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AN IMPROVED METHOD FOR ESTABLISHING FUSS' RELATIONS FOR BICENTRIC n-GONS WHERE $n \geq 4$ IS AN EVEN INTEGER

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Abstract. In [7] we have given one relatively simple and practical method for establishing Fuss' relations for bicentric n-gons where $n \geq 3$ is an odd integer. In the present article we give one relatively simple and practical method for establishing Fuss' relations for bicentric n-gon where $n \geq 4$ is an even integer. In [7] the rotation numbers for bicentric n-gons have the key role, while in the present article tangent lengths of bicentric n-gons have the key role. So in the present article is described an algorithm to obtain Fuss' relation for bicentric n-gons where $n \geq 4$ is an even integer. Several yet unknown Fuss' relations are established.

1. Introduction

One of the very famous results inspiring mathematicians during centuries is Poncelet's celebrated closure theorem [5] which can be stated as follows.

Let C and D be two nested conics such that there is an n-sided polygon inscribed in C and circumscribed around D. Then, for every point x on C there is an n-sided polygon inscribed in C and circumscribed around D such that the point x is one of its vertices. Hence, for every starting point x there is a polygon with the same n-periodicity.

In this article we restrict ourselves to the case when the conics are circles. The pair of conics C and D can be taken to be a pair of circles by a projective transformation. Let us denote by C_1 and C_2 the resulting circles, and let R, r and d be, respectively, the radius of C_1, the radius of C_2 and the distance between centers of C_1 and C_2. The n-periodicity of Poncelet's configuration then implies algebraic relations on R, r and d. For $n \leq 8$, these relations were found by N. Fuss [2, 3] and they referred to as Fuss' relations for all values of n. A general condition on n-periodicity in terms of given conics is the content of the important Cayley's theorem [1] (which implies Fuss' relations; however the deduction of the latter from the former may be a non-trivial task).

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In the present article we show how Fuss' relation for bicentric n-gon where $n \geq 4$ is an even integer one can obtain in a relatively simple way. First about the notation which will be used. Denote by

$$F_n(R, r, d) = 0 \tag{1.1}$$

Fuss' relation for bicentric n-gon. Let (R_0, r_0, d_0) be a (positive) solution of Fuss' relation (1.1) and let the corresponding class of bicentric n-gons be denoted by

$$C_n(R_0, r_0, d_0). \tag{1.2}$$

An important role in the following will be played by the lengths t_M and t_m given by

$$t_M = \sqrt{(R_0 + d_0)^2 - r_0^2}, \quad t_m = \sqrt{(R_0 - d_0)^2 - r_0^2}. \tag{1.3}$$

See Figure 1. The lengths t_M and t_m can be called the *maximal* and *minimal* tangent length of the class $C_n(R_0, r_0, d_0)$.

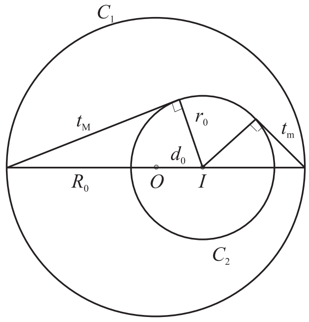

FIGURE 1. $t_m = \sqrt{(R_0 - d_0)^2 - r_0^2}, \quad t_M = \sqrt{(R_0 + d_0)^2 - r_0^2}$.

From Poncelet's closure theorem it is clear that the following holds. If t_1 is any given length such that $t_M \geq t_1 \geq t_m$, where t_M and t_m are given by (1.3), then there exists a bicentric n-gon from the class $C_n(R_0, r_0, d_0)$ such that its *first tangent* has length t_1.

In [6, Lemma 1] it is proved that for calculation of tangent lengths of bicentric polygon the following formula

$$(t_2)_{1,2} = \frac{(R_0^2 - d_0^2)t_1 \pm r_0\sqrt{D_1}}{r_0^2 + t_1^2} \tag{1.4}$$

can be used, where $D_1 = (t_M^2 - t_1^2)(t_1^2 - t_m^2)$. If t_1 is given then its follower is $(t_2)_1$ or $(t_2)_2$.

Concerning the signs + and - in the expression $\pm\sqrt{D_i}$, where $D_i = (t_M^2 - t_i^2)(t_i^2 - t_m^2)$, $i = 1, \ldots, n$, the following is valid

$$t_{i+1} = \frac{(R_0^2 - d_0^2)t_i + r_0\sqrt{D_i}}{r_0^2 + t_i^2} \iff t_{i-1} = \frac{(R_0^2 - d_0^2)t_i - r_0\sqrt{D_i}}{r_0^2 + t_i^2}. \tag{1.5}$$

Using this property the following algorithm can be used. Let t_1 be any given length such that $t_m \leq t_1 \leq t_M$ where t_m and t_M are given by (1.3). Then there exists a bicentric n-gon $A_1 \cdots A_n$ from the class $C_n(R_0, d_0, r_0)$ such that its first tangent has the length t_1. The other tangent lengths can be calculated as follows.

For t_2 one can use $(t_2)_1$ or $(t_2)_2$ given by (1.4). Depending on which of $(t_2)_1$ and $(t_2)_2$ is taken for t_2 we get the ordering of tangent lengths $t_1, t_2, ..., t_n$ clockwise or counterclockwise. Let $(t_2)_1$ be taken for t_2 that is, let

$$t_2 = \frac{(R_0^2 - d_0^2)t_1 + r_0\sqrt{D_1}}{r_0^2 + t_1^2}. \tag{1.6}$$

The following notation will be used

$$t_{i+2}^+ = \frac{(R_0^2 - d_0^2)t_{i+1} + r_0\sqrt{D_{i+1}}}{r_0^2 + t_{i+1}^2}, \quad t_{i+2}^- = \frac{(R_0^2 - d_0^2)t_{i+1} - r_0\sqrt{D_{i+1}}}{r_0^2 + t_{i+1}^2}. \tag{1.7}$$

From this and (1.5) one can conclude that $\left\{t_{i+2}^+, t_{i+2}^-\right\} = \{t_i, t_{i+2}\}$. Thus, we have the equality $t_{i+2}^+ \cdot t_{i+2}^- = t_i \cdot t_{i+2}$, from which it follows $t_{i+2} = \frac{t_{i+2}^+ \cdot t_{i+2}^-}{t_i}$. So we have the following sequence

$$t_1, t_2, \frac{t_3^+ t_3^-}{t_1}, \frac{t_4^+ t_4^-}{t_2}, \tag{1.8}$$

and so on.

Although the closure in Poncelet's closure theorem is a topological property, the formula (1.4), as can be seen, may be very useful in some problems concerning bicentric polygons.

2. An improved method for establishing Fuss' relations for bicentric n-gons where $n \geq 4$ is an even integer.

In the following will be displayed an algorithm for obtaining Fuss' relation for bicentric n-gons where $n \geq 4$ is an even integer.

Let $(R_0, r_0, d_0) \in \mathbb{R}_+^3$ and let $R_0 > r_0 + d_0$. Let t_1 be a length (in fact positive number) such that $t_m \leq t_1 \leq t_M$, where $t_M = \sqrt{(R_0 + d_0)^2 - r_0^2}$, $t_m = \sqrt{(R_0 - d_0)^2 - r_0^2}$. Let t_2 be given by (1.6) and let $t_3, t_4, ...$ be given by

$$t_3 = \frac{(R_0^2 - d_0^2)^2 t_2^2 - r_0^2 D_2}{\left(r_0^2 + t_2^2\right)^2 t_1}, \quad t_4 = \frac{(R_0^2 - d_0^2)^2 t_3^2 - r_0^2 D_3}{\left(r_0^2 + t_3^2\right)^2 t_2},$$

and so on. (The rule given by sequence (1.8) is used.)

Now let $t_1 = t_M$, that is, let each of t_1, t_2, t_3,... be expressed by R_0, r_0, d_0. Finally, let $n \geq 4$ be an even integer such that $t_{1+\frac{n}{2}} = t_m$. Then from $t_{1+\frac{n}{2}} = t_m$ it follows $F_n(R_0, r_0, d_0) = 0$, where $F_n(R, r, d) = 0$ is Fuss' relation for bicentric n-gons. Thus the relation $t_{1+\frac{n}{2}} = t_m$ has the key role in the article since this relation becomes Fuss' relation $F_n(R, r, d) = 0$ by expression of $F_n(R, r, d) \cdot t_M$.

Using computer algebra we have the following results.

1. Let $n = 4$. Then from $t_3 - t_m = 0$, it follows

$$4d_0r_0(d_0 - r_0 + R_0)(d_0 + r_0 + R_0)\left[F_4(R_0, r_0, d_0)\right]^2 = 0, \tag{2.1}$$

   where $F_4(R, r, d) = 0$ is Fuss' relation for bicentric quadrilaterals.
2. Let $n = 6$. Then $4d_0r_0^2R_0\left[F_6(R_0, r_0, d_0)\right]^2 = 0$, where, for brevity, we write only the final result.
3. Let $n = 8$. Then $4d_0r_0\left[F_8(R_0, r_0, d_0)\right]^2 = 0$.
4. Let $n = 10$. Then $4d_0r_0^2R_0\left[F_{10}(R_0, r_0, d_0)\right]^2 = 0$.
5. Let $n = 12$. Then $4d_0R_0\left[F_4(R_0, r_0, d_0)\right]^2\left[F_{12}(R_0, r_0, d_0)\right]^2 = 0$.
6. Let $n = 14$. Then $4d_0r_0^2R_0\left[F_{14}(R_0, r_0, d_0)\right]^2 = 0$.
7. Let $n = 16$. Then $4d_0R_0\left[F_{16}(R_0, r_0, d_0)\right]^2 = 0$.
8. Let $n = 18$. Then $4d_0r_0^2R_0\left[F_6(R_0, r_0, d_0)\right]^2\left[F_{18}(R_0, r_0, d_0)\right]^2 = 0$.
9. Let $n = 20$. Then $4d_0R_0\left[F_4(R_0, r_0, d_0)\right]^2\left[F_{20}(R_0, r_0, d_0)\right]^2 = 0$.

Here let us remark that $t_{1+n/2} = t_m$ is indeed one equality if we start from $t_1 = t_M$ and the triple (R_0, r_0, d_0) has n-closure, that is, $t_{1+n} = t_1$. This equality plays the key role in the article since $t_{1+n/2} - t_m = 0 \iff F_n(R_0, r_0, d_0) = 0$.

Also let us remark that each of the relations 1. – 9. is obtained from the equality $(t_{1+n/2})^2 - (t_m)^2 = 0$ using its factorization.

Since $t_{1+n/2}$ can be written as $F_n(R_0, r_0, d_0) \cdot t_M$, where $F_n(R_0, r_0, d_0)$ is the corresponding expression in R_0, r_0, d_0, we can also write Fuss relation $F_n(R, r, d) = 0$ as $F_n(R, r, d) \cdot t_M = t_m$.

The reason why we can get for some n two or more Fuss relations from $t_{1+\frac{n}{2}} = t_m$ lies in the fact that there can exist an even integer $k \geq 4$ and an integer $j \geq 1$ such that $1 + \frac{n}{2} = 1 + \frac{k}{2} + jk$.

The following examples and their consideration may be useful for better comprehension of the algorithm.

In order that the consideration be convenient for reading we here also list some notation which will be often used. So by

$$t_M = \sqrt{(R_0 + d_0)^2 - r_0^2}, \quad t_m = \sqrt{(R_0 - d_0)^2 - r_0^2} \tag{2.2}$$

we denote, respectively, maximal and minimal tangent length of the class $C_n(R_0, r_0, d_0)$. By

$$F_n(R, r, d) = 0 \tag{2.3}$$

is denoted Fuss' relation for bicentric n-gons. In writing Fuss' relations in one of the very short form the letters p and q have the following meaning

$$p=\frac{R+d}{r},\quad q=\frac{R-d}{r}. \tag{2.4}$$

For calculation of tangent lengths $t_1,t_2,\dots,t_n$ of bicentric n-gon we can use rule given by (1.8). Using this rule we can write

$$t_m\le t_1\le t_M,$$

$$t_2=\frac{(R_0^2-d_0^2)t_1+r_0\sqrt{D_1}}{r_0^2+t_1^2},\qquad t_3=\frac{(R_0^2-d_0^2)^2t_2^2-r_0^2D_2}{\left(r_0^2+t_2^2\right)^2t_1}, \tag{2.5}$$

$$t_4=\frac{(R_0^2-d_0^2)^2t_3^2-r_0^2D_3}{\left(r_0^2+t_3^2\right)^2t_2},\qquad t_5=\frac{(R_0^2-d_0^2)^2t_4^2-r_0^2D_4}{\left(r_0^2+t_4^2\right)^2t_3}, \tag{2.6}$$

$$t_6=\frac{(R_0^2-d_0^2)^2t_5^2-r_0^2D_5}{\left(r_0^2+t_5^2\right)^2t_4},\qquad t_7=\frac{(R_0^2-d_0^2)^2t_6^2-r_0^2D_6}{\left(r_0^2+t_6^2\right)^2t_5}, \tag{2.7}$$

and so on, where $D_i=\left(t_M^2-t_i^2\right)\left(t_i^2-t_m^2\right),\ i=1,2,\dots,\frac{n}{2}$.

