} & \overline{a_{01}} & \overline{a_{02}} \\ \overline{a_{01}} & \overline{a_{11}} & \overline{a_{12}} \\ \overline{a_{02}} & \overline{a_{12}} & \overline{a_{22}} \end{vmatrix} \\ &= -\overline{a_{00}}\,\overline{a_{12}}^2 + 2\overline{a_{01}}\,\overline{a_{12}}\,\overline{a_{02}} - \overline{a_{11}}\,\overline{a_{02}}^2 - \overline{a_{01}}^2\overline{a_{22}} + \overline{a_{00}}\,\overline{a_{11}}\,\overline{a_{22}} \\ &= -a_{00}{a_{12}}^2\cosh^4\varphi + 2a_{01}a_{12}a_{02}\cosh^4\varphi - a_{11}{a_{02}}^2\cosh^4\varphi - {a_{01}}^2a_{22}\cosh^4\varphi \\ &\quad + a_{00}a_{11}a_{22}\cosh^4\varphi + 2a_{00}a_{12}^2\cosh^2\varphi\sinh^2\varphi - 4a_{01}a_{12}a_{02}\cosh^2\varphi\sinh^2\varphi \\ &\quad + 2a_{11}{a_{02}}^2\cosh^2\varphi\sinh^2\varphi + 2a_{01}^2a_{22}\cosh^2\varphi\sinh^2\varphi \\ &\quad - 2a_{00}a_{11}a_{22}\cosh^2\varphi\sinh^2\varphi - a_{00}{a_{12}}^2\sinh^4\varphi + 2a_{01}a_{12}a_{02}\sinh^4\varphi \\ &\quad - a_{11}{a_{02}}^2\sinh^4\varphi - {a_{01}}^2a_{22}\sinh^4\varphi + a_{00}a_{11}a_{22}\sinh^4\varphi \\ &= -a_{00}a_{12}^2 + 2a_{01}a_{12}a_{02} - a_{11}a_{02}^2 - a_{01}^2a_{22} + a_{00}a_{11}a_{22} = I_3.\end{aligned}$$

The same can be proved for I_1, I_2, I_4 and I_5, as well.
Taking translations from (1.6) given by

$$\begin{aligned} x &= \overline{x} + x_0 \\ y &= \overline{y} + y_0 \end{aligned} \tag{2.9}$$

the equation (2.1) turns into (2.7) where

$$\begin{aligned} \overline{a_{11}} &= a_{11} \\ \overline{a_{12}} &= a_{12} \\ \overline{a_{22}} &= a_{22} \\ \overline{a_{01}} &= a_{11}x_0 + a_{12}y_0 + a_{01} \\ \overline{a_{02}} &= a_{12}x_0 + a_{22}y_0 + a_{02} \\ \overline{a_{00}} &= a_{11}{x_0}^2 + 2a_{12}x_0y_0 + a_{22}{y_0}^2 + 2a_{01}x_0 + 2a_{02}y_0 + a_{00}. \end{aligned} \tag{2.10}$$

It is easy to show that I_1, I_2, I_3 are invariants under (2.9). One concludes that I_1, I_2, and I_3 are invariants of conics with respect to the group of motions B_3.
The observation given above regarding the invariants I_1, I_2, I_3, I_4, I_5 can be found in [5]. In addition, Reveruk [5] defines conics with respect to their relationship to the absolute figure, relying on the fact that the focus points (foci) are the points of intersection of the isotropic tangents at the conic. The paper, however, showed to be incomplete (see Tables 1-5, where the conics added from us are written in italic) and not possible to cite in further studies of the properties of conics in the pseudo-Euclidean plane.

3. Diagonalization of the quadratic form

In the chapters that follows, based on the methods of linear algebra, we give a complete classification of conic sections, divide them into families and define types, giving both of them geometrical meaning.
The quadratic form within the equation (2.1) is a second degree homogenous polynomial

$$Q(x,y) := a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{3.1}$$

The question is whether and when it is possible to obtain $\overline{a_{12}} = 0$ using transformations of the group B_3. It can be seen from (2.8) that $\overline{a_{12}} = 0$ implies

$$\begin{aligned} &(a_{11} + a_{22})\cosh\varphi\sinh\varphi + a_{12}(\cosh^2\varphi + \sinh^2\varphi) = 0, \\ &\frac{1}{2}(a_{11} + a_{22})\sinh 2\varphi + a_{12}\cosh 2\varphi = 0, \\ &\text{i.e.} \qquad \tanh 2\varphi = -\frac{2a_{12}}{a_{11} + a_{22}}, \quad a_{11} + a_{22} \neq 0. \end{aligned} \tag{3.2}$$

From (3.2) we read:

(i) $-1 < \tanh 2\varphi < 1, -\infty < 2\varphi < \infty$ is fulfilled when $|a_{11}+a_{22}| > 2|a_{12}|$;
(ii) $\tanh 2\varphi = 1$, $2\varphi = \infty$ is fulfilled when $a_{11}+a_{22} = -2a_{12}$,
$\tanh 2\varphi = -1$, $2\varphi = -\infty$ is fulfilled when $a_{11}+a_{22} = 2a_{12}$,
(iii) $\tanh 2\varphi < -1$ and $\tanh 2\varphi > 1$ is impossible. This follows when $|a_{11}+a_{22}| < 2|a_{12}|$.