Now let $t_1=t_M$, that is, let each of the tangent lengths $t_1,t_2,t_3,\dots$ be expressed by R_0,r_0,d_0 since t_M is given by (2.2). So we have

($\mathbf{i}_1$)

$$t_1=t_M, \tag{2.8a}$$

$$t_2=\frac{q}{p}t_M, \tag{2.8b}$$

$$t_3=\frac{p^2q^2-p^2+q^2}{p^2q^2+p^2-q^2}t_M, \tag{2.8c}$$

$$t_4=\frac{q\left(p^4q^4+2p^4q^2-3p^4-2p^2q^4+2p^2q^2+q^4\right)}{p\left(p^4q^4-2p^4q^2+p^4+2p^2q^4+2p^2q^2-3q^4\right)}t_M. \tag{2.8d}$$

($\mathbf{i}_2$)

$$t_5=\frac{n_5}{d_5}t_M, \tag{2.8e}$$

where

$$\begin{aligned}n_5&=p^8q^8-4p^8q^6+6p^8q^4-4p^8q^2+p^8+4p^6q^8+4p^6q^6-4p^6q^4\\&\quad-4p^6q^2-10p^4q^8+4p^4q^6+6p^4q^4+4p^2q^8-4p^2q^6+q^8,\\d_5&=p^8q^8+4p^8q^6-10p^8q^4+4p^8q^2+p^8-4p^6q^8+4p^6q^6+4p^6q^4\\&\quad-4p^6q^2+6p^4q^8-4p^4q^6+6p^4q^4-4p^2q^8-4p^2q^6+q^8.\end{aligned}$$

($\mathbf{i}_3$)

$$t_6 = \frac{n_6}{d_6} t_M, \tag{2.8f}$$

where

$$\begin{aligned}
n_6 = q\,\big(&p^{12}q^{12} + 6p^{12}q^{10} - 29p^{12}q^8 + 36p^{12}q^6 - 9p^{12}q^4 \\
&- 10p^{12}q^2 + 5p^{12} - 6p^{10}q^{12} + 14p^{10}q^{10} + 4p^{10}q^8 - 36p^{10}q^6 \\
&+ 34p^{10}q^4 - 10p^{10}q^2 + 15p^8q^{12} - 20p^8q^{10} + 50p^8q^8 - 36p^8q^6 \\
&- 9p^8q^4 - 20p^6q^{12} - 20p^6q^{10} + 4p^6q^8 + 36p^6q^6 + 15p^4q^{12} + 14p^4 \\
&q^{10} - 29p^4q^8 - 6p^2q^{12} + 6p^2q^{10} + q^{12}\big)\,, \\
d_6 = p\,\big(&p^{12}q^{12} - 6p^{12}q^{10} + 15p^{12}q^8 - 20p^{12}q^6 + 15p^{12}q^4 \\
&- 6p^{12}q^2 + p^{12} + 6p^{10}q^{12} + 14p^{10}q^{10} - 20p^{10}q^8 - 20p^{10}q^6 \\
&+ 14p^{10}q^4 + 6p^{10}q^2 - 29p^8q^{12} + 4p^8q^{10} + 50p^8q^8 + 4p^8q^6 \\
&- 29p^8q^4 + 36p^6q^{12} - 36p^6q^{10} - 36p^6q^8 + 36p^6q^6 - 9p^4q^{12} \\
&+34p^4q^{10} - 9p^4q^8 - 10p^2q^{12} - 10p^2q^{10} + 5q^{12}\big)\,.
\end{aligned}$$

($\mathbf{i}_4$)

$$t_7 = \frac{n_7}{d_7} t_M, \tag{2.8g}$$

where

$$\begin{aligned}
n_7 = {}&p^{18}q^{18} - 9p^{18}q^{16} + 36p^{18}q^{14} - 84p^{18}q^{12} + 126p^{18}q^{10} \\
&- 126p^{18}q^8 + 84p^{18}q^6 - 36p^{18}q^4 + 9p^{18}q^2 - p^{18} + 9p^{16}q^{18} \\
&+ 24p^{16}q^{16} - 68p^{16}q^{14} - 88p^{16}q^{12} + 246p^{16}q^{10} - 88p^{16}q^8 \\
&- 68p^{16}q^6 + 24p^{16}q^4 + 9p^{16}q^2 - 60p^{14}q^{18} + 36p^{14}q^{16} \\
&+ 244p^{14}q^{14} - 172p^{14}q^{12} - 372p^{14}q^{10} + 428p^{14}q^8 - 68p^{14}q^6 \\
&- 36p^{14}q^4 + 116p^{12}q^{18} - 216p^{12}q^{16} - 212p^{12}q^{14} + 688p^{12}q^{12} \\
&- 372p^{12}q^{10} - 88p^{12}q^8 + 84p^{12}q^6 - 66p^{10}q^{18} + 330p^{10}q^{16} \\
&- 212p^{10}q^{14} - 172p^{10}q^{12} + 246p^{10}q^{10} - 126p^{10}q^8 - 66p^8q^{18} \\
&- 216p^8q^{16} + 244p^8q^{14} - 88p^8q^{12} + 126p^8q^{10} + 116p^6q^{18} \\
&+ 36p^6q^{16} - 68p^6q^{14} - 84p^6q^{12} - 60p^4q^{18} + 24p^4q^{16} + 36p^4q^{14} \\
&+ 9p^2q^{18} - 9p^2q^{16} + q^{18},
\end{aligned}$$

$$\begin{aligned}
d_7 = {} & p^{18}q^{18} + 9p^{18}q^{16} - 60p^{18}q^{14} + 116p^{18}q^{12} - 66p^{18}q^{10} \\
& - 66p^{18}q^{8} + 116p^{18}q^{6} - 60p^{18}q^{4} + 9p^{18}q^{2} + p^{18} - 9p^{16}q^{18} \\
& + 24p^{16}q^{16} + 36p^{16}q^{14} - 216p^{16}q^{12} + 330p^{16}q^{10} - 216p^{16}q^{8} \\
& + 36p^{16}q^{6} + 24p^{16}q^{4} - 9p^{16}q^{2} + 36p^{14}q^{18} - 68p^{14}q^{16} \\
& + 244p^{14}q^{14} - 212p^{14}q^{12} - 212p^{14}q^{10} + 244p^{14}q^{8} - 68p^{14}q^{6} \\
& + 36p^{14}q^{4} - 84p^{12}q^{18} - 88p^{12}q^{16} - 172p^{12}q^{14} + 688p^{12}q^{12} \\
& - 172p^{12}q^{10} - 88p^{12}q^{8} - 84p^{12}q^{6} + 126p^{10}q^{18} + 246p^{10}q^{16} \\
& - 372p^{10}q^{14} - 372p^{10}q^{12} + 246p^{10}q^{10} + 126p^{10}q^{8} - 126p^{8}q^{18} \\
& - 88p^{8}q^{16} + 428p^{8}q^{14} - 88p^{8}q^{12} - 126p^{8}q^{10} + 84p^{6}q^{18} - 68p^{6}q^{16} \\
& - 68p^{6}q^{14} + 84p^{6}q^{12} - 36p^{4}q^{18} + 24p^{4}q^{16} - 36p^{4}q^{14} \\
& + 9p^{2}q^{18} + 9p^{2}q^{16} - q^{18}.
\end{aligned}$$

($\mathbf{i}_5$)

$$t_8 = \frac{n_8}{d_8} t_M, \tag{2.8h}$$

where

$$\begin{aligned}
n_8 = {} & q(p^{24}q^{24} + 12p^{24}q^{22} - 118p^{24}q^{20} + 364p^{24}q^{18} - 441p^{24}q^{16} - 168p^{24}q^{14} \\
& + 1260p^{24}q^{12} - 1800p^{24}q^{10} + 1311p^{24}q^{8} - 484p^{24}q^{6} + 42p^{24}q^{4} + 28p^{24}q^{2} - 7p^{24} \\
& - 12p^{22}q^{24} + 52p^{22}q^{22} + 44p^{22}q^{20} - 756p^{22}q^{18} + 2184p^{22}q^{16} - 3192p^{22}q^{14} \\
& + 2520p^{22}q^{12} - 744p^{22}q^{10} - 444p^{22}q^{8} + 516p^{22}q^{6} - 196p^{22}q^{4} + 28p^{22}q^{2} + 66p^{20}q^{24} \\
& - 188p^{20}q^{22} + 962p^{20}q^{20} - 1680p^{20}q^{18} - 700p^{20}q^{16} + 3928p^{20}q^{14} - 3756p^{20}q^{12} \\
& + 2544p^{20}q^{10} - 1734p^{20}q^{8} + 516p^{20}q^{6} + 42p^{20}q^{4} - 220p^{18}q^{24} - 260p^{18}q^{22} \\
& - 752p^{18}q^{20} + 5936p^{18}q^{18} - 5704p^{18}q^{16} - 568p^{18}q^{14} - 48p^{18}q^{12} + 2544p^{18}q^{10} \\
& - 444p^{18}q^{8} - 484p^{18}q^{6} + 495p^{16}q^{24} + 1208p^{16}q^{22} - 3404p^{16}q^{20} - 3864p^{16}q^{18} \\
& + 9322p^{16}q^{16} - 568p^{16}q^{14} - 3756p^{16}q^{12} - 744p^{16}q^{10} + 1311p^{16}q^{8} - 792p^{14}q^{24} \\
& - 824p^{14}q^{22} + 6536p^{14}q^{20} - 3864p^{14}q^{18} - 5704p^{14}q^{16} + 3928p^{14}q^{14} + 2520p^{14}q^{12} \\
& - 1800p^{14}q^{10} + 924p^{12}q^{24} - 824p^{12}q^{22} - 3404p^{12}q^{20} + 5936p^{12}q^{18} - 700p^{12}q^{16} \\
& - 3192p^{12}q^{14} + 1260p^{12}q^{12} - 792p^{10}q^{24} + 1208p^{10}q^{22} - 752p^{10}q^{20} - 1680p^{10}q^{18} \\
& + 2184p^{10}q^{16} - 168p^{10}q^{14} + 495p^{8}q^{24} - 260p^{8}q^{22} + 962p^{8}q^{20} - 756p^{8}q^{18} - 441p^{8}q^{16} \\
& - 220p^{6}q^{24} - 188p^{6}q^{22} + 44p^{6}q^{20} + 364p^{6}q^{18} + 66p^{4}q^{24} + 52p^{4}q^{22} - 118p^{4}q^{20} \\
& - 12p^{2}q^{24} + 12p^{2}q^{22} + q^{24}),
\end{aligned}$$

$$\begin{aligned}d_8 &= p^{25}q^{24} - 12p^{25}q^{22} + 66p^{25}q^{20} - 220p^{25}q^{18} + 495p^{25}q^{16} - 792p^{25}q^{14}\\ &+ 924p^{25}q^{12} - 792p^{25}q^{10} + 495p^{25}q^{8} - 220p^{25}q^{6} + 66p^{25}q^{4} - 12p^{25}q^{2} + p^{25}\\ &+ 12p^{23}q^{24} + 52p^{23}q^{22} - 188p^{23}q^{20} - 260p^{23}q^{18} + 1208p^{23}q^{16} - 824p^{23}q^{14}\\ &- 824p^{23}q^{12} + 1208p^{23}q^{10} - 260p^{23}q^{8} - 188p^{23}q^{6} + 52p^{23}q^{4} + 12p^{23}q^{2} - 118p^{21}q^{24}\\ &+ 44p^{21}q^{22} + 962p^{21}q^{20} - 752p^{21}q^{18} - 3404p^{21}q^{16} + 6536p^{21}q^{14} - 3404p^{21}q^{12}\\ &- 752p^{21}q^{10} + 962p^{21}q^{8} + 44p^{21}q^{6} - 118p^{21}q^{4} + 364p^{19}q^{24} - 756p^{19}q^{22}\\ &- 1680p^{19}q^{20} + 5936p^{19}q^{18} - 3864p^{19}q^{16} - 3864p^{19}q^{14} + 5936p^{19}q^{12}\\ &- 1680p^{19}q^{10} - 756p^{19}q^{8} + 364p^{19}q^{6} - 441p^{17}q^{24} + 2184p^{17}q^{22} - 700p^{17}q^{20}\\ &- 5704p^{17}q^{18} + 9322p^{17}q^{16} - 5704p^{17}q^{14} - 700p^{17}q^{12} + 2184p^{17}q^{10} - 441p^{17}q^{8}\\ &- 168p^{15}q^{24} - 3192p^{15}q^{22} + 3928p^{15}q^{20} - 568p^{15}q^{18} - 568p^{15}q^{16} + 3928p^{15}q^{14}\\ &- 3192p^{15}q^{12} - 168p^{15}q^{10} + 1260p^{13}q^{24} + 2520p^{13}q^{22} - 3756p^{13}q^{20} - 48p^{13}q^{18}\\ &- 3756p^{13}q^{16} + 2520p^{13}q^{14} + 1260p^{13}q^{12} - 1800p^{11}q^{24} - 744p^{11}q^{22}\\ &+ 2544p^{11}q^{20} + 2544p^{11}q^{18} - 744p^{11}q^{16} - 1800p^{11}q^{14} + 1311p^{9}q^{24} - 444p^{9}q^{22}\\ &- 1734p^{9}q^{20} - 444p^{9}q^{18} + 1311p^{9}q^{16} - 484p^{7}q^{24} + 516p^{7}q^{22} + 516p^{7}q^{20} - 484p^{7}q^{18}\\ &+ 42p^{5}q^{24} - 196p^{5}q^{22} + 42p^{5}q^{20} + 28p^{3}q^{24} + 28p^{3}q^{22} - 7pq^{24}.\end{aligned}$$

($\mathbf{i}_6$)

(2.8i) $$t_9 = \frac{n_9}{d_9} t_M,$$

where

$$\begin{aligned}n_9 &= q^{32}p^{32} - 16q^{30}p^{32} + 120q^{28}p^{32} - 560q^{26}p^{32} + 1820q^{24}p^{32} - 4368q^{22}p^{32}\\ &+ 8008q^{20}p^{32} - 11440q^{18}p^{32} + 12870q^{16}p^{32} - 11440q^{14}p^{32} + 8008q^{12}p^{32}\\ &- 4368q^{10}p^{32} + 1820q^{8}p^{32} - 560q^{6}p^{32} + 120q^{4}p^{32} - 16q^{2}p^{32} + p^{32} + 16q^{32}p^{30}\\ &+ 80q^{30}p^{30} - 432q^{28}p^{30} - 752q^{26}p^{30} + 6096q^{24}p^{30} - 9712q^{22}p^{30} + 16q^{20}p^{30}\\ &+ 16080q^{18}p^{30} - 16080q^{16}p^{30} - 16q^{14}p^{30} + 9712q^{12}p^{30} - 6096q^{10}p^{30} + 752q^{8}p^{30}\\ &+ 432q^{6}p^{30} - 80q^{4}p^{30} - 16q^{2}p^{30} - 200q^{32}p^{28} + 176q^{30}p^{28} + 2856q^{28}p^{28}\\ &- 4896q^{26}p^{28} - 16008q^{24}p^{28} + 63312q^{22}p^{28} - 81240q^{20}p^{28} + 29760q^{18}p^{28}\\ &+ 31464q^{16}p^{28} - 33712q^{14}p^{28} + 2488q^{12}p^{28} + 10464q^{10}p^{28} - 5016q^{8}p^{28} + 432q^{6}p^{28}\\ &+ 120q^{4}p^{28} + 816q^{32}p^{26} - 2544q^{30}p^{26} - 5856q^{28}p^{26} + 36832q^{26}p^{26} - 51568q^{24}p^{26}\\ &- 18000q^{22}p^{26} + 127680q^{20}p^{26} - 136896q^{18}p^{26} + 34128q^{16}p^{26} + 45168q^{14}p^{26}\\ &- 40416q^{12}p^{26} + 10464q^{10}p^{26} + 752q^{8}p^{26} - 560q^{6}p^{26} - 1380q^{32}p^{24} + 9520q^{30}p^{24}\\ &- 7240q^{28}p^{24} - 45200q^{26}p^{24} + 133668q^{24}p^{24} - 163232q^{22}p^{24} + 52752q^{20}p^{24}\\ &+ 102496q^{18}p^{24} - 124764q^{16}p^{24} + 45168q^{14}p^{24} + 2488q^{12}p^{24} - 6096q^{10}p^{24}\end{aligned}$$

$$+ 1820q^8p^{24} - 496q^{32}p^{22} - 19184q^{30}p^{22} + 39600q^{28}p^{22} - 13392q^{26}p^{22} - 32352q^{24}p^{22}$$
$$+ 132000q^{22}p^{22} - 214432q^{20}p^{22} + 102496q^{18}p^{22} + 34128q^{16}p^{22} - 33712q^{14}p^{22}$$
$$+ 9712q^{12}p^{22} - 4368q^{10}p^{22} + 7304q^{32}p^{20} + 22000q^{30}p^{20} - 61272q^{28}p^{20}$$
$$+ 27968q^{26}p^{20} - 83312q^{24}p^{20} + 132000q^{22}p^{20} + 52752q^{20}p^{20} - 136896q^{18}p^{20}$$
$$+ 31464q^{16}p^{20} - 16q^{14}p^{20} + 8008q^{12}p^{20} - 16720q^{32}p^{18} - 10032q^{30}p^{18}$$
$$+ 64448q^{28}p^{18} + 27968q^{26}p^{18} - 32352q^{24}p^{18} - 163232q^{22}p^{18} + 127680q^{20}p^{18}$$
$$+ 29760q^{18}p^{18} - 16080q^{16}p^{18} - 11440q^{14}p^{18} + 21318q^{32}p^{16} - 10032q^{30}p^{16}$$
$$- 61272q^{28}p^{16} - 13392q^{26}p^{16} + 133668q^{24}p^{16} - 18000q^{22}p^{16} - 81240q^{20}p^{16}$$
$$+ 16080q^{18}p^{16} + 12870q^{16}p^{16} - 16720q^{32}p^{14} + 22000q^{30}p^{14} + 39600q^{28}p^{14}$$
$$- 45200q^{26}p^{14} - 51568q^{24}p^{14} + 63312q^{22}p^{14} + 16q^{20}p^{14} - 11440q^{18}p^{14}$$
$$+ 7304q^{32}p^{12} - 19184q^{30}p^{12} - 7240q^{28}p^{12} + 36832q^{26}p^{12} - 16008q^{24}p^{12}$$
$$- 9712q^{22}p^{12} + 8008q^{20}p^{12} - 496q^{32}p^{10} + 9520q^{30}p^{10} - 5856q^{28}p^{10} - 4896q^{26}p^{10}$$
$$+ 6096q^{24}p^{10} - 4368q^{22}p^{10} - 1380q^{32}p^8 - 2544q^{30}p^8 + 2856q^{28}p^8 - 752q^{26}p^8$$
$$+ 1820q^{24}p^8 + 816q^{32}p^6 + 176q^{30}p^6 - 432q^{28}p^6 - 560q^{26}p^6 - 200q^{32}p^4 + 80q^{30}p^4$$
$$+ 120q^{28}p^4 + 16q^{32}p^2 - 16q^{30}p^2 + q^{32},$$

$$d_9 = q^{32}p^{32} + 16q^{30}p^{32} - 200q^{28}p^{32} + 816q^{26}p^{32} - 1380q^{24}p^{32} - 496q^{22}p^{32}$$
$$+ 7304q^{20}p^{32} - 16720q^{18}p^{32} + 21318q^{16}p^{32} - 16720q^{14}p^{32} + 7304q^{12}p^{32}$$
$$- 496q^{10}p^{32} - 1380q^8p^{32} + 816q^6p^{32} - 200q^4p^{32} + 16q^2p^{32} + p^{32} - 16q^{32}p^{30}$$
$$+ 80q^{30}p^{30} + 176q^{28}p^{30} - 2544q^{26}p^{30} + 9520q^{24}p^{30} - 19184q^{22}p^{30} + 22000q^{20}p^{30}$$
$$- 10032q^{18}p^{30} - 10032q^{16}p^{30} + 22000q^{14}p^{30} - 19184q^{12}p^{30} + 9520q^{10}p^{30}$$
$$- 2544q^8p^{30} + 176q^6p^{30} + 80q^4p^{30} - 16q^2p^{30} + 120q^{32}p^{28} - 432q^{30}p^{28}$$
$$+ 2856q^{28}p^{28} - 5856q^{26}p^{28} - 7240q^{24}p^{28} + 39600q^{22}p^{28} - 61272q^{20}p^{28}$$
$$+ 64448q^{18}p^{28} - 61272q^{16}p^{28} + 39600q^{14}p^{28} - 7240q^{12}p^{28} - 5856q^{10}p^{28}$$
$$+ 2856q^8p^{28} - 432q^6p^{28} + 120q^4p^{28} - 560q^{32}p^{26} - 752q^{30}p^{26} - 4896q^{28}p^{26}$$
$$+ 36832q^{26}p^{26} - 45200q^{24}p^{26} - 13392q^{22}p^{26} + 27968q^{20}p^{26} + 27968q^{18}p^{26}$$
$$- 13392q^{16}p^{26} - 45200q^{14}p^{26} + 36832q^{12}p^{26} - 4896q^{10}p^{26} - 752q^8p^{26} - 560q^6p^{26}$$
$$+ 1820q^{32}p^{24} + 6096q^{30}p^{24} - 16008q^{28}p^{24} - 51568q^{26}p^{24} + 133668q^{24}p^{24}$$
$$- 32352q^{22}p^{24} - 83312q^{20}p^{24} - 32352q^{18}p^{24} + 133668q^{16}p^{24} - 51568q^{14}p^{24}$$
$$- 16008q^{12}p^{24} + 6096q^{10}p^{24} + 1820q^8p^{24} - 4368q^{32}p^{22} - 9712q^{30}p^{22} + 63312q^{28}p^{22}$$
$$- 18000q^{26}p^{22} - 163232q^{24}p^{22} + 132000q^{22}p^{22} + 132000q^{20}p^{22} - 163232q^{18}p^{22}$$
$$- 18000q^{16}p^{22} + 63312q^{14}p^{22} - 9712q^{12}p^{22} - 4368q^{10}p^{22} + 8008q^{32}p^{20} + 16q^{30}p^{20}$$
$$- 81240q^{28}p^{20} + 127680q^{26}p^{20} + 52752q^{24}p^{20} - 214432q^{22}p^{20} + 52752q^{20}p^{20}$$
$$+ 127680q^{18}p^{20} - 81240q^{16}p^{20} + 16q^{14}p^{20} + 8008q^{12}p^{20} - 11440q^{32}p^{18}$$
$$+ 16080q^{30}p^{18} + 29760q^{28}p^{18} - 136896q^{26}p^{18} + 102496q^{24}p^{18} + 102496q^{22}p^{18}$$
$$- 136896q^{20}p^{18} + 29760q^{18}p^{18} + 16080q^{16}p^{18} - 11440q^{14}p^{18} + 12870q^{32}p^{16}$$

$$
\begin{aligned}
&- 16080q^{30}p^{16} + 31464q^{28}p^{16} + 34128q^{26}p^{16} - 124764q^{24}p^{16} + 34128q^{22}p^{16}\\
&+ 31464q^{20}p^{16} - 16080q^{18}p^{16} + 12870q^{16}p^{16} - 11440q^{32}p^{14} - 16q^{30}p^{14}\\
&- 33712q^{28}p^{14} + 45168q^{26}p^{14} + 45168q^{24}p^{14} - 33712q^{22}p^{14} - 16q^{20}p^{14}\\
&- 11440q^{18}p^{14} + 8008q^{32}p^{12} + 9712q^{30}p^{12} + 2488q^{28}p^{12} - 40416q^{26}p^{12}\\
&+ 2488q^{24}p^{12} + 9712q^{22}p^{12} + 8008q^{20}p^{12} - 4368q^{32}p^{10} - 6096q^{30}p^{10}\\
&+ 10464q^{28}p^{10} + 10464q^{26}p^{10} - 6096q^{24}p^{10} - 4368q^{22}p^{10} + 1820q^{32}p^{8} + 752q^{30}p^{8}\\
&- 5016q^{28}p^{8} + 752q^{26}p^{8} + 1820q^{24}p^{8} - 560q^{32}p^{6} + 432q^{30}p^{6} + 432q^{28}p^{6} - 560q^{26}p^{6}\\
&+ 120q^{32}p^{4} - 80q^{30}p^{4} + 120q^{28}p^{4} - 16q^{32}p^{2} - 16q^{30}p^{2} + q^{32}.
\end{aligned}
$$

($\mathbf{i}_7$)

(2.8j)
$$t_{10} = \frac{n_{10}}{d_{10}} t_M,$$

where

$$
\begin{aligned}
n_{10} = q(&q^{40}p^{40} + 20q^{38}p^{40} - 330q^{36}p^{40} + 1828q^{34}p^{40} - 4555q^{32}p^{40} + 528q^{30}p^{40}\\
&+ 34120q^{28}p^{40} - 125680q^{26}p^{40} + 259506q^{24}p^{40} - 356200q^{22}p^{40} + 328900q^{20}p^{40}\\
&- 179400q^{18}p^{40} + 8450q^{16}p^{40} + 84304q^{14}p^{40} - 85560q^{12}p^{40} + 47312q^{10}p^{40}\\
&- 16115q^{8}p^{40} + 3060q^{6}p^{40} - 138q^{4}p^{40} - 60q^{2}p^{40} + 9p^{40} - 20q^{40}p^{38} + 140q^{38}p^{38}\\
&+ 228q^{36}p^{38} - 6428q^{34}p^{38} + 35120q^{32}p^{38} - 106320q^{30}p^{38} + 201040q^{28}p^{38}\\
&- 221104q^{26}p^{38} + 45864q^{24}p^{38} + 314600q^{22}p^{38} - 657800q^{20}p^{38} + 756600q^{18}p^{38}\\
&- 581776q^{16}p^{38} + 303856q^{14}p^{38} - 98160q^{12}p^{38} + 10640q^{10}p^{38} + 6220q^{8}p^{38}\\
&- 3348q^{6}p^{38} + 708q^{4}p^{38} - 60q^{2}p^{38} + 190q^{40}p^{36} - 916q^{38}p^{36} + 7878q^{36}p^{36}\\
&- 24416q^{34}p^{36} - 16168q^{32}p^{36} + 236112q^{30}p^{36} - 609128q^{28}p^{36} + 1016160q^{26}p^{36}\\
&- 1470492q^{24}p^{36} + 1742600q^{22}p^{36} - 1301900q^{20}p^{36} + 318048q^{18}p^{36} + 328760q^{16}p^{36}\\
&- 342704q^{14}p^{36} + 157368q^{12}p^{36} - 57952q^{10}p^{36} + 20046q^{8}p^{36} - 3348q^{6}p^{36} - 138q^{4}p^{36}\\
&- 1140q^{40}p^{34} - 1740q^{38}p^{34} - 15264q^{36}p^{34} + 170016q^{34}p^{34} - 353904q^{32}p^{34}\\
&+ 65520q^{30}p^{34} + 366112q^{28}p^{34} + 164960q^{26}p^{34} - 695160q^{24}p^{34} - 760200q^{22}p^{34}\\
&+ 2707872q^{20}p^{34} - 2421792q^{18}p^{34} + 816144q^{16}p^{34} - 45456q^{14}p^{34} + 52704q^{12}p^{34}\\
&- 57952q^{10}p^{34} + 6220q^{8}p^{34} + 3060q^{6}p^{34} + 4845q^{40}p^{32} + 19536q^{38}p^{32} - 78984q^{36}p^{32}\\
&- 275664q^{34}p^{32} + 1222860q^{32}p^{32} - 1002288q^{30}p^{32} - 984440q^{28}p^{32} + 469040q^{26}p^{32}\\
&+ 4039470q^{24}p^{32} - 5846928q^{22}p^{32} + 1942728q^{20}p^{32} + 1526544q^{18}p^{32} - 1143156q^{16}p^{32}\\
&- 45456q^{14}p^{32} + 157368q^{12}p^{32} + 10640q^{10}p^{32} - 16115q^{8}p^{32} - 15504q^{40}p^{30}\\
&- 41424q^{38}p^{30} + 397296q^{36}p^{30} - 274512q^{34}p^{30} - 1801168q^{32}p^{30} + 3140592q^{30}p^{30}\\
&+ 1244080q^{28}p^{30} - 6602640q^{26}p^{30} + 3137616q^{24}p^{30} + 4906128q^{22}p^{30} - 6039600q^{20}p^{30}\\
&+ 1526544q^{18}p^{30} + 816144q^{16}p^{30} - 342704q^{14}p^{30} - 98160q^{12}p^{30} + 47312q^{10}p^{30}\\
&+ 38760q^{40}p^{28} - 1680q^{38}p^{28} - 716328q^{36}p^{28} + 1692128q^{34}p^{28} + 389608q^{32}p^{28}
\end{aligned}
$$