So, under the condition (i) one obtains

$$Q(\overline{x},\overline{y}) = \left[\begin{array}{cc} \overline{x} & \overline{y} \end{array}\right] \left[\begin{array}{cc} \overline{a_{11}} & 0 \\ 0 & \overline{a_{22}} \end{array}\right] \left[\begin{array}{c} \overline{x} \\ \overline{y} \end{array}\right], \tag{3.3}$$

where $\overline{a_{11}} - \overline{a_{22}} = I_1$, $\overline{a_{11}} \cdot \overline{a_{22}} = I_2$.

Definition 3.1. *Let* $A := \left[\begin{array}{cc} a_{11} & a_{12} \\ a_{12} & a_{22} \end{array}\right]$ *be any real symmetric matrix. Then the values* λ_1, λ_2,

$$\lambda_1 - \lambda_2 = a_{11} - a_{22}, \quad \lambda_1 \cdot \lambda_2 = a_{11}a_{22} - {a_{12}}^2$$

are called pseudo-Euclidean values of the matrix A.

Definition 3.2. *We say that the real symmetric* 2×2 *matrix* A *allows the pseudo-Euclidean diagonalization if there is a matrix*

$$R = \left[\begin{array}{cc} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{array}\right]$$

such that RAR *is a diagonal matrix, i.e.*

$$RAR = \left[\begin{array}{cc} \lambda_1 & 0 \\ 0 & \lambda_2 \end{array}\right],$$

where λ_1, λ_2 *are the pseudo-Euclidean values of the matrix* A. *We say that the matrix* R *diagonalizes* A *in a pseudo-Euclidean way.*

From the results obtained in Section 2 related to the invariants (2.4) it follows:

Proposition 3.3. *The difference* $\lambda_1 - \lambda_2$ *of the pseudo-Euclidean values as well as their product* $\lambda_1 \cdot \lambda_2$ *are invariant with respect to the group* B_3 *of motions in the pseudo-Euclidean plane.*

Out of (3.1), (i), (ii), (iii), and (3.3), Propositions 3.4 and 3.5 are valid:

Proposition 3.4. *Let* A *be a matrix from Definition 3.1. Then there is a matrix* $R = \left[\begin{array}{cc} \cosh\varphi & \sinh\varphi \\ \sinh\varphi & \cosh\varphi \end{array}\right]$ *with* $\tanh 2\varphi = -\dfrac{2a_{12}}{a_{11}+a_{22}}$ *which under the conditions* $a_{11}+a_{22} \neq 0$ *and* $|a_{11}+a_{22}| > 2|a_{12}|$ *diagonalizes* A *in the pseudo-Euclidean way.*

Proposition 3.5. *It is always possible to reduce the quadratic form (3.1) by a pseudo-Euclidean motion to the canonical form (3.3) except for: (ii) and (iii).*

Next we divide the conics in the pseudo-Euclidean plane in four families according to their geometrical properties. First, we define the families:

Definition 3.6. *1st family conics in the pseudo-Euclidean plane are conics with no real isotropic directions while their isotropic points are spacelike or timelike.*
2nd family conics are conics having one real isotropic direction.
3rd family conics are conics with two real isotropic points, one being spacelike and the other being timelike.
4th family conics are ones incident with both absolute points.

Taking in consideration the range of angles in the pseudo-Euclidean plane [4], [6], the significance of the conditions (i), (ii), (iii) as well as that of the equality $a_{11} + a_{22} = 0$ is given in the proposition that follows.

Proposition 3.7. *Any conic that satisfies the condition (i) within its equation (2.1) represents a conic with no real isotropic directions while their isotropic points are spacelike or timelike. When one of the conditions (ii) is fulfilled, (2.1) represents a conic having one real isotropic direction. For (iii) (2.1) represents a conic with two real isotropic points, one being spacelike and the other being timelike. Finally, when $a_{11} + a_{22} = 0$ is fulfilled, (2.1) is a conic incident with both absolute points.*

Let us now discuss the geometrical meaning of the invariants as follows:

1. $I_3 \neq 0$ represents a proper conic while $I_3 = 0$ represents a degenerate conic.

2. $I_2 \neq 0$ represents a conic with center and $I_2 = 0$ a conic without center. As it is well known 1. and 2. are affine conditions for conics.

3. Conics belonging to the 1st family with $I_1 \neq 0$ are conics without real isotropic directions while those with $I_1 = 0$ have imaginary isotropic directions.

4. Conics belonging to the 2nd family with $I_1 \neq 0$ are conics with one isotropic direction. If $I_1 = 0$ is valid the considered conic is a conic with double isotropic direction.

5. Conics belonging to the 4th family with $I_1 \neq 0$ are conics with two isotropic directions. If $I_1 = 0$ is valid the considered conic is a conic consisting of an absolute line and one more line.

Furthermore, for conics with isotropic points of the same type we have introduced the following notations:

- *first type conic* is a conic with spacelike isotropic points;
- *second type conic* is a conic with timelike isotropic points.

4. Pseudo-Euclidean classification of conics

In anticipation of classifying conics based on their isometric invariants, we give the pseudo-Euclidean classification based on families and types of conics in the projective model, in order to point out the need for our investigation. The projective representations of conics from [5] are given in black, while the ones we have completed Reveruk's classification with are drawn in gray (see Figures 1, 2, and 3).