$$
\begin{aligned}
&- 5386992q^{30}p^{28} + 4344728q^{28}p^{28} + 5299264q^{26}p^{28} - 10633608q^{24}p^{28} + 4906128q^{22}p^{28}\\
&+ 1942728q^{20}p^{28} - 2421792q^{18}p^{28} + 328760q^{16}p^{28} + 303856q^{14}p^{28} - 85560q^{12}p^{28}\\
&- 77520q^{40}p^{26} + 147216q^{38}p^{26} + 461472q^{36}p^{26} - 2596000q^{34}p^{26} + 3086608q^{32}p^{26}\\
&+ 3052848q^{30}p^{26} - 9193024q^{28}p^{26} + 5299264q^{26}p^{26} + 3137616q^{24}p^{26} - 5846928q^{22}p^{26}\\
&+ 2707872q^{20}p^{26} + 318048q^{18}p^{26} - 581776q^{16}p^{26} + 84304q^{14}p^{26} + 125970q^{40}p^{24}\\
&- 254120q^{38}p^{24} + 394548q^{36}p^{24} + 1313048q^{34}p^{24} - 5116802q^{32}p^{24} + 3052848q^{30}p^{24}\\
&+ 4344728q^{28}p^{24} - 6602640q^{26}p^{24} + 4039470q^{24}p^{24} - 760200q^{22}p^{24} - 1301900q^{20}p^{24}\\
&+ 756600q^{18}p^{24} + 8450q^{16}p^{24} - 167960q^{40}p^{22} + 132968q^{38}p^{22} - 901032q^{36}p^{22}\\
&+ 1313048q^{34}p^{22} + 3086608q^{32}p^{22} - 5386992q^{30}p^{22} + 1244080q^{28}p^{22} + 469040q^{26}p^{22}\\
&- 695160q^{24}p^{22} + 1742600q^{22}p^{22} - 657800q^{20}p^{22} - 179400q^{18}p^{22} + 184756q^{40}p^{20}\\
&+ 132968q^{38}p^{20} + 394548q^{36}p^{20} - 2596000q^{34}p^{20} + 389608q^{32}p^{20} + 3140592q^{30}p^{20}\\
&- 984440q^{28}p^{20} + 164960q^{26}p^{20} - 1470492q^{24}p^{20} + 314600q^{22}p^{20} + 328900q^{20}p^{20}\\
&- 167960q^{40}p^{18} - 254120q^{38}p^{18} + 461472q^{36}p^{18} + 1692128q^{34}p^{18} - 1801168q^{32}p^{18}\\
&- 1002288q^{30}p^{18} + 366112q^{28}p^{18} + 1016160q^{26}p^{18} + 45864q^{24}p^{18} - 356200q^{22}p^{18}\\
&+ 125970q^{40}p^{16} + 147216q^{38}p^{16} - 716328q^{36}p^{16} - 274512q^{34}p^{16} + 1222860q^{32}p^{16}\\
&+ 65520q^{30}p^{16} - 609128q^{28}p^{16} - 221104q^{26}p^{16} + 259506q^{24}p^{16} - 77520q^{40}p^{14}\\
&- 1680q^{38}p^{14} + 397296q^{36}p^{14} - 275664q^{34}p^{14} - 353904q^{32}p^{14} + 236112q^{30}p^{14}\\
&+ 201040q^{28}p^{14} - 125680q^{26}p^{14} + 38760q^{40}p^{12} - 41424q^{38}p^{12} - 78984q^{36}p^{12}\\
&+ 170016q^{34}p^{12} - 16168q^{32}p^{12} - 106320q^{30}p^{12} + 34120q^{28}p^{12} - 15504q^{40}p^{10}\\
&+ 19536q^{38}p^{10} - 15264q^{36}p^{10} - 24416q^{34}p^{10} + 35120q^{32}p^{10} + 528q^{30}p^{10}\\
&+ 4845q^{40}p^{8} - 1740q^{38}p^{8} + 7878q^{36}p^{8} - 6428q^{34}p^{8} - 4555q^{32}p^{8} - 1140q^{40}p^{6}\\
&- 916q^{38}p^{6} + 228q^{36}p^{6} + 1828q^{34}p^{6} + 190q^{40}p^{4} + 140q^{38}p^{4} - 330q^{36}p^{4} - 20q^{40}p^{2}\\
&+ 20q^{38}p^{2} + q^{40}),
\end{aligned}
$$

$$
\begin{aligned}
d_{10} = p(&q^{40}p^{40} - 20q^{38}p^{40} + 190q^{36}p^{40} - 1140q^{34}p^{40} + 4845q^{32}p^{40} - 15504q^{30}p^{40}\\
&+ 38760q^{28}p^{40} - 77520q^{26}p^{40} + 125970q^{24}p^{40} - 167960q^{22}p^{40} + 184756q^{20}p^{40}\\
&- 167960q^{18}p^{40} + 125970q^{16}p^{40} - 77520q^{14}p^{40} + 38760q^{12}p^{40} - 15504q^{10}p^{40}\\
&+ 4845q^{8}p^{40} - 1140q^{6}p^{40} + 190q^{4}p^{40} - 20q^{2}p^{40} + p^{40} + 20q^{40}p^{38} + 140q^{38}p^{38}\\
&- 916q^{36}p^{38} - 1740q^{34}p^{38} + 19536q^{32}p^{38} - 41424q^{30}p^{38} - 1680q^{28}p^{38}\\
&+ 147216q^{26}p^{38} - 254120q^{24}p^{38} + 132968q^{22}p^{38} + 132968q^{20}p^{38} - 254120q^{18}p^{38}\\
&+ 147216q^{16}p^{38} - 1680q^{14}p^{38} - 41424q^{12}p^{38} + 19536q^{10}p^{38} - 1740q^{8}p^{38} - 916q^{6}p^{38}\\
&+ 140q^{4}p^{38} + 20q^{2}p^{38} - 330q^{40}p^{36} + 228q^{38}p^{36} + 7878q^{36}p^{36} - 15264q^{34}p^{36}\\
&- 78984q^{32}p^{36} + 397296q^{30}p^{36} - 716328q^{28}p^{36} + 461472q^{26}p^{36} + 394548q^{24}p^{36}\\
&- 901032q^{22}p^{36} + 394548q^{20}p^{36} + 461472q^{18}p^{36} - 716328q^{16}p^{36} + 397296q^{14}p^{36}\\
&- 78984q^{12}p^{36} - 15264q^{10}p^{36} + 7878q^{8}p^{36} + 228q^{6}p^{36} - 330q^{4}p^{36} + 1828q^{40}p^{34}\\
&- 6428q^{38}p^{34} - 24416q^{36}p^{34} + 170016q^{34}p^{34} - 275664q^{32}p^{34} - 274512q^{30}p^{34}
\end{aligned}
$$

$$+ 1692128q^{28}p^{34} - 2596000q^{26}p^{34} + 1313048q^{24}p^{34} + 1313048q^{22}p^{34} - 2596000q^{20}p^{34}$$
$$+ 1692128q^{18}p^{34} - 274512q^{16}p^{34} - 275664q^{14}p^{34} + 170016q^{12}p^{34} - 24416q^{10}p^{34}$$
$$- 6428q^{8}p^{34} + 1828q^{6}p^{34} - 4555q^{40}p^{32} + 35120q^{38}p^{32} - 16168q^{36}p^{32} - 353904q^{34}p^{32}$$
$$+ 1222860q^{32}p^{32} - 1801168q^{30}p^{32} + 389608q^{28}p^{32} + 3086608q^{26}p^{32} - 5116802q^{24}p^{32}$$
$$+ 3086608q^{22}p^{32} + 389608q^{20}p^{32} - 1801168q^{18}p^{32} + 1222860q^{16}p^{32} - 353904q^{14}p^{32}$$
$$- 16168q^{12}p^{32} + 35120q^{10}p^{32} - 4555q^{8}p^{32} + 528q^{40}p^{30} - 106320q^{38}p^{30} + 236112q^{36}p^{30}$$
$$+ 65520q^{34}p^{30} - 1002288q^{32}p^{30} + 3140592q^{30}p^{30} - 5386992q^{28}p^{30} + 3052848q^{26}p^{30}$$
$$+ 3052848q^{24}p^{30} - 5386992q^{22}p^{30} + 3140592q^{20}p^{30} - 1002288q^{18}p^{30} + 65520q^{16}p^{30}$$
$$+ 236112q^{14}p^{30} - 106320q^{12}p^{30} + 528q^{10}p^{30} + 34120q^{40}p^{28} + 201040q^{38}p^{28}$$
$$- 609128q^{36}p^{28} + 366112q^{34}p^{28} - 984440q^{32}p^{28} + 1244080q^{30}p^{28} + 4344728q^{28}p^{28}$$
$$- 9193024q^{26}p^{28} + 4344728q^{24}p^{28} + 1244080q^{22}p^{28} - 984440q^{20}p^{28} + 366112q^{18}p^{28}$$
$$- 609128q^{16}p^{28} + 201040q^{14}p^{28} + 34120q^{12}p^{28} - 125680q^{40}p^{26} - 221104q^{38}p^{26}$$
$$+ 1016160q^{36}p^{26} + 164960q^{34}p^{26} + 469040q^{32}p^{26} - 6602640q^{30}p^{26} + 5299264q^{28}p^{26}$$
$$+ 5299264q^{26}p^{26} - 6602640q^{24}p^{26} + 469040q^{22}p^{26} + 164960q^{20}p^{26} + 1016160q^{18}p^{26}$$
$$- 221104q^{16}p^{26} - 125680q^{14}p^{26} + 259506q^{40}p^{24} + 45864q^{38}p^{24} - 1470492q^{36}p^{24}$$
$$- 695160q^{34}p^{24} + 4039470q^{32}p^{24} + 3137616q^{30}p^{24} - 10633608q^{28}p^{24} + 3137616q^{26}p^{24}$$
$$+ 4039470q^{24}p^{24} - 695160q^{22}p^{24} - 1470492q^{20}p^{24} + 45864q^{18}p^{24} + 259506q^{16}p^{24}$$
$$- 356200q^{40}p^{22} + 314600q^{38}p^{22} + 1742600q^{36}p^{22} - 760200q^{34}p^{22} - 5846928q^{32}p^{22}$$
$$+ 4906128q^{30}p^{22} + 4906128q^{28}p^{22} - 5846928q^{26}p^{22} - 760200q^{24}p^{22} + 1742600q^{22}p^{22}$$
$$+ 314600q^{20}p^{22} - 356200q^{18}p^{22} + 328900q^{40}p^{20} - 657800q^{38}p^{20} - 1301900q^{36}p^{20}$$
$$+ 2707872q^{34}p^{20} + 1942728q^{32}p^{20} - 6039600q^{30}p^{20} + 1942728q^{28}p^{20} + 2707872q^{26}p^{20}$$
$$- 1301900q^{24}p^{20} - 657800q^{22}p^{20} + 328900q^{20}p^{20} - 179400q^{40}p^{18} + 756600q^{38}p^{18}$$
$$+ 318048q^{36}p^{18} - 2421792q^{34}p^{18} + 1526544q^{32}p^{18} + 1526544q^{30}p^{18} - 2421792q^{28}p^{18}$$
$$+ 318048q^{26}p^{18} + 756600q^{24}p^{18} - 179400q^{22}p^{18} + 8450q^{40}p^{16} - 581776q^{38}p^{16}$$
$$+ 328760q^{36}p^{16} + 816144q^{34}p^{16} - 1143156q^{32}p^{16} + 816144q^{30}p^{16} + 328760q^{28}p^{16}$$
$$- 581776q^{26}p^{16} + 8450q^{24}p^{16} + 84304q^{40}p^{14} + 303856q^{38}p^{14} - 342704q^{36}p^{14}$$
$$- 45456q^{34}p^{14} - 45456q^{32}p^{14} - 342704q^{30}p^{14} + 303856q^{28}p^{14} + 84304q^{26}p^{14}$$
$$- 85560q^{40}p^{12} - 98160q^{38}p^{12} + 157368q^{36}p^{12} + 52704q^{34}p^{12} + 157368q^{32}p^{12}$$
$$- 98160q^{30}p^{12} - 85560q^{28}p^{12} + 47312q^{40}p^{10} + 10640q^{38}p^{10} - 57952q^{36}p^{10}$$
$$- 57952q^{34}p^{10} + 10640q^{32}p^{10} + 47312q^{30}p^{10} - 16115q^{40}p^{8} + 6220q^{38}p^{8} + 20046q^{36}p^{8}$$
$$+ 6220q^{34}p^{8} - 16115q^{32}p^{8} + 3060q^{40}p^{6} - 3348q^{38}p^{6} - 3348q^{36}p^{6} + 3060q^{34}p^{6}$$
$$- 138q^{40}p^{4} + 708q^{38}p^{4} - 138q^{36}p^{4} - 60q^{40}p^{2} - 60q^{38}p^{2} + 9q^{40}).$$

$(\mathbf{i}_8)$

(2.8k) $$t_{11} = \frac{n_{11}}{d_{11}} t_M,$$

where

$$
\begin{aligned}
n_{11} &= q^{50}p^{50} - 25q^{48}p^{50} + 300q^{46}p^{50} - 2300q^{44}p^{50} + 12650q^{42}p^{50} - 53130q^{40}p^{50} \\
&+ 177100q^{38}p^{50} - 480700q^{36}p^{50} + 1081575q^{34}p^{50} - 2042975q^{32}p^{50} + 3268760q^{30}p^{50} \\
&- 4457400q^{28}p^{50} + 5200300q^{26}p^{50} - 5200300q^{24}p^{50} + 4457400q^{22}p^{50} - 3268760q^{20}p^{50} \\
&+ 2042975q^{18}p^{50} - 1081575q^{16}p^{50} + 480700q^{14}p^{50} - 177100q^{12}p^{50} + 53130q^{10}p^{50} \\
&- 12650q^{8}p^{50} + 2300q^{6}p^{50} - 300q^{4}p^{50} + 25q^{2}p^{50} - p^{50} + 25q^{50}p^{48} + 200q^{48}p^{48} \\
&- 1740q^{46}p^{48} - 4040q^{44}p^{48} + 64306q^{42}p^{48} - 206696q^{40}p^{48} + 149860q^{38}p^{48} \\
&+ 759400q^{36}p^{48} - 2428585q^{34}p^{48} + 2669392q^{32}p^{48} + 1087848q^{30}p^{48} - 7412560q^{28}p^{48} \\
&+ 10645180q^{26}p^{48} - 7412560q^{24}p^{48} + 1087848q^{22}p^{48} + 2669392q^{20}p^{48} - 2428585q^{18}p^{48} \\
&+ 759400q^{16}p^{48} + 149860q^{14}p^{48} - 206696q^{12}p^{48} + 64306q^{10}p^{48} - 4040q^{8}p^{48} \\
&- 1740q^{6}p^{48} + 200q^{4}p^{48} + 25q^{2}p^{48} - 500q^{50}p^{46} + 620q^{48}p^{46} + 18300q^{46}p^{46} \\
&- 55204q^{44}p^{46} - 258476q^{42}p^{46} + 2022900q^{40}p^{46} - 5585020q^{38}p^{46} + 7037860q^{36}p^{46} \\
&+ 394872q^{34}p^{46} - 15047112q^{32}p^{46} + 20287320q^{30}p^{46} - 1695720q^{28}p^{46} - 29683160q^{26}p^{46} \\
&+ 46020968q^{24}p^{46} - 35872248q^{22}p^{46} + 14437960q^{20}p^{46} - 518980q^{18}p^{46} - 2428260q^{16}p^{46} \\
&+ 1005164q^{14}p^{46} + 5836q^{12}p^{46} - 117180q^{10}p^{46} + 32100q^{8}p^{46} - 1740q^{6}p^{46} \\
&- 300q^{4}p^{46} + 3420q^{50}p^{44} - 16200q^{48}p^{44} - 65244q^{46}p^{44} + 664784q^{44}p^{44} \\
&- 1669100q^{42}p^{44} - 665640q^{40}p^{44} + 13654060q^{38}p^{44} - 35295808q^{36}p^{44} + 39902808q^{34}p^{44} \\
&+ 2375920q^{32}p^{44} - 76899160q^{30}p^{44} + 120188640q^{28}p^{44} - 90236952q^{26}p^{44} \\
&+ 19483632q^{24}p^{44} + 28292760q^{22}p^{44} - 31013440q^{20}p^{44} + 13270540q^{18}p^{44} \\
&- 976296q^{16}p^{44} - 1635724q^{14}p^{44} + 755920q^{12}p^{44} - 117180q^{10}p^{44} - 4040q^{8}p^{44} \\
&+ 2300q^{6}p^{44} - 10710q^{50}p^{42} + 107150q^{48}p^{42} - 100172q^{46}p^{42} - 1639156q^{44}p^{42} \\
&+ 8207274q^{42}p^{42} - 18257346q^{40}p^{42} + 13418416q^{38}p^{42} + 36444112q^{36}p^{42} \\
&- 120569836q^{34}p^{42} + 154838044q^{32}p^{42} - 71534856q^{30}p^{42} - 74228216q^{28}p^{42} \\
&+ 161992548q^{26}p^{42} - 142747284q^{24}p^{42} + 66574896q^{22}p^{42} - 6004144q^{20}p^{42} \\
&- 12365950q^{18}p^{42} + 7453462q^{16}p^{42} - 1635724q^{14}p^{42} + 5836q^{12}p^{42} + 64306q^{10}p^{42} \\
&- 12650q^{8}p^{42} + 2122q^{50}p^{40} - 404200q^{48}p^{40} + 1271916q^{46}p^{40} + 27096q^{44}p^{40} \\
&- 8553534q^{42}p^{40} + 37031904q^{40}p^{40} - 91661296q^{38}p^{40} + 103363936q^{36}p^{40} \\
&+ 29963316q^{34}p^{40} - 236543664q^{32}p^{40} + 313092360q^{30}p^{40} - 221648304q^{28}p^{40} \\
&+ 84440244q^{26}p^{40} + 20291424q^{24}p^{40} - 64540656q^{22}p^{40} + 46357984q^{20}p^{40} \\
&- 12365950q^{18}p^{40} - 976296q^{16}p^{40} + 1005164q^{14}p^{40} - 206696q^{12}p^{40} + 53130q^{10}p^{40} \\
&+ 127660q^{50}p^{38} + 967100q^{48}p^{38} - 4215420q^{46}p^{38} + 4434900q^{44}p^{38} - 9748176q^{42}p^{38} \\
&+ 16508016q^{40}p^{38} + 75728080q^{38}p^{38} - 280626160q^{36}p^{38} + 313362600q^{34}p^{38} \\
&- 53109368q^{32}p^{38} - 159756232q^{30}p^{38} + 159286680q^{28}p^{38} - 142358160q^{26}p^{38} \\
&+ 139128240q^{24}p^{38} - 64540656q^{22}p^{38} - 6004144q^{20}p^{38} + 13270540q^{18}p^{38} \\
&- 2428260q^{16}p^{38} + 149860q^{14}p^{38} - 177100q^{12}p^{38} - 622820q^{50}p^{36} - 1347480q^{48}p^{36} \\
&+ 9269340q^{46}p^{36} - 2934336q^{44}p^{36} + 7125936q^{42}p^{36} - 120304800q^{40}p^{36}
\end{aligned}
$$