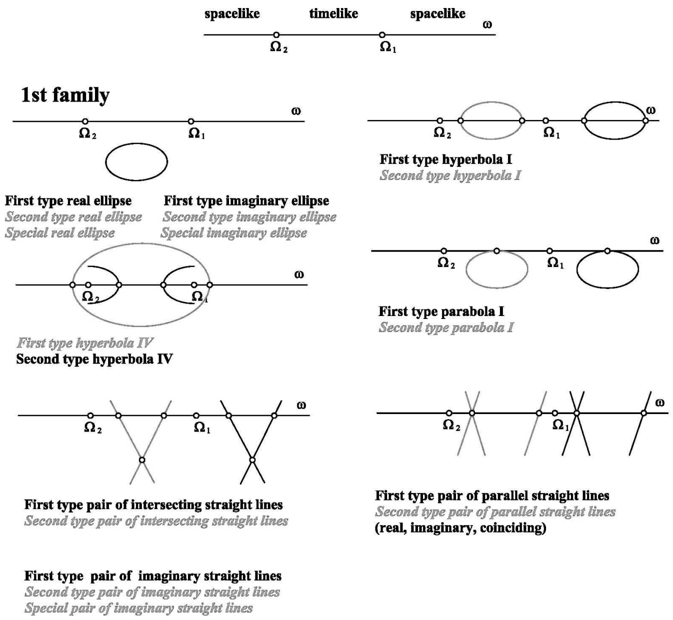

Figure 1. 1st family conics

4.1. *1st family conics.* Let's assume that it is possible to reduce the quadratic form in the conic equation (2.1) to the canonical form (3.3). This implies according to Propositions 3.4 and 3.5 that $|a_{11} + a_{22}| > 2|a_{12}|$, and that it is possible to write down the conic equation (2.1) in the form

$$F(\overline{x}, \overline{y}) \equiv \overline{a_{11}x}^2 + \overline{a_{22}y}^2 + 2\overline{a_{01}x} + 2\overline{a_{02}y} + \overline{a_{00}} = 0. \tag{4.1}$$

Let's consider conics with center ($I_2 \neq 0$).
After a translation of the coordinate system in $\overline{x}$-and $\overline{y}$-direction we have

$$F(\overline{x}, \overline{y}) \equiv \overline{a_{11}x}^2 + \overline{a_{22}y}^2 + \overline{\overline{a_{00}}} = 0. \tag{4.2}$$

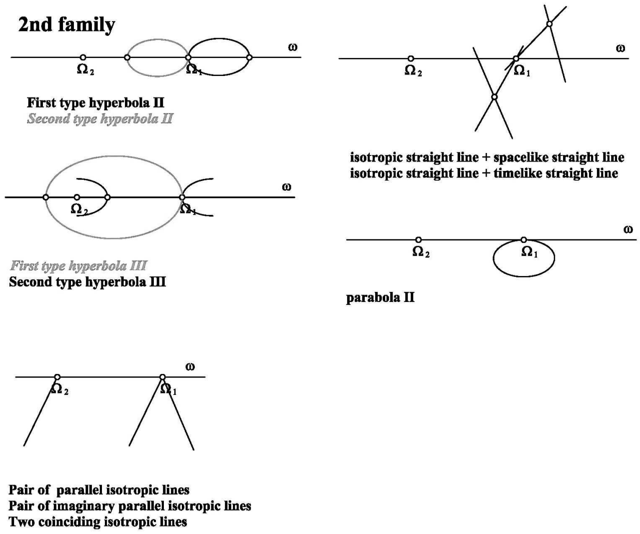

FIGURE 2. 2nd family conics

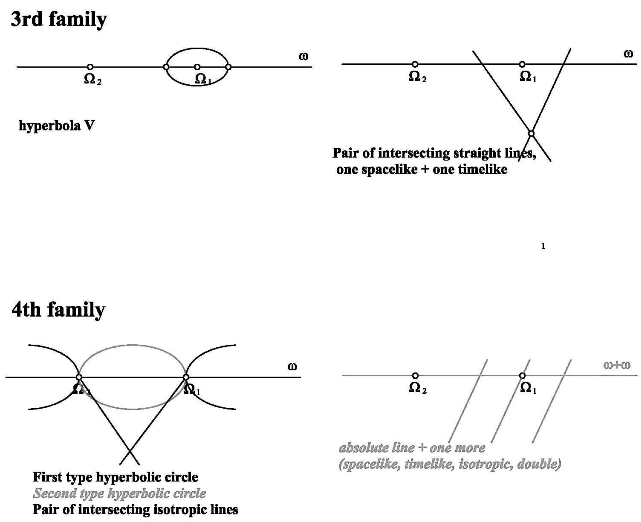

FIGURE 3. 3rd and 4th family conics

One computes

$$I_1 = \overline{a_{11}} - \overline{a_{22}}, \quad I_2 = \overline{a_{11}} \cdot \overline{a_{22}}, \quad I_3 = \overline{a_{11}} \cdot \overline{a_{22}} \cdot \overline{\overline{a_{00}}} \Rightarrow \overline{\overline{a_{00}}} = \frac{I_3}{I_2}. \tag{4.3}$$

Let's introduce:

$$a := \sqrt{\left|\frac{\overline{\overline{a_{00}}}}{\overline{a_{11}}}\right|}, \qquad b := \sqrt{\left|\frac{\overline{\overline{a_{00}}}}{\overline{a_{22}}}\right|}. \tag{4.4}$$

The values a and b shall be called *pseudo-Euclidean semiaxes*.
In Table 1 we give the possibilities for the conic sections with equation (4.2) depending on the signs of the coefficients. The italic cases are those we added to Reveruk's classification.

Table 1

$\overline{a_{11} a_{22} a_{00}}$	canonical form	conic
$+++$ $---$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$	first type imaginary ellipse ($a > b$) *second type imaginary ellipse* ($a < b$) *special imaginary ellipse* ($a = b$)
$++-$ $--+$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	first type real ellipse ($a > b$) *second type real ellipse* ($a < b$) *special real ellipse* ($a = b$)
$+--$ $-++$	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	first type hyperbola I ($a > b$) second type hyperbola IV ($a < b$)
$-+-$ $+-+$	$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	*second type hyperbola I* ($a < b$) *first type hyperbola IV* ($a > b$)
$++0$ $--0$	$a_{11}x^2 + a_{22}y^2 = 0$	first type pair of imaginary straight lines ($\|a_{11}\| < \|a_{22}\|$) *second type pair of imaginary straight lines* ($\|a_{11}\| > \|a_{22}\|$) *special pair of imaginary straight lines* ($\|a_{11}\| = \|a_{22}\|$)
$+-0$ $-+0$	$a_{11}x^2 + a_{22}y^2 = 0$	first type pair of intersecting straight lines ($\|a_{11}\| < \|a_{22}\|$) *second type pair of intersecting straight lines* ($\|a_{11}\| > \|a_{22}\|$)

The question that naturally arises is why the curves with the canonical equations given in Table 1 in the pseudo - Euclidean plane are called as it is given in the same table and what is the connection between the signs of the coefficients and the conditions based on the invariants (2.4). We answer by demonstrating on the case of hyperbola I the procedure conducted for all the curves from this family.

4.1.1. *First and second type hyperbola I.*

Definition 4.1. *The locus of points in the pseudo-Euclidean plane for which the difference of their distances from two different fixed points (foci) in this plane is constant will be called hyperbola I.*

We distinguish two cases: first and second type hyperbola I.
Let $F_1 = (c, 0), F_2 = (-c, 0), F_3 = (0, c), F_4 = (0, -c), c \neq 0$ be the given points. For any point $M = (x, y)$ for which $\overrightarrow{F_1M}$ and $\overrightarrow{F_2M}$ are spacelike vectors, according to definition 4.1

$$|F_1M| - |F_2M| = 2a, \quad a \in \mathbb{R}, a \neq 0, \tag{4.5}$$

$$\text{i.e.} \quad \sqrt{(x-c)^2 - y^2} - \sqrt{(x+c)^2 - y^2} = 2a. \tag{4.6}$$

After computing (4.6) we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad b^2 = a^2 - c^2, a > b, a > c. \tag{4.7}$$

Out of which, according to the affine classification of the second order curves, because of $a_{11}a_{22} - a_{12}^2 = -a^2b^2 < 0, a^4b^4 \neq 0$, we conclude that the considered conic is a hyperbola. The symbol I denotes that the foci are real points, i. e., the points $(0 : 1 : 1)$ and $(0 : 1 : -1)$ are lying outside the hyperbola. It is easy to check that the isotropic points are spacelike points, being property of a first type conic and achieved when $a > b$.
Equation (4.7) can be obtained in much the same way carrying out a calculation for the points F_3 and F_4, $\overrightarrow{F_3M}$ and $\overrightarrow{F_4M}$ being again spacelike vectors, i. e. $|F_3M| - |F_4M| = 2b, \quad b \in \mathbb{R}, b \neq 0$.
It is easy to show the opposite direction of the above statement as well, i.e., for any point $M(x, y)$ whose coordinates fulfill the equation (4.7) the equality $|F_1M| - |F_2M| = 2a$ is valid, i.e. the point M is incident to the hyperbola I.

Let's presume next $\overrightarrow{F_1M}$ and $\overrightarrow{F_2M}$ are timelike vectors,

$$|F_1M| - |F_2M| = 2ai, \quad a \in \mathbb{R}, a \neq 0, \tag{4.8}$$

$$\text{i.e.} \quad \sqrt{(x-c)^2 - y^2} - \sqrt{(x+c)^2 - y^2} = 2ai \tag{4.9}$$

From (4.9) we get

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a^2 + c^2 = b^2, b > a, \tag{4.10}$$

being a hyperbola I, of the second type.

The connection between the signs of the coefficients in the canonical forms of the discussed conics and the (meeting) conditions based on the invariants I_1, I_2, I_3 is given next:
For first type hyperbola I the signs of the coefficients $\overline{a_{11}}$, $\overline{a_{22}}$, $\overline{\overline{a_{00}}}$ are $+$, $-$, $-$ or $-$, $+$, $+$, respectively, and $a > b$. This results in

$$I_2 < 0 \quad \wedge \quad ((I_1 > 0 \wedge I_3 > 0) \vee (I_1 < 0 \wedge I_3 < 0)) \quad \wedge \quad |\overline{a_{11}}| < |\overline{a_{22}}|$$

$$\text{i.e.} \quad I_2 < 0 \wedge I_1I_3 > 0 \wedge |\overline{a_{11}}| < |\overline{a_{22}}|.$$

The opposite direction holds as well.
For second type hyperbola I the signs for $\overline{a_{11}}$, $\overline{a_{22}}$, $\overline{\overline{a_{00}}}$ are $-$, $+$, $-$ or $+$, $-$, $+$, and $a < b$. This results in

$$I_2 < 0 \quad \wedge \quad ((I_1 > 0 \wedge I_3 < 0) \vee (I_1 < 0 \wedge I_3 > 0)) \quad \wedge \quad |\overline{a_{11}}| > |\overline{a_{22}}|$$

$$\text{i.e.} \quad I_2 < 0 \wedge I_1 I_3 < 0 \wedge |\overline{a_{11}}| > |\overline{a_{22}}|.$$

Conics of the 1st family with $I_2 = 0$ may be considered in a similar way. They are included in Table 5. If it is deemed necessary, those cases can be discuss as well.
We conclude subsection 4.1 with the following proposition:

PROPOSITION 4.2. *In the pseudo-Euclidean plane there are 23 (12 proper + 11 degenerate) different types of conic sections of the 1st family to distinguish with respect to the group B_3 of motions (see Tables 5 and 6).*

4.2. *2nd family conics.* Let's assume furtheron that it is not possible to diagonalize the quadratic form in the conic equation. Then according to Proposition 3.5 we have to distinguish (ii) $|a_{11}+a_{22}| = 2|a_{12}|$ and (iii) $|a_{11}+a_{22}| < 2|a_{12}|$, that is 2nd and 3rd family conics.