$$+ 184649200q^{38}p^{36} + 125331520q^{36}p^{36} - 491940088q^{34}p^{36} + 346169968q^{32}p^{36}$$
$$- 29546040q^{30}p^{36} + 59933760q^{28}p^{36} - 142358160q^{26}p^{36} + 20291424q^{24}p^{36}$$
$$+ 66574896q^{22}p^{36} - 31013440q^{20}p^{36} - 518980q^{18}p^{36} + 759400q^{16}p^{36} + 480700q^{14}p^{36}$$
$$+ 1738215q^{50}p^{34} + 218025q^{48}p^{34} - 17339976q^{46}p^{34} - 1572504q^{44}p^{34} + 72561300q^{42}p^{34}$$
$$+ 63313740q^{40}p^{34} - 436698520q^{38}p^{34} + 375512632q^{36}p^{34} + 230233338q^{34}p^{34}$$
$$- 398620410q^{32}p^{34} - 29546040q^{30}p^{34} + 159286680q^{28}p^{34} + 84440244q^{26}p^{34}$$
$$- 142747284q^{24}p^{34} + 28292760q^{22}p^{34} + 14437960q^{20}p^{34} - 2428585q^{18}p^{34}$$
$$- 1081575q^{16}p^{34} - 3331745q^{50}p^{32} + 4005200q^{48}p^{32} + 27096072q^{46}p^{32} - 22635024q^{44}p^{32}$$
$$- 154557084q^{42}p^{32} + 222032976q^{40}p^{32} + 246168120q^{38}p^{32} - 664093584q^{36}p^{32}$$
$$+ 230233338q^{34}p^{32} + 346169968q^{32}p^{32} - 159756232q^{30}p^{32} - 221648304q^{28}p^{32}$$
$$+ 161992548q^{26}p^{32} + 19483632q^{24}p^{32} - 35872248q^{22}p^{32} + 2669392q^{20}p^{32}$$
$$+ 2042975q^{18}p^{32} + 4529752q^{50}p^{30} - 11305000q^{48}p^{30} - 29014696q^{46}p^{30}$$
$$+ 82720344q^{44}p^{30} + 86814904q^{42}p^{30} - 402843848q^{40}p^{30} + 246168120q^{38}p^{30}$$
$$+ 375512632q^{36}p^{30} - 491940088q^{34}p^{30} - 53109368q^{32}p^{30} + 313092360q^{30}p^{30}$$
$$- 74228216q^{28}p^{30} - 90236952q^{26}p^{30} + 46020968q^{24}p^{30} + 1087848q^{22}p^{30}$$
$$- 3268760q^{20}p^{30} - 4121480q^{50}p^{28} + 18734000q^{48}p^{28} + 13081320q^{46}p^{28}$$
$$- 118009120q^{44}p^{28} + 86814904q^{42}p^{28} + 222032976q^{40}p^{28} - 436698520q^{38}p^{28}$$
$$+ 125331520q^{36}p^{28} + 313362600q^{34}p^{28} - 236543664q^{32}p^{28} - 71534856q^{30}p^{28}$$
$$+ 120188640q^{28}p^{28} - 29683160q^{26}p^{28} - 7412560q^{24}p^{28} + 4457400q^{22}p^{28}$$
$$+ 1686060q^{50}p^{26} - 21918780q^{48}p^{26} + 13081320q^{46}p^{26} + 82720344q^{44}p^{26}$$
$$- 154557084q^{42}p^{26} + 63313740q^{40}p^{26} + 184649200q^{38}p^{26} - 280626160q^{36}p^{26}$$
$$+ 29963316q^{34}p^{26} + 154838044q^{32}p^{26} - 76899160q^{30}p^{26} - 1695720q^{28}p^{26}$$
$$+ 10645180q^{26}p^{26} - 5200300q^{24}p^{26} + 1686060q^{50}p^{24} + 18734000q^{48}p^{24}$$
$$- 29014696q^{46}p^{24} - 22635024q^{44}p^{24} + 72561300q^{42}p^{24} - 120304800q^{40}p^{24}$$
$$+ 75728080q^{38}p^{24} + 103363936q^{36}p^{24} - 120569836q^{34}p^{24} + 2375920q^{32}p^{24}$$
$$+ 20287320q^{30}p^{24} - 7412560q^{28}p^{24} + 5200300q^{26}p^{24} - 4121480q^{50}p^{22} - 11305000q^{48}p^{22}$$
$$+ 27096072q^{46}p^{22} - 1572504q^{44}p^{22} + 7125936q^{42}p^{22} + 16508016q^{40}p^{22}$$
$$- 91661296q^{38}p^{22} + 36444112q^{36}p^{22} + 39902808q^{34}p^{22} - 15047112q^{32}p^{22}$$
$$+ 1087848q^{30}p^{22} - 4457400q^{28}p^{22} + 4529752q^{50}p^{20}4005200q^{48}p^{20} - 17339976q^{46}p^{20}$$
$$- 2934336q^{44}p^{20} - 9748176q^{42}p^{20} + 37031904q^{40}p^{20} + 13418416q^{38}p^{20}$$
$$- 35295808q^{36}p^{20} + 394872q^{34}p^{20} + 2669392q^{32}p^{20} + 3268760q^{30}p^{20} - 3331745q^{50}p^{18}$$
$$+ 218025q^{48}p^{18} + 9269340q^{46}p^{18} + 4434900q^{44}p^{18} - 8553534q^{42}p^{18} - 18257346q^{40}p^{18}$$
$$+ 13654060q^{38}p^{18} + 7037860q^{36}p^{18} - 2428585q^{34}p^{18} - 2042975q^{32}p^{18} + 1738215q^{50}p^{16}$$
$$- 1347480q^{48}p^{16} - 4215420q^{46}p^{16} + 27096q^{44}p^{16} + 8207274q^{42}p^{16} - 665640q^{40}p^{16}$$
$$- 5585020q^{38}p^{16} + 759400q^{36}p^{16} + 1081575q^{34}p^{16} - 622820q^{50}p^{14} + 967100q^{48}p^{14}$$

$$+ 1271916q^{46}p^{14} - 1639156q^{44}p^{14} - 1669100q^{42}p^{14} + 2022900q^{40}p^{14} + 149860q^{38}p^{14} - 480700q^{36}p^{14} + 127660q^{50}p^{12} - 404200q^{48}p^{12} - 100172q^{46}p^{12} + 664784q^{44}p^{12} - 258476q^{42}p^{12} - 206696q^{40}p^{12} + 177100q^{38}p^{12} + 2122q^{50}p^{10} + 107150q^{48}p^{10} - 65244q^{46}p^{10} - 55204q^{44}p^{10} + 64306q^{42}p^{10} - 53130q^{40}p^{10} - 10710q^{50}p^{8} - 16200q^{48}p^{8} + 18300q^{46}p^{8} - 4040q^{44}p^{8} + 12650q^{42}p^{8} + 3420q^{50}p^{6} + 620q^{48}p^{6} - 1740q^{46}p^{6} - 2300q^{44}p^{6} - 500q^{50}p^{4} + 200q^{48}p^{4} + 300q^{46}p^{4} + 25q^{50}p^{2} - 25q^{48}p^{2} + q^{50},$$

$$\begin{aligned}
d_{11} = {} & q^{50}p^{50} + 25q^{48}p^{50} - 500q^{46}p^{50} + 3420q^{44}p^{50} - 10710q^{42}p^{50} + 2122q^{40}p^{50} \\
& + 127660q^{38}p^{50} - 622820q^{36}p^{50} + 1738215q^{34}p^{50} - 3331745q^{32}p^{50} + 4529752q^{30}p^{50} \\
& - 4121480q^{28}p^{50} + 1686060q^{26}p^{50} + 1686060q^{24}p^{50} - 4121480q^{22}p^{50} + 4529752q^{20}p^{50} \\
& - 3331745q^{18}p^{50} + 1738215q^{16}p^{50} - 622820q^{14}p^{50} + 127660q^{12}p^{50} + 2122q^{10}p^{50} \\
& - 10710q^{8}p^{50} + 3420q^{6}p^{50} - 500q^{4}p^{50} + 25q^{2}p^{50} + p^{50} - 25q^{50}p^{48} + 200q^{48}p^{48} \\
& + 620q^{46}p^{48} - 16200q^{44}p^{48} + 107150q^{42}p^{48} - 404200q^{40}p^{48} + 967100q^{38}p^{48} \\
& - 1347480q^{36}p^{48} + 218025q^{34}p^{48} + 4005200q^{32}p^{48} - 11305000q^{30}p^{48} + 18734000q^{28}p^{48} \\
& - 21918780q^{26}p^{48} + 18734000q^{24}p^{48} - 11305000q^{22}p^{48} + 4005200q^{20}p^{48} + 218025q^{18}p^{48} \\
& - 1347480q^{16}p^{48} + 967100q^{14}p^{48} - 404200q^{12}p^{48} + 107150q^{10}p^{48} - 16200q^{8}p^{48} \\
& + 620q^{6}p^{48} + 200q^{4}p^{48} - 25q^{2}p^{48} + 300q^{50}p^{46} - 1740q^{48}p^{46} + 18300q^{46}p^{46} \\
& - 65244q^{44}p^{46} - 100172q^{42}p^{46} + 1271916q^{40}p^{46} - 4215420q^{38}p^{46} \\
& + 9269340q^{36}p^{46} - 17339976q^{34}p^{46} + 27096072q^{32}p^{46} - 29014696q^{30}p^{46} \\
& + 13081320q^{28}p^{46} + 13081320q^{26}p^{46} - 29014696q^{24}p^{46} + 27096072q^{22}p^{46} \\
& - 17339976q^{20}p^{46} + 9269340q^{18}p^{46} - 4215420q^{16}p^{46} + 1271916q^{14}p^{46} \\
& - 100172q^{12}p^{46} - 65244q^{10}p^{46} + 18300q^{8}p^{46} - 1740q^{6}p^{46} + 300q^{4}p^{46} - 2300q^{50}p^{44} \\
& - 4040q^{48}p^{44} - 55204q^{46}p^{44} + 664784q^{44}p^{44} - 1639156q^{42}p^{44} + 27096q^{40}p^{44} \\
& + 4434900q^{38}p^{44} - 2934336q^{36}p^{44} - 1572504q^{34}p^{44} - 22635024q^{32}p^{44} + 82720344q^{30}p^{44} \\
& - 118009120q^{28}p^{44} + 82720344q^{26}p^{44} - 22635024q^{24}p^{44} - 1572504q^{22}p^{44} \\
& - 2934336q^{20}p^{44} + 4434900q^{18}p^{44} + 27096q^{16}p^{44} - 1639156q^{14}p^{44} + 664784q^{12}p^{44} \\
& - 55204q^{10}p^{44} - 4040q^{8}p^{44} - 2300q^{6}p^{44} + 12650q^{50}p^{42} + 64306q^{48}p^{42} \\
& - 258476q^{46}p^{42} - 1669100q^{44}p^{42} + 8207274q^{42}p^{42} - 8553534q^{40}p^{42} \\
& - 9748176q^{38}p^{42} + 7125936q^{36}p^{42} + 72561300q^{34}p^{42} - 154557084q^{32}p^{42} \\
& + 86814904q^{30}p^{42} + 86814904q^{28}p^{42} - 154557084q^{26}p^{42} + 72561300q^{24}p^{42} \\
& + 7125936q^{22}p^{42} - 9748176q^{20}p^{42} - 8553534q^{18}p^{42} + 8207274q^{16}p^{42} \\
& - 1669100q^{14}p^{42} - 258476q^{12}p^{42} + 64306q^{10}p^{42} + 12650q^{8}p^{42} - 53130q^{50}p^{40} \\
& - 206696q^{48}p^{40} + 2022900q^{46}p^{40} - 665640q^{44}p^{40} - 18257346q^{42}p^{40} \\
& + 37031904q^{40}p^{40} + 16508016q^{38}p^{40} - 120304800q^{36}p^{40} + 63313740q^{34}p^{40} \\
& + 222032976q^{32}p^{40} - 402843848q^{30}p^{40} + 222032976q^{28}p^{40} + 63313740q^{26}p^{40}
\end{aligned}$$