The conditions $|a_{11} + a_{22}| = 2|a_{12}|$ and $a_{12} \neq 0$ imply $a_{11} + a_{22} \neq 0$. The conic equation is of the initial form (2.1).

Let's consider conics with center ($I_2 \neq 0$).
After a translation of a coordinate system in $\overline{x}$- and $\overline{y}$- direction we have

$$F(\overline{x}, \overline{y}) \equiv a_{11}\overline{x}^2 + 2a_{12}\overline{xy} + a_{22}\overline{y}^2 + \overline{a_{00}} = 0. \tag{4.11}$$

One computes

$$\overline{a_{00}} = \frac{I_3}{I_2}.$$

The possibilities for the conic sections with equation (4.11) are given in Table 2.

We point out that Reveruk makes difference by name but not by the invariants between the degenerate conics from Table 2, as well as between hyperbolas II and III from the same table.

4.2.1. *First and second type hyperbola II.* Let us next turn our attention to, for example, hyperbolas II. We will demonstrate how their names has been derived from their canonical equations. In addition we provide a link between the signs of the coefficients within their canonical equations and the conditions based on the invariants (2.4) for a conic to represent first, i. e. second type hyperbola II.

TABLE 2

$a_{11}a_{12}a_{22}\overline{a_{00}}$	canonical form	conic
$++--$ $--++$	$x^2(a^2-c^2)+2xyc^2-y^2(a^2+c^2)-a^4=0$	first type hyperbola II
$+---$ $-+++$	$x^2(a^2-c^2)-2xyc^2-y^2(a^2+c^2)-a^4=0$	
$+--+$ $-++-$	$x^2(a^2+c^2)-2xyc^2-y^2(a^2-c^2)+a^4=0$	*second type hyperbola II*
$++-+$ $--+-$	$x^2(a^2+c^2)+2xyc^2-y^2(a^2-c^2)+a^4=0$	
$++-+$ $--+-$	$x^2(a^2-c^2)+2xyc^2-y^2(a^2+c^2)+a^4=0$	*first type hyperbola III*
$+--+$ $-++-$	$x^2(a^2-c^2)-2xyc^2-y^2(a^2+c^2)+a^4=0$	
$+---$ $-+++$	$x^2(a^2+c^2)-2xyc^2-y^2(a^2-c^2)-a^4=0$	second type hyperbola III
$++--$ $--++$	$x^2(a^2+c^2)+2xyc^2-y^2(a^2-c^2)-a^4=0$	
$++-0$ $--+0$	$x^2(a^2-c^2)+2xyc^2-y^2(a^2+c^2)=0$	pair of lines, one isotropic and one spacelike
$+--0$ $-++0$	$x^2(a^2+c^2)-2xyc^2-y^2(a^2-c^2)=0$	pair of lines, one isotropic and one timelike

DEFINITION 4.3. *The locus of points in the pseudo-Euclidean plane for which the difference from two fixed points (foci) lying on one of the isotropic lines is constant is called hyperbola II.*

We distinguish 2 cases: first and second type hyperbola II.
Let $F_1=(c,c), F_2=(-c,-c)$ be the given points. For any point $M=(x,y)$ for which $\overrightarrow{F_1M}$ and $\overrightarrow{F_2M}$ are spacelike vectors,

$$|F_1M|-|F_2M|=2a, \quad a\in\mathbb{R}, a\neq 0, \tag{4.12}$$

$$\text{i.e.} \quad \sqrt{(x-c)^2-(y-c)^2}-\sqrt{(x+c)^2-(y+c)^2}=2a. \tag{4.13}$$

After computing (4.13) we get

$$x^2(a^2-c^2)+2xyc^2-y^2(a^2+c^2)-a^4=0, \quad a>c. \tag{4.14}$$

Out of (4.14), according to the affine classification of the second order curves $a_{11}a_{22}-a_{12}{}^2=-(a^2-c^2)(a^2+c^2)-c^4=-a^4<0$, $(a_{11}a_{22}-a_{12}{}^2)a_{00}=a^8\neq 0$; it is a matter of a hyperbola [1]. Further, $\Omega_1=(0:1:1)$ is lying on while $\Omega_2=(0:1:-1)$ is lying outside the hyperbola, being properties of II. As the second isotropic point of the curve belongs to the spacelike area, it is a

matter of a first type curve.
Carrying out a calculation for the points $F_1 = (-c, c), F_2 = (c, -c)$, lying on the isotropic line $x + y = 0$, one gets

$$x^2(a^2 - c^2) - 2xyc^2 - y^2(a^2 + c^2) - a^4 = 0, \quad a > c, \tag{4.15}$$

being again first type hyperbola II.

Presuming that for $F_1 = (c, c)$, $F_2 = (-c, -c)$ and $M = (x, y)$ $\overrightarrow{F_1M}$ and $\overrightarrow{F_2M}$ are timelike vectors, we start from

$$|F_1M| - |F_2M| = 2ai \tag{4.16}$$

which leads to

$$x^2(a^2 + c^2) - 2c^2xy - y^2(a^2 - c^2) + a^4 = 0, \quad a > c. \tag{4.17}$$

The equation (4.17) represents a *second type hyperbola II.*
Repeating the calculation for $F_1 = (-c, c)$, $F_2(c, -c)$ we get

$$x^2(a^2 + c^2) + 2c^2xy - y^2(a^2 - c^2) + a^4 = 0, \tag{4.18}$$

being again a *second type hyperbola II.*
Same as in the case of hyperbolas I, the opposite direction holds as well.
If we discuss the signs of the coefficients for first type hyperbola II we get: there are two possibilities for the signs of the coefficients a_{11}, a_{12}, a_{22}, $\overline{a_{00}}$

$$\left(\begin{array}{c} ++-- \\ --++ \end{array} \quad \text{or} \quad \begin{array}{c} +--- \\ -+++ \end{array} \right) \quad \text{and} \quad |a_{11}| < |a_{22}|.$$

Both combinations of signs yield

$$I_2 < 0 \quad \wedge \quad ((I_1 > 0 \wedge I_3 > 0) \vee (I_1 < 0 \wedge I_3 < 0))$$

which result in

$$I_2 < 0, \quad I_1 I_3 > 0, \quad |a_{11}| < |a_{22}|.$$

For second type hyperbola II we start from

$$\left(\begin{array}{c} +--+ \\ -++- \end{array} \quad \text{or} \quad \begin{array}{c} ++-+ \\ --+- \end{array} \right) \quad \text{and} \quad |a_{11}| > |a_{22}|,$$

which leads to

$$I_2 < 0, \quad I_1 I_3 > 0, \quad |a_{11}| > |a_{22}|.$$

The opposite direction is valid in both cases. In a very similar way conics of the 2nd family with $I_2 = 0$ are considered. We conclude the analysis within this family with

PROPOSITION 4.4. *In the pseudo-Euclidean plane there are 10 (5 proper + 5 degenerate) different types of conic sections of the 2nd family to distinguish with respect to the group B_3 of motions (see Tables 5 and 6).*

4.3. *3rd family conics.* Conic sections of the 3rd family are those with two real isotropic points, one being spacelike and the other being timelike. Due to this property conics has to be with a center, of hyperbolic type, which is provided by $I_2 < 0$. Apart from that according to Proposition 3.7 the condition $|a_{11} + a_{22}| < 2|a_{12}|$ has to be fulfilled within equation (2.1).
After a translation

$$\overline{x} = x - \frac{a_{12}a_{02} - a_{22}a_{01}}{a_{11}a_{22} - {a_{12}}^2}, \qquad \overline{y} = y - \frac{a_{12}a_{01} - a_{11}a_{02}}{a_{11}a_{22} - {a_{12}}^2}$$

of the coordinate system in $x-$ and $y-$direction, obtained from $\dfrac{\partial F}{\partial x} = 0$ and $\dfrac{\partial F}{\partial y} = 0$, for the conic equation (2.1) we have

$$F(\overline{x}, \overline{y}) \equiv a_{11}\overline{x}^2 + 2a_{12}\overline{xy} + a_{22}\overline{y}^2 + \overline{a_{00}}, \tag{4.19}$$

where $\overline{a_{00}} = \dfrac{I_3}{I_2}$.
The possibilities for the conic sections with the equation (4.19), according to [5] are given in Table 3.

Table 3

$a_{11}a_{12}a_{22}\overline{a_{00}}$	canonical form	conic
$+-+-$ $-+-+$	$(a^2 - c^2)x^2 - 2(a^2 + c^2)xy + (a^2 - c^2)y^2 - a^4 = 0$	hyperbola V
$+-+0$ $-+-0$	$(a^2 - c^2)x^2 - 2(a^2 + c^2)xy + (a^2 - c^2)y^2 = 0$	pair of lines, one spacelike and one timelike

As we didn't have to interfere in Reveruk's classification concerning conics of the 3rd family, for details on obtaining the canonical forms in Table 3 one can consult [5].
However, we note that in this case the foci of a hyperbola are complex conjugate, and in order to comply the canonical form of a hyperbola with those of the hyperbolas of the 1st and 2nd family for the asymptotes were selected straight lines of the form

$$x(a - c) - y(a + c) = 0, \qquad x(a + c) - y(a - c) = 0. \tag{4.20}$$

For the conditions based on the invariants (2.4) to represent conics of this family see Tables 5 and 6.

Proposition 4.5. *In the pseudo-Euclidean plane there are 2 (1 proper + 1 degenerated) different types of conic sections of the 3rd family to distinguish with respect to the group B_3 of motions (see Table 5).*

4.4. *4th family conics.* For a complete classification of conic sections in the pseudo-Euclidean plane it is necessary to take into account the conic sections incident with both absolute points. According to Definition 3.6, such curves belong to the 4th family. On the other hand, according to Proposition 3.7, the condition $a_{11}+a_{22}=0$ has to be fulfilled within the conic equation (2.1). The conic section equation in homogeneous coordinates $(x_0 : x_1 : x_2)$ is of the form

(4.21)
$$F(x_0,x_1,x_2) \equiv a_{11}{x_1}^2+2a_{12}x_1x_2+a_{22}{x_2}^2+2a_{01}x_1x_0+2a_{02}x_2x_0+a_{00}{x_0}^2=0.$$