$$
\begin{aligned}
&- 120304800q^{24}p^{40} + 16508016q^{22}p^{40} + 37031904q^{20}p^{40} - 18257346q^{18}p^{40}\\
&- 665640q^{16}p^{40} + 2022900q^{14}p^{40} - 206696q^{12}p^{40} - 53130q^{10}p^{40}\\
&+ 177100q^{50}p^{38} + 149860q^{48}p^{38} - 5585020q^{46}p^{38} + 13654060q^{44}p^{38}\\
&+ 13418416q^{42}p^{38} - 91661296q^{40}p^{38} + 75728080q^{38}p^{38} + 184649200q^{36}p^{38}\\
&- 436698520q^{34}p^{38} + 246168120q^{32}p^{38} + 246168120q^{30}p^{38} - 436698520q^{28}p^{38}\\
&+ 184649200q^{26}p^{38} + 75728080q^{24}p^{38} - 91661296q^{22}p^{38} + 13418416q^{20}p^{38}\\
&+ 13654060q^{18}p^{38} - 5585020q^{16}p^{38} + 149860q^{14}p^{38} + 177100q^{12}p^{38}\\
&- 480700q^{50}p^{36} + 759400q^{48}p^{36} + 7037860q^{46}p^{36} - 35295808q^{44}p^{36}\\
&+ 36444112q^{42}p^{36} + 103363936q^{40}p^{36} - 280626160q^{38}p^{36} + 125331520q^{36}p^{36}\\
&+ 375512632q^{34}p^{36} - 664093584q^{32}p^{36} + 375512632q^{30}p^{36} + 125331520q^{28}p^{36}\\
&- 280626160q^{26}p^{36} + 103363936q^{24}p^{36} + 36444112q^{22}p^{36} - 35295808q^{20}p^{36}\\
&+ 7037860q^{18}p^{36} + 759400q^{16}p^{36} - 480700q^{14}p^{36} + 1081575q^{50}p^{34} - 2428585q^{48}p^{34}\\
&+ 394872q^{46}p^{34} + 39902808q^{44}p^{34} - 120569836q^{42}p^{34} + 29963316q^{40}p^{34}\\
&+ 313362600q^{38}p^{34} - 491940088q^{36}p^{34} + 230233338q^{34}p^{34} + 230233338q^{32}p^{34}\\
&- 491940088q^{30}p^{34} + 313362600q^{28}p^{34} + 29963316q^{26}p^{34} - 120569836q^{24}p^{34}\\
&+ 39902808q^{22}p^{34} + 394872q^{20}p^{34} - 2428585q^{18}p^{34} + 1081575q^{16}p^{34} - 2042975q^{50}p^{32}\\
&+ 2669392q^{48}p^{32} - 15047112q^{46}p^{32} + 2375920q^{44}p^{32} + 154838044q^{42}p^{32}\\
&- 236543664q^{40}p^{32} - 53109368q^{38}p^{32} + 346169968q^{36}p^{32} - 398620410q^{34}p^{32}\\
&+ 346169968q^{32}p^{32} - 53109368q^{30}p^{32} - 236543664q^{28}p^{32} + 154838044q^{26}p^{32}\\
&+ 2375920q^{24}p^{32} - 15047112q^{22}p^{32} + 2669392q^{20}p^{32} - 2042975q^{18}p^{32} + 3268760q^{50}p^{30}\\
&+ 1087848q^{48}p^{30} + 20287320q^{46}p^{30} - 76899160q^{44}p^{30} - 71534856q^{42}p^{30}\\
&+ 313092360q^{40}p^{30} - 159756232q^{38}p^{30} - 29546040q^{36}p^{30} - 29546040q^{34}p^{30}\\
&- 159756232q^{32}p^{30} + 313092360q^{30}p^{30} - 71534856q^{28}p^{30} - 76899160q^{26}p^{30}\\
&+ 20287320q^{24}p^{30} + 1087848q^{22}p^{30} + 3268760q^{20}p^{30} - 4457400q^{50}p^{28} - 7412560q^{48}p^{28}\\
&- 1695720q^{46}p^{28} + 120188640q^{44}p^{28} - 74228216q^{42}p^{28} - 221648304q^{40}p^{28}\\
&+ 159286680q^{38}p^{28} + 59933760q^{36}p^{28} + 159286680q^{34}p^{28} - 221648304q^{32}p^{28}\\
&- 74228216q^{30}p^{28} + 120188640q^{28}p^{28} - 1695720q^{26}p^{28} - 7412560q^{24}p^{28}\\
&- 4457400q^{22}p^{28} + 5200300q^{50}p^{26} + 10645180q^{48}p^{26} - 29683160q^{46}p^{26}\\
&- 90236952q^{44}p^{26} + 161992548q^{42}p^{26} + 84440244q^{40}p^{26} - 142358160q^{38}p^{26}\\
&- 142358160q^{36}p^{26} + 84440244q^{34}p^{26} + 161992548q^{32}p^{26} - 90236952q^{30}p^{26}\\
&- 29683160q^{28}p^{26} + 10645180q^{26}p^{26} + 5200300q^{24}p^{26} - 5200300q^{50}p^{24}\\
&- 7412560q^{48}p^{24} + 46020968q^{46}p^{24} + 19483632q^{44}p^{24} - 142747284q^{42}p^{24}\\
&+ 20291424q^{40}p^{24} + 139128240q^{38}p^{24} + 20291424q^{36}p^{24} - 142747284q^{34}p^{24}\\
&+ 19483632q^{32}p^{24} + 46020968q^{30}p^{24} - 7412560q^{28}p^{24} - 5200300q^{26}p^{24}\\
&+ 4457400q^{50}p^{22} + 1087848q^{48}p^{22} - 35872248q^{46}p^{22} + 28292760q^{44}p^{22}\\
&+ 66574896q^{42}p^{22} - 64540656q^{40}p^{22} - 64540656q^{38}p^{22} + 66574896q^{36}p^{22}
\end{aligned}
$$

$$+ 28292760q^{34}p^{22} - 35872248q^{32}p^{22} + 1087848q^{30}p^{22} + 4457400q^{28}p^{22}$$
$$- 3268760q^{50}p^{20} + 2669392q^{48}p^{20} + 14437960q^{46}p^{20} - 31013440q^{44}p^{20}$$
$$- 6004144q^{42}p^{20} + 46357984q^{40}p^{20} - 6004144q^{38}p^{20} - 31013440q^{36}p^{20}$$
$$+ 14437960q^{34}p^{20} + 2669392q^{32}p^{20} - 3268760q^{30}p^{20} + 2042975q^{50}p^{18}$$
$$- 2428585q^{48}p^{18} - 518980q^{46}p^{18} + 13270540q^{44}p^{18} - 12365950q^{42}p^{18}$$
$$- 12365950q^{40}p^{18} + 13270540q^{38}p^{18} - 518980q^{36}p^{18} - 2428585q^{34}p^{18}$$
$$+ 2042975q^{32}p^{18} - 1081575q^{50}p^{16} + 759400q^{48}p^{16} - 2428260q^{46}p^{16} - 976296q^{44}p^{16}$$
$$+ 7453462q^{42}p^{16} - 976296q^{40}p^{16} - 2428260q^{38}p^{16} + 759400q^{36}p^{16} - 1081575q^{34}p^{16}$$
$$+ 480700q^{50}p^{14} + 149860q^{48}p^{14} + 1005164q^{46}p^{14} - 1635724q^{44}p^{14} - 1635724q^{42}p^{14}$$
$$+ 1005164q^{40}p^{14} + 149860q^{38}p^{14} + 480700q^{36}p^{14} - 177100q^{50}p^{12} - 206696q^{48}p^{12}$$
$$+ 5836q^{46}p^{12} + 755920q^{44}p^{12} + 5836q^{42}p^{12} - 206696q^{40}p^{12} - 177100q^{38}p^{12}$$
$$+ 53130q^{50}p^{10} + 64306q^{48}p^{10} - 117180q^{46}p^{10} - 117180q^{44}p^{10} + 64306q^{42}p^{10}$$
$$+ 53130q^{40}p^{10} - 12650q^{50}p^{8} - 4040q^{48}p^{8} + 32100q^{46}p^{8} - 4040q^{44}p^{8} - 12650q^{42}p^{8}$$
$$+ 2300q^{50}p^{6} - 1740q^{48}p^{6} - 1740q^{46}p^{6} + 2300q^{44}p^{6} - 300q^{50}p^{4} + 200q^{48}p^{4} - 300q^{46}p^{4}$$
$$+ 25q^{50}p^{2} + 25q^{48}p^{2} - q^{50}.$$

And so on.

Of course, starting from $t_1 = t_M$ and using the rule given by (1.8) in the article, we can obtain directly the above expressions for tangent lengths.

Now, using the above expressions we list obtained Fuss' relations for bicentric n-gons where $n \geq 4$ is an even integer and $n \leq 20$.

1. Let $n = 4$. Then from $t_3 - t_m = 0$, that is, from

$$\frac{\left(R_0^2 - d_0^2\right)^2 - 4R_0 d_0 r_0^2}{\left(R_0^2 - d_0^2\right)^2 + 4R_0 d_0 r_0^2} \cdot t_M = t_m,$$

it follows

$$t_M^2 \left(\frac{\left(R_0^2 - d_0^2\right)^2 - 4R_0 d_0 r_0^2}{\left(R_0^2 - d_0^2\right)^2 + 4R_0 d_0 r_0^2} \right)^2 = t_m^2,$$

which can be written as

$$4R_0 d_0 \left[\left(R_0^2 - d_0\right)^4 + 16R_0^2 d_0^2 r_0^4 - 4r_0^2 \left(R_0^2 - d_0^2\right)^2 \left(R_0^2 + d_0^2 - r_0^2\right) \right] = 0$$

or

$$4R_0 d_0 \left[\left(R_0^2 - d_0^2\right)^2 - 2r_0^2 \left(R_0^2 + d_0^2\right) \right]^2 = 0,$$

where $\left(R^2 - d^2\right)^2 - 2r^2\left(R^2 + d^2\right) = 0$ is Fuss' relation $F_4(R, d, r) = 0$.

Thus in the case when $n = 4$, Fuss' relation $F_4(R, d, r) = 0$ can be easily obtained even by hand (without using computer algebra). It seems that for even $n > 20$ needs a powerful computer.

Using notation given by (2.4), Fuss' relation $F_4(R, r, d) = 0$ can be written as $p^2 + q^2 - p^2q^2 = 0$.

2. Let $n = 6$. Then from

$$t_4 - t_m = 0$$

it follows

$$4dr^2R \cdot F_6(R,d,r)^2 = 0,$$

where

$$F_6(R,d,r) = p^4 - 2p^2q^2 + 2p^4q^2 + q^4 + 2p^2q^4 - 3p^4q^4.$$

3. Let $n = 8$. Then from

$$t_5 - t_m = 0$$

it follows

$$4dR \cdot F_8(R,d,r)^2 = 0,$$

where

$$\begin{aligned} F_8(R,d,r) = {} & p^8q^8 - 4p^8q^6 + 6p^8q^4 - 4p^8q^2 + p^8 - 4p^6q^8 - 4p^6q^6 + 4p^6q^4 \\ & + 4p^6q^2 + 6p^4q^8 + 4p^4q^6 - 10p^4q^4 - 4p^2q^8 + 4p^2q^6 + q^8. \end{aligned}$$

4. Let $n = 10$. Then from

$$t_6 - t_m = 0$$

it follows

$$4dr^2R \cdot F_{10}(R,d,r)^2 = 0,$$

where

$$\begin{aligned} F_{10}(R,d,r) = {} & -5p^{12}q^{12} + 10p^{12}q^{10} + 9p^{12}q^8 - 36p^{12}q^6 + 29p^{12}q^4 - 6p^{12}q^2 \\ & - p^{12} + 10p^{10}q^{12} - 34p^{10}q^{10} + 36p^{10}q^8 - 4p^{10}q^6 - 14p^{10}q^4 + 6p^{10}q^2 \\ & + 9p^8q^{12} + 36p^8q^{10} - 50p^8q^8 + 20p^8q^6 - 15p^8q^4 - 36p^6q^{12} - 4p^6q^{10} \\ & + 20p^6q^8 + 20p^6q^6 + 29p^4q^{12} - 14p^4q^{10} - 15p^4q^8 - 6p^2q^{12} + 6p^2q^{10} - q^{12}. \end{aligned}$$

5. Let $n = 12$. Then from

$$t_7 - t_m = 0$$

it follows

$$4dR \cdot F_4(R,d,r)^2 \cdot F_{12}(R,d,r)^2 = 0,$$

where

$$
\begin{aligned}
F_{12}(R,d,r) &= p^{16}q^{16} - 8p^{16}q^{14} + 28p^{16}q^{12} - 56p^{16}q^{10} + 70p^{16}q^{8} - 56p^{16}q^{6} \\
&+ 28p^{16}q^{4} - 8p^{16}q^{2} + p^{16} - 8p^{14}q^{16} - 40p^{14}q^{14} + 56p^{14}q^{12} \\
&+ 88p^{14}q^{10} - 88p^{14}q^{8} - 56p^{14}q^{6} + 40p^{14}q^{4} + 8p^{14}q^{2} + 28p^{12}q^{16} \\
&+ 56p^{12}q^{14} - 316p^{12}q^{12} + 144p^{12}q^{10} + 228p^{12}q^{8} - 72p^{12}q^{6} \\
&- 68p^{12}q^{4} - 56p^{10}q^{16} + 88p^{10}q^{14} + 144p^{10}q^{12} - 400p^{10}q^{10} \\
&+ 40p^{10}q^{8} + 184p^{10}q^{6} + 70p^{8}q^{16} - 88p^{8}q^{14} + 228p^{8}q^{12} + 40p^{8}q^{10} \\
&- 250p^{8}q^{8} - 56p^{6}q^{16} - 56p^{6}q^{14} - 72p^{6}q^{12} + 184p^{6}q^{10} + 28p^{4}q^{16} \\
&+ 40p^{4}q^{14} - 68p^{4}q^{12} - 8p^{2}q^{16} + 8p^{2}q^{14} + q^{16}.
\end{aligned}
$$