From the requirement that the conic with equation (4.21) is incident with the absolute points $\Omega_1=(0:1:1)$ and $\Omega_2=(0:1:-1)$ is easy to show that, apart from $a_{11}+a_{22}=0, a_{12}=0$ holds as well. The equation (2.1) now turns into

$$F(x,y) \equiv a_{11}x_1^2+a_{22}y^2+2a_{01}x+2a_{02}y+a_{00}=0 \tag{4.22}$$

Presuming that $I_2 \neq 0$, both linear terms can be eliminated by a translation in direction of the $x-$ and $y-$ axes, which gives us

$$F(\overline{x},\overline{y}) \equiv a_{11}\overline{x}^2+a_{22}\overline{y}^2+\overline{a_{00}}=0. \tag{4.23}$$

One computes

$$I_1=a_{11}-a_{22}, \quad I_2=a_{11}a_{22}, \quad I_3=a_{11}a_{22}\overline{a_{00}} \Rightarrow \overline{a_{00}}=\frac{I_3}{I_2}. \tag{4.24}$$

The possibilities for the conic sections with equation (4.23) are given in Table 4.

TABLE 4

$a_{11}a_{22}\overline{a_{00}}$	canonical form	conic
$+-+$ $-+-$	$x^2-y^2+a^2=0$	*second type hyperbolic circle*
$+--$ $-++$	$x^2-y^2-a^2=0$	first type hyperbolic circle
$+-0$ $-+0$	$x^2-y^2=0$	pair of isotropic lines

Links among the canonical equations and the corresponding names of conics from Table 4 are obvious. For the conditions based on the invariants (2.4) to represent those conics see Tables 5 and 6.

We continue our study by analyzing conics consisting of two straight lines including the absolute line ω. This is achieved when $I_2=0$. Indeed, $a_{11}+$

Table 5

<table>
<tr><th>Family</th><th colspan="4">Conditions on invariants</th><th>Conic</th></tr>
<tr><td rowspan="23">$1^{\mathrm{st}}: \vert a_{11}+a_{22}\vert > 2\vert a_{12}\vert$</td><td rowspan="12">$I_3 \neq 0$</td><td rowspan="6">$I_2 > 0$</td><td>$I_1 I_3 < 0$</td><td>$a_{11} I_1 < 0$</td><td>first type imaginary ellipse</td></tr>
<tr><td>$I_1 I_3 > 0$</td><td>$a_{11} I_1 > 0$</td><td>second type imaginary ellipse</td></tr>
<tr><td>$I_1 = 0$</td><td>$a_{11} I_3 > 0$</td><td>special imaginary ellipse</td></tr>
<tr><td>$I_1 I_3 > 0$</td><td>$a_{11} I_1 < 0$</td><td>first type real ellipse</td></tr>
<tr><td>$I_1 I_3 < 0$</td><td>$a_{11} I_1 > 0$</td><td>second type real ellipse</td></tr>
<tr><td>$I_1 = 0$</td><td>$a_{11} I_3 < 0$</td><td>special real ellipse</td></tr>
<tr><td rowspan="4">$I_2 < 0$</td><td>$I_1 I_3 > 0$</td><td>$\vert\overline{a_{11}}\vert < \vert\overline{a_{22}}\vert$</td><td>first type hyperbola I</td></tr>
<tr><td>$I_1 I_3 < 0$</td><td>$\vert\overline{a_{11}}\vert > \vert\overline{a_{22}}\vert$</td><td>second type hyperbola I</td></tr>
<tr><td>$I_1 I_3 < 0$</td><td>$\vert\overline{a_{11}}\vert < \vert\overline{a_{22}}\vert$</td><td>first type hyperbola IV</td></tr>
<tr><td>$I_1 I_3 > 0$</td><td>$\vert\overline{a_{11}}\vert > \vert\overline{a_{22}}\vert$</td><td>second type hyperbola IV</td></tr>
<tr><td rowspan="2">$I_2 = 0$</td><td>$I_1 I_3 > 0$</td><td colspan="2">first type parabola I</td></tr>
<tr><td>$I_1 I_3 < 0$</td><td colspan="2">second type parabola I</td></tr>
<tr><td rowspan="11">$I_3 = 0$</td><td rowspan="3">$I_2 > 0$</td><td>$a_{11} I_1 < 0$</td><td colspan="2">first type pair of imaginary straight lines</td></tr>
<tr><td>$a_{11} I_1 > 0$</td><td colspan="2">second type pair of imaginary straight lines</td></tr>
<tr><td>$I_1 = 0$</td><td colspan="2">special pair of imaginary straight lines</td></tr>
<tr><td rowspan="2">$I_2 < 0$</td><td>$\vert\overline{a_{11}}\vert < \vert\overline{a_{22}}\vert$</td><td colspan="2">first type pair of intersecting straight lines</td></tr>
<tr><td>$\vert\overline{a_{11}}\vert > \vert\overline{a_{22}}\vert$</td><td colspan="2">second type pair of intersecting straight lines</td></tr>
<tr><td rowspan="6">$I_2 = 0$</td><td>$I_4 > 0$</td><td rowspan="3">$\vert a_{11}\vert < \vert a_{22}\vert$</td><td>first type pair of parallel lines</td></tr>
<tr><td>$I_4 < 0$</td><td>first type pair of imaginary parallel lines</td></tr>
<tr><td>$I_4 = 0$</td><td>first type two coinciding parallel lines</td></tr>
<tr><td>$I_4 < 0$</td><td rowspan="3">$\vert a_{11}\vert > \vert a_{22}\vert$</td><td>second type pair of parallel lines</td></tr>
<tr><td>$I_4 > 0$</td><td>second type pair of imaginary parallel lines</td></tr>
<tr><td>$I_4 = 0$</td><td>second type two coinciding parallel lines</td></tr>
</table>

$a_{22} = 0$ and $I_2 = a_{11} \cdot a_{22} = 0$ entails $a_{11} = a_{22} = 0$.
The conic section equation (4.21) turns into
(4.25)
$F(x_0, x_1, x_2) \equiv 2a_{01}x_1x_0 + 2a_{02}x_2x_0 + a_{00}{x_0}^2 = x_0(2a_{01}x_1 + 2a_{02}x_2 + a_{00}x_0) = 0,$

out of which we read the invariants (2.4):

$$I_1 = 0, \quad I_2 = 0, \quad I_3 = 0, \quad I_4 = -{a_{01}}^2 + {a_{02}}^2, \quad I_5 = a_{00}.$$

According to (1.2) and (1.3) the possibilities for the other line, besides ω ($x_0 = 0$), are the following:

- $I_4 > 0$ yields the second line in (4.25) is spacelike;
- $I_4 < 0$ yields it is a timelike straight line;
- $I_4 = 0, \quad a_{01} \neq 0$ reveals the line is isotropic.

To end this subsection, for

- $I_4 = 0, \quad a_{01} = 0, \quad I_5 \neq 0$ (4.25) represents a double absolute line ω;
- $I_4 = 0, \quad a_{01} = 0, \quad I_5 = 0$ yields from (4.25) a zero polynomial.

TABLE 6

Family	Conditions on invariants						Conic
2nd: $\lvert a_{11}+a_{22}\rvert = 2\lvert a_{12}\rvert$	$I_3 \neq 0$	$I_2 < 0$	$I_1 I_3 > 0$	$\lvert a_{11}\rvert < \lvert a_{22}\rvert$			first type hyperbola II
			$I_1 I_3 < 0$	$\lvert a_{11}\rvert > \lvert a_{22}\rvert$			*second type hyperbola II*
			$I_1 I_3 < 0$	$\lvert a_{11}\rvert < \lvert a_{22}\rvert$			*first type hyperbola III*
			$I_1 I_3 > 0$	$\lvert a_{11}\rvert > \lvert a_{22}\rvert$			second type hyperbola III
		$I_2 = 0$	$I_1 = 0$				parabola II
	$I_3 = 0$	$I_2 < 0$	$I_1 \neq 0$	$\lvert a_{11}\rvert < \lvert a_{22}\rvert$			pair of lines, one isotropic + one spacelike
				$\lvert a_{11}\rvert < \lvert a_{22}\rvert$			pair of lines, one isotropic + one timelike
		$I_2 = 0$	$I_1 = 0$	$I_4 = 0$	$I_5 < 0$		pair of parallel isotropic lines
					$I_5 > 0$		*pair of imaginary parallel isotropic lines*
					$I_5 = 0$		two coinciding isotropic lines
3rd: $\lvert a_{11}+a_{22}\rvert < 2\lvert a_{12}\rvert$	$I_3 \neq 0$	$I_2 < 0$					hyperbola V
	$I_3 = 0$	$I_2 < 0$					pair of lines, one spacelike + one timelike
4th: $a_{11}+a_{22} = 0$	$I_3 \neq 0$	$I_2 < 0$	$I_1 I_3 > 0$				first type hyperbolic circle
			$I_1 I_3 < 0$				*second type hyperbolic circle*
	$I_3 = 0$	$I_2 < 0$					pair of intersecting isotropic lines
		$I_2 = 0$	$I_1 = 0$	$I_4 > 0$			*spacelike straight line + ω*
				$I_4 < 0$			*timelike straight line + ω*
				$I_4 = 0$	$a_{01} \neq 0$		*isotropic line + ω*
					$a_{01} = 0$	$I_5 \neq 0$	*double ω*
						$I_5 = 0$	*all points in PE_2*

PROPOSITION 4.6. *In the pseudo-Euclidean plane there are* 8 *(*2 *proper +* 6 *degenerated) different types of conic sections of the* 4*th family to distinguish with respect to the group B_3 of motions (see Tables 5 and 6).*

We conclude with

THEOREM 4.7. *In the pseudo-Euclidean plane there are* 43 *(*20 *proper +* 23 *degenerated) different types of conic sections to distinguish with respect to the group B_3 of motions (see Tables 5 and 6).*

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Klasifikacija konika u $PE_2(\mathbb{R})$

Jelena Beban-Brkić i Marija Šimić Horvath

Sažetak. U članku se prikazuje potpuna klasifikacija konika u $PE_2(\mathbb{R})$. Iako je klasifikacija napravljena i ranije (Reveruk [5]), pokazala se nepotpunom i kao takva neadekvatna u daljnjem proučavanju konika, pramena konika te kvadratnih formi u pseudo-Euklidskim prostorima. Ovaj članak upravo to omogućava. Uvode se pojmovi pseudo-ortogonalne matrice, pseudo-Euklidskih svojstava matrice te dijagonalizacija matrice na pseudo-Euklidski način. Dajući im geometrijsko značenje, konike se dijele po obiteljima i po tipovima. Određuju se i invarijante konika s obzirom na grupu gibanja u $PE_2(\mathbb{R})$ što omogućava da se odredi konika bez prevođenja njezine jednadžbe u kanonski oblik. Dana je i pregledna tablica.

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