6. Let $n = 14$. Then from

$$t_8 - t_m = 0$$

it follows

$$4dr^2R \cdot F_{14}(R,d,r)^2 = 0,$$

where

$$
\begin{aligned}
F_{14}(R,d,r) &= 7p^{24}q^{24} - 28p^{24}q^{22} - 42p^{24}q^{20} + 484p^{24}q^{18} - 1311p^{24}q^{16} \\
&+ 1800p^{24}q^{14} - 1260p^{24}q^{12} + 168p^{24}q^{10} + 441p^{24}q^{8} - 364p^{24}q^{6} \\
&+ 118p^{24}q^{4} - 12p^{24}q^{2} - p^{24} - 28p^{22}q^{24} + 196p^{22}q^{22} - 516p^{22}q^{20} \\
&+ 444p^{22}q^{18} + 744p^{22}q^{16} - 2520p^{22}q^{14} + 3192p^{22}q^{12} - 2184p^{22}q^{10} \\
&+ 756p^{22}q^{8} - 44p^{22}q^{6} - 52p^{22}q^{4} + 12p^{22}q^{2} - 42p^{20}q^{24} - 516p^{20}q^{22} \\
&+ 1734p^{20}q^{20} - 2544p^{20}q^{18} + 3756p^{20}q^{16} - 3928p^{20}q^{14} + 700p^{20}q^{12} \\
&+ 1680p^{20}q^{10} - 962p^{20}q^{8} + 188p^{20}q^{6} - 66p^{20}q^{4} + 484p^{18}q^{24} \\
&+ 444p^{18}q^{22} - 2544p^{18}q^{20} + 48p^{18}q^{18} + 568p^{18}q^{16} + 5704p^{18}q^{14} \\
&- 5936p^{18}q^{12} + 752p^{18}q^{10} + 260p^{18}q^{8} + 220p^{18}q^{6} - 1311p^{16}q^{24} \\
&+ 744p^{16}q^{22} + 3756p^{16}q^{20} + 568p^{16}q^{18} - 9322p^{16}q^{16} + 3864p^{16}q^{14} \\
&+ 3404p^{16}q^{12} - 1208p^{16}q^{10} - 495p^{16}q^{8} + 1800p^{14}q^{24} - 2520p^{14}q^{22} \\
&- 3928p^{14}q^{20} + 5704p^{14}q^{18} + 3864p^{14}q^{16} - 6536p^{14}q^{14} + 824p^{14}q^{12} \\
&+ 792p^{14}q^{10} - 1260p^{12}q^{24} + 3192p^{12}q^{22} + 700p^{12}q^{20} - 5936p^{12}q^{18} \\
&+ 3404p^{12}q^{16} + 824p^{12}q^{14} - 924p^{12}q^{12} + 168p^{10}q^{24} - 2184p^{10}q^{22} \\
&+ 1680p^{10}q^{20} + 752p^{10}q^{18} - 1208p^{10}q^{16} + 792p^{10}q^{14} + 441p^{8}q^{24} \\
&+ 756p^{8}q^{22} - 962p^{8}q^{20} + 260p^{8}q^{18} - 495p^{8}q^{16} - 364p^{6}q^{24} - 44p^{6}q^{22} \\
&+ 188p^{6}q^{20} + 220p^{6}q^{18} + 118p^{4}q^{24} - 52p^{4}q^{22} - 66p^{4}q^{20} - 12p^{2}q^{24} \\
&+ 12p^{2}q^{22} - q^{24}.
\end{aligned}
$$

7. Let $n = 16$. Then from

$$t_9 - t_m = 0$$

it follows

$$4dR \cdot F_{16}(R,d,r)^2 = 0,$$

where

$$\begin{aligned}
F_{16}(R,d,r) = {} & q^{32}p^{32} - 16q^{30}p^{32} + 120q^{28}p^{32} - 560q^{26}p^{32} + 1820q^{24}p^{32} - 4368q^{22}p^{32} \\
& + 8008q^{20}p^{32} - 11440q^{18}p^{32} + 12870q^{16}p^{32} - 11440q^{14}p^{32} + 8008q^{12}p^{32} \\
& - 4368q^{10}p^{32} + 1820q^{8}p^{32} - 560q^{6}p^{32} + 120q^{4}p^{32} - 16q^{2}p^{32} + p^{32} \\
& - 16q^{32}p^{30} - 80q^{30}p^{30} + 432q^{28}p^{30} + 752q^{26}p^{30} - 6096q^{24}p^{30} \\
& + 9712q^{22}p^{30} - 16q^{20}p^{30} - 16080q^{18}p^{30} + 16080q^{16}p^{30} + 16q^{14}p^{30} \\
& - 9712q^{12}p^{30} + 6096q^{10}p^{30} - 752q^{8}p^{30} - 432q^{6}p^{30} + 80q^{4}p^{30} \\
& + 16q^{2}p^{30} + 120q^{32}p^{28} + 432q^{30}p^{28} - 5016q^{28}p^{28} + 10464q^{26}p^{28} \\
& + 2488q^{24}p^{28} - 33712q^{22}p^{28} + 31464q^{20}p^{28} + 29760q^{18}p^{28} \\
& - 81240q^{16}p^{28} + 63312q^{14}p^{28} - 16008q^{12}p^{28} - 4896q^{10}p^{28} \\
& + 2856q^{8}p^{28} + 176q^{6}p^{28} - 200q^{4}p^{28} - 560q^{32}p^{26} + 752q^{30}p^{26} \\
& + 10464q^{28}p^{26} - 40416q^{26}p^{26} + 45168q^{24}p^{26} + 34128q^{22}p^{26} \\
& - 136896q^{20}p^{26} + 127680q^{18}p^{26} - 18000q^{16}p^{26} - 51568q^{14}p^{26} \\
& + 36832q^{12}p^{26} - 5856q^{10}p^{26} - 2544q^{8}p^{26} + 816q^{6}p^{26} + 1820q^{32}p^{24} \\
& - 6096q^{30}p^{24} + 2488q^{28}p^{24} + 45168q^{26}p^{24} - 124764q^{24}p^{24} \\
& + 102496q^{22}p^{24} + 52752q^{20}p^{24} - 163232q^{18}p^{24} + 133668q^{16}p^{24} \\
& - 45200q^{14}p^{24} - 7240q^{12}p^{24} + 9520q^{10}p^{24} - 1380q^{8}p^{24} \\
& - 4368q^{32}p^{22} + 9712q^{30}p^{22} - 33712q^{28}p^{22} + 34128q^{26}p^{22} \\
& + 102496q^{24}p^{22} - 214432q^{22}p^{22} + 132000q^{20}p^{22} - 32352q^{18}p^{22} \\
& - 13392q^{16}p^{22} + 39600q^{14}p^{22} - 19184q^{12}p^{22} - 496q^{10}p^{22} \\
& + 8008q^{32}p^{20} - 16q^{30}p^{20} + 31464q^{28}p^{20} - 136896q^{26}p^{20} \\
& + 52752q^{24}p^{20} + 132000q^{22}p^{20} - 83312q^{20}p^{20} + 27968q^{18}p^{20} \\
& - 61272q^{16}p^{20} + 22000q^{14}p^{20} + 7304q^{12}p^{20} - 11440q^{32}p^{18} \\
& - 16080q^{30}p^{18} + 29760q^{28}p^{18} + 127680q^{26}p^{18} - 163232q^{24}p^{18} \\
& - 32352q^{22}p^{18} + 27968q^{20}p^{18} + 64448q^{18}p^{18} - 10032q^{16}p^{18}
\end{aligned}$$

$$
\begin{aligned}
&- 16720q^{14}p^{18} + 12870q^{32}p^{16} + 16080q^{30}p^{16} - 81240q^{28}p^{16} \\
&- 18000q^{26}p^{16} + 133668q^{24}p^{16} - 13392q^{22}p^{16} - 61272q^{20}p^{16} \\
&- 10032q^{18}p^{16} + 21318q^{16}p^{16} - 11440q^{32}p^{14} + 16q^{30}p^{14} \\
&+ 63312q^{28}p^{14} - 51568q^{26}p^{14} - 45200q^{24}p^{14} + 39600q^{22}p^{14} \\
&+ 22000q^{20}p^{14} - 16720q^{18}p^{14} + 8008q^{32}p^{12} - 9712q^{30}p^{12} \\
&- 16008q^{28}p^{12} + 36832q^{26}p^{12} - 7240q^{24}p^{12} - 19184q^{22}p^{12} \\
&+ 7304q^{20}p^{12} - 4368q^{32}p^{10} + 6096q^{30}p^{10} - 4896q^{28}p^{10} \\
&- 5856q^{26}p^{10} + 9520q^{24}p^{10} - 496q^{22}p^{10} + 1820q^{32}p^{8} \\
&- 752q^{30}p^{8} + 2856q^{28}p^{8} - 2544q^{26}p^{8} - 1380q^{24}p^{8} - 560q^{32}p^{6} \\
&- 432q^{30}p^{6} + 176q^{28}p^{6} + 816q^{26}p^{6} + 120q^{32}p^{4} + 80q^{30}p^{4} \\
&- 200q^{28}p^{4} - 16q^{32}p^{2} + 16q^{30}p^{2} + q^{32}.
\end{aligned}
$$

8. Let $n = 18$. Then from

$$t_{10} - t_m = 0$$

it follows

$$4dr^2R \cdot F_6(R,d,r)^2 \cdot F_{18}(R,d,r)^2 = 0,$$

where

$$
\begin{aligned}
F_{18}(R,d,r) = {}& 3q^{36}p^{36} - 18q^{34}p^{36} - 57q^{32}p^{36} + 976q^{30}p^{36} - 4740q^{28}p^{36} + 12936q^{26}p^{36} \\
&- 21476q^{24}p^{36} + 18096q^{22}p^{36} + 7722q^{20}p^{36} - 48620q^{18}p^{36} + 79794q^{16}p^{36} \\
&- 81744q^{14}p^{36} + 58604q^{12}p^{36} - 30072q^{10}p^{36} + 10860q^{8}p^{36} - 2608q^{6}p^{36} \\
&+ 363q^{4}p^{36} - 18q^{2}p^{36} - p^{36} - 18q^{36}p^{34} + 210q^{34}p^{34} - 1008q^{32}p^{34} \\
&+ 2160q^{30}p^{34} + 840q^{28}p^{34} - 19656q^{26}p^{34} + 65520q^{24}p^{34} - 130416q^{22}p^{34} \\
&+ 180180q^{20}p^{34} - 180180q^{18}p^{34} + 130416q^{16}p^{34} - 65520q^{14}p^{34} + 19656q^{12}p^{34} \\
&- 840q^{10}p^{34} - 2160q^{8}p^{34} + 1008q^{6}p^{34} - 210q^{4}p^{34} + 18q^{2}p^{34} - 57q^{36}p^{32} \\
&- 1008q^{34}p^{32} + 5160q^{32}p^{32} - 13776q^{30}p^{32} + 42532q^{28}p^{32} - 99824q^{26}p^{32} \\
&+ 106840q^{24}p^{32} + 19376q^{22}p^{32} - 175798q^{20}p^{32} + 213680q^{18}p^{32} \\
&- 172648q^{16}p^{32} + 136976q^{14}p^{32} - 92956q^{12}p^{32} + 38704q^{10}p^{32} - 7832q^{8}p^{32} \\
&+ 784q^{6}p^{32} - 153q^{4}p^{32} + 976q^{36}p^{30} + 2160q^{34}p^{30} - 13776q^{32}p^{30} \\
&- 2800q^{30}p^{30} + 16208q^{28}p^{30} + 180464q^{26}p^{30} - 521936q^{24}p^{30} + 513040q^{22}p^{30} \\
&- 155408q^{20}p^{30} + 35280q^{18}p^{30} - 187376q^{16}p^{30} + 193456q^{14}p^{30} - 58384q^{12}p^{30} \\
&- 4912q^{10}p^{30} + 2192q^{8}p^{30} + 816q^{6}p^{30} - 4740q^{36}p^{28} + 840q^{34}p^{28} \\
&+ 42532q^{32}p^{28} + 16208q^{30}p^{28} - 329220q^{28}p^{28} + 371000q^{26}p^{28} + 352516q^{24}p^{28} \\
&- 893856q^{22}p^{28} + 363860q^{20}p^{28} + 299320q^{18}p^{28} - 213300q^{16}p^{28} - 76976q^{14}p^{28}
\end{aligned}
$$

$$+ 86324q^{12}p^{28} - 11448q^{10}p^{28} - 3060q^{8}p^{28} + 12936q^{36}p^{26} - 19656q^{34}p^{26}$$
$$- 99824q^{32}p^{26} + 180464q^{30}p^{26} + 371000q^{28}p^{26} - 1178744q^{26}p^{26} + 786912q^{24}p^{26}$$
$$+ 517664q^{22}p^{26} - 806792q^{20}p^{26} + 18120q^{18}p^{26} + 385808q^{16}p^{26} - 181776q^{14}p^{26}$$
$$+ 5320q^{12}p^{26} + 8568q^{10}p^{26} - 21476q^{36}p^{24} + 65520q^{34}p^{24} + 106840q^{32}p^{24}$$
$$- 521936q^{30}p^{24} + 352516q^{28}p^{24} + 786912q^{26}p^{24} - 1474480q^{24}p^{24} + 568288q^{22}p^{24}$$
$$+ 573924q^{20}p^{24} - 573904q^{18}p^{24} + 112600q^{16}p^{24} + 43760q^{14}p^{24} - 18564q^{12}p^{24}$$
$$+ 18096q^{36}p^{22} - 130416q^{34}p^{22} + 19376q^{32}p^{22} + 513040q^{30}p^{22} - 893856q^{28}p^{22}$$
$$+ 517664q^{26}p^{22} + 568288q^{24}p^{22} - 978528q^{22}p^{22} + 289520q^{20}p^{22} + 136528q^{18}p^{22}$$
$$- 91536q^{16}p^{22} + 31824q^{14}p^{22} + 7722q^{36}p^{20} + 180180q^{34}p^{20} - 175798q^{32}p^{20}$$
$$- 155408q^{30}p^{20} + 363860q^{28}p^{20} - 806792q^{26}p^{20} + 573924q^{24}p^{20} + 289520q^{22}p^{20}$$
$$- 284606q^{20}p^{20} + 51156q^{18}p^{20} - 43758q^{16}p^{20} - 48620q^{36}p^{18} - 180180q^{34}p^{18}$$
$$+ 213680q^{32}p^{18} + 35280q^{30}p^{18} + 299320q^{28}p^{18} + 18120q^{26}p^{18} - 573904q^{24}p^{18}$$
$$+ 136528q^{22}p^{18} + 51156q^{20}p^{18} + 48620q^{18}p^{18} + 79794q^{36}p^{16} + 130416q^{34}p^{16}$$
$$- 172648q^{32}p^{16} - 187376q^{30}p^{16} - 213300q^{28}p^{16} + 385808q^{26}p^{16} + 112600q^{24}p^{16}$$
$$- 91536q^{22}p^{16} - 43758q^{20}p^{16} - 81744q^{36}p^{14} - 65520q^{34}p^{14} + 136976q^{32}p^{14}$$
$$+ 193456q^{30}p^{14} - 76976q^{28}p^{14} - 181776q^{26}p^{14} + 43760q^{24}p^{14} + 31824q^{22}p^{14}$$
$$+ 58604q^{36}p^{12} + 19656q^{34}p^{12} - 92956q^{32}p^{12} - 58384q^{30}p^{12} + 86324q^{28}p^{12}$$
$$+ 5320q^{26}p^{12} - 18564q^{24}p^{12} - 30072q^{36}p^{10} - 840q^{34}p^{10} + 38704q^{32}p^{10}$$
$$- 4912q^{30}p^{10} - 11448q^{28}p^{10} + 8568q^{26}p^{10} + 10860q^{36}p^{8} - 2160q^{34}p^{8}$$
$$- 7832q^{32}p^{8} + 2192q^{30}p^{8} - 3060q^{28}p^{8} - 2608q^{36}p^{6} + 1008q^{34}p^{6} + 784q^{32}p^{6}$$
$$+ 816q^{30}p^{6} + 363q^{36}p^{4} - 210q^{34}p^{4} - 153q^{32}p^{4} - 18q^{36}p^{2} + 18q^{34}.$$

9. Let $n = 20$. Then from

$$t_{11} - t_m = 0$$

it follows

$$4dR \cdot F_4(R,d,r)^2 \cdot F_{20}(R,d,r)^2 = 0,$$

where

$$F_{20}(R,d,r) = q^{48}p^{48} - 24q^{46}p^{48} + 276q^{44}p^{48} - 2024q^{42}p^{48} + 10626q^{40}p^{48} - 42504q^{38}p^{48}$$
$$+ 134596q^{36}p^{48} - 346104q^{34}p^{48} + 735471q^{32}p^{48} - 1307504q^{30}p^{48}$$
$$+ 1961256q^{28}p^{48} - 2496144q^{26}p^{48} + 2704156q^{24}p^{48} - 2496144q^{22}p^{48}$$
$$+ 1961256q^{20}p^{48} - 1307504q^{18}p^{48} + 735471q^{16}p^{48} - 346104q^{14}p^{48}$$
$$+ 134596q^{12}p^{48} - 42504q^{10}p^{48} + 10626q^{8}p^{48} - 2024q^{6}p^{48} + 276q^{4}p^{48}$$
$$- 24q^{2}p^{48} + p^{48} - 24q^{48}p^{46} - 248q^{46}p^{46} + 1768q^{44}p^{46} + 3784q^{42}p^{46}$$
$$- 49896q^{40}p^{46} + 114296q^{38}p^{46} + 99032q^{36}p^{46} - 1006472q^{34}p^{46}$$
$$+ 2157584q^{32}p^{46} - 1819312q^{30}p^{46} - 945904q^{28}p^{46} + 3970512q^{26}p^{46}$$
$$- 3970512q^{24}p^{46} + 945904q^{22}p^{46} + 1819312q^{20}p^{46} - 2157584q^{18}p^{46}$$

$$
\begin{aligned}
&+1006472q^{16}p^{46}-99032q^{14}p^{46}-114296q^{12}p^{46}+49896q^{10}p^{46}\\
&-3784q^{8}p^{46}-1768q^{6}p^{46}+248q^{4}p^{46}+24q^{2}p^{46}+276q^{48}p^{44}+1768q^{46}p^{44}\\
&-28564q^{44}p^{44}+92400q^{42}p^{44}+36668q^{40}p^{44}-854200q^{38}p^{44}+1673092q^{36}p^{44}\\
&+1185600q^{34}p^{44}-11094776q^{32}p^{44}+22958160q^{30}p^{44}-24008712q^{28}p^{44}\\
&+9644960q^{26}p^{44}+7370168q^{24}p^{44}-11971248q^{22}p^{44}+4895176q^{20}p^{44}\\
&+2342720q^{18}p^{44}-3688668q^{16}p^{44}+1797320q^{14}p^{44}-339876q^{12}p^{44}\\
&-31504q^{10}p^{44}+19916q^{8}p^{44}-152q^{6}p^{44}-524q^{4}p^{44}-2024q^{48}p^{42}+3784q^{46}p^{42}\\
&+92400q^{44}p^{42}-571120q^{42}p^{42}+1101272q^{40}p^{42}+1223368q^{38}p^{42}\\
&-10374080q^{36}p^{42}+21824960q^{34}p^{42}-17562576q^{32}p^{42}-14088048q^{30}p^{42}\\
&+52140192q^{28}p^{42}-58403488q^{26}p^{42}+25865840q^{24}p^{42}+11518672q^{22}p^{42}\\
&-23488960q^{20}p^{42}+14149568q^{18}p^{42}-3193160q^{16}p^{42}-730200q^{14}p^{42}\\
&+599024q^{12}p^{42}-97264q^{10}p^{42}-12104q^{8}p^{42}+3944q^{6}p^{42}+10626q^{48}p^{40}\\
&-49896q^{46}p^{40}+36668q^{44}p^{40}+1101272q^{42}p^{40}-5250918q^{40}p^{40}\\
&+8338400q^{38}p^{40}+3968464q^{36}p^{40}-40781472q^{34}p^{40}+84403236q^{32}p^{40}\\
&-91677360q^{30}p^{40}+34691048q^{28}p^{40}+47822416q^{26}p^{40}-81149788q^{24}p^{40}\\
&+50938720q^{22}p^{40}-8994352q^{20}p^{40}-8263200q^{18}p^{40}+6800986q^{16}p^{40}\\
&-2136488q^{14}p^{40}+101692q^{12}p^{40}+104600q^{10}p^{40}-14654q^{8}p^{40}-42504q^{48}p^{38}\\
&+114296q^{46}p^{38}-854200q^{44}p^{38}+1223368q^{42}p^{38}+8338400q^{40}p^{38}\\
&-29681184q^{38}p^{38}+38827936q^{36}p^{38}-22244960q^{34}p^{38}-22281968q^{32}p^{38}\\
&+107688976q^{30}p^{38}-170712336q^{28}p^{38}+113653744q^{26}p^{38}+2540640q^{24}p^{38}\\
&-49884576q^{22}p^{38}+32782368q^{20}p^{38}-12512736q^{18}p^{38}+2841784q^{16}p^{38}\\
&+678200q^{14}p^{38}-492024q^{12}p^{38}+16776q^{10}p^{38}+134596q^{48}p^{36}+99032q^{46}p^{36}\\
&+1673092q^{44}p^{36}-10374080q^{42}p^{36}+3968464q^{40}p^{36}+38827936q^{38}p^{36}\\
&-61472368q^{36}p^{36}+58640832q^{34}p^{36}-122927816q^{32}p^{36}+144517392q^{30}p^{36}\\
&+26914424q^{28}p^{36}-172794432q^{26}p^{36}+110372368q^{24}p^{36}-15240288q^{22}p^{36}\\
&+1034064q^{20}p^{36}-1730496q^{18}p^{36}-3323612q^{16}p^{36}+1570008q^{14}p^{36}\\
&+110884q^{12}p^{36}-346104q^{48}p^{34}-1006472q^{46}p^{34}+1185600q^{44}p^{34}\\
&+21824960q^{42}p^{34}-40781472q^{40}p^{34}-22244960q^{38}p^{34}+58640832q^{36}p^{34}\\
&+57347904q^{34}p^{34}-36033872q^{32}p^{34}-237686448q^{30}p^{34}+281168064q^{28}p^{34}\\
&-16957888q^{26}p^{34}-91234720q^{24}p^{34}+13829792q^{22}p^{34}+7737920q^{20}p^{34}\\
&+8941760q^{18}p^{34}-3651192q^{16}p^{34}-733704q^{14}p^{34}+735471q^{48}p^{32}\\
&+2157584q^{46}p^{32}-11094776q^{44}p^{32}-17562576q^{42}p^{32}+84403236q^{40}p^{32}\\
&-22281968q^{38}p^{32}-122927816q^{36}p^{32}-36033872q^{34}p^{32}+326552666q^{32}p^{32}\\
&-141367120q^{30}p^{32}-235711688q^{28}p^{32}+184028944q^{26}p^{32}+29480484q^{24}p^{32}\\
&-29251024q^{22}p^{32}-19940600q^{20}p^{32}+6341136q^{18}p^{32}+2471919q^{16}p^{32}\\
&-1307504q^{48}p^{30}-1819312q^{46}p^{30}+22958160q^{44}p^{30}-14088048q^{42}p^{30}
\end{aligned}
$$

$$
\begin{aligned}
&- 91677360q^{40}p^{30} + 107688976q^{38}p^{30} + 144517392q^{36}p^{30} - 237686448q^{34}p^{30}\\
&- 141367120q^{32}p^{30} + 381359344q^{30}p^{30} - 100520464q^{28}p^{30} - 138524496q^{26}p^{30}\\
&+ 45513072q^{24}p^{30} + 38897072q^{22}p^{30} - 8139600q^{20}p^{30} - 5803664q^{18}p^{30}\\
&+ 1961256q^{48}p^{28} - 945904q^{46}p^{28} - 24008712q^{44}p^{28} + 52140192q^{42}p^{28}\\
&+ 34691048q^{40}p^{28} - 170712336q^{38}p^{28} + 26914424q^{36}p^{28} + 281168064q^{34}p^{28}\\
&- 235711688q^{32}p^{28} - 100520464q^{30}p^{28} + 201802920q^{28}p^{28} - 23536480q^{26}p^{28}\\
&- 60743752q^{24}p^{28} + 7168016q^{22}p^{28} + 10333416q^{20}p^{28} - 2496144q^{48}p^{26}\\
&+ 3970512q^{46}p^{26} + 9644960q^{44}p^{26} - 58403488q^{42}p^{26} + 47822416q^{40}p^{26}\\
&+ 113653744q^{38}p^{26} - 172794432q^{36}p^{26} - 16957888q^{34}p^{26} + 184028944q^{32}p^{26}\\
&- 138524496q^{30}p^{26} - 23536480q^{28}p^{26} + 70936160q^{26}p^{26} - 2888912q^{24}p^{26}\\
&- 14454896q^{22}p^{26} + 2704156q^{48}p^{24} - 3970512q^{46}p^{24} + 7370168q^{44}p^{24}\\
&+ 25865840q^{42}p^{24} - 81149788q^{40}p^{24} + 2540640q^{38}p^{24} + 110372368q^{36}p^{24}\\
&- 91234720q^{34}p^{24} + 29480484q^{32}p^{24} + 45513072q^{30}p^{24} - 60743752q^{28}p^{24}\\
&- 2888912q^{26}p^{24} + 16140956q^{24}p^{24} - 2496144q^{48}p^{22} + 945904q^{46}p^{22}\\
&- 11971248q^{44}p^{22} + 11518672q^{42}p^{22} + 50938720q^{40}p^{22} - 49884576q^{38}p^{22}\\
&- 15240288q^{36}p^{22} + 13829792q^{34}p^{22} - 29251024q^{32}p^{22} + 38897072q^{30}p^{22}\\
&+ 7168016q^{28}p^{22} - 14454896q^{26}p^{22} + 1961256q^{48}p^{20} + 1819312q^{46}p^{20}\\
&+ 4895176q^{44}p^{20} - 23488960q^{42}p^{20} - 8994352q^{40}p^{20} + 32782368q^{38}p^{20}\\
&+ 1034064q^{36}p^{20} + 7737920q^{34}p^{20} - 19940600q^{32}p^{20} - 8139600q^{30}p^{20}\\
&+ 10333416q^{28}p^{20} - 1307504q^{48}p^{18} - 2157584q^{46}p^{18} + 2342720q^{44}p^{18}\\
&+ 14149568q^{42}p^{18} - 8263200q^{40}p^{18} - 12512736q^{38}p^{18} - 1730496q^{36}p^{18}\\
&+ 8941760q^{34}p^{18} + 6341136q^{32}p^{18} - 5803664q^{30}p^{18} + 735471q^{48}p^{16}\\
&+ 1006472q^{46}p^{16} - 3688668q^{44}p^{16} - 3193160q^{42}p^{16} + 6800986q^{40}p^{16}\\
&+ 2841784q^{38}p^{16} - 3323612q^{36}p^{16} - 3651192q^{34}p^{16} + 2471919q^{32}p^{16}\\
&- 346104q^{48}p^{14} - 99032q^{46}p^{14} + 1797320q^{44}p^{14} - 730200q^{42}p^{14}\\
&- 2136488q^{40}p^{14} + 678200q^{38}p^{14} + 1570008q^{36}p^{14} - 733704q^{34}p^{14}\\
&+ 134596q^{48}p^{12} - 114296q^{46}p^{12} - 339876q^{44}p^{12} + 599024q^{42}p^{12} + 101692q^{40}p^{12}\\
&- 492024q^{38}p^{12} + 110884q^{36}p^{12} - 42504q^{48}p^{10} + 49896q^{46}p^{10} - 31504q^{44}p^{10}\\
&- 97264q^{42}p^{10} + 104600q^{40}p^{10} + 16776q^{38}p^{10} + 10626q^{48}p^{8} - 3784q^{46}p^{8}\\
&+ 19916q^{44}p^{8} - 12104q^{42}p^{8} - 14654q^{40}p^{8} - 2024q^{48}p^{6} - 1768q^{46}p^{6} - 152q^{44}p^{6}\\
&+ 3944q^{42}p^{6} + 276q^{48}p^{4} + 248q^{46}p^{4} - 524q^{44}p^{4} - 24q^{48}p^{2} + 24q^{46}p^{2} + q^{48}.
\end{aligned}
$$

REMARK 2.1. From $t_{1+n/2} = t_m$ we can get two or more Fuss' relations. The reason is the following. Let there exist an even integer $k \geq 4$ and an integer $j \geq 1$ such that $1 + n/2 = 1 + k/2 + jk$ or $n = (1 + 2j)k$. Then we get not only Fuss' relation $F_n(R, r, d) = 0$ but also Fuss' relation $F_k(R, r, d) = 0$. So, for example, if $n = 12$ then $12 = k(1 + 2j)$ where $k = 4$ and $j = 1$, namely in this case we have $1+6=7$ and $1+2+4=7$, that is,

$$t_M, t_2, t_3, t_4, t_5, t_6, t_m, \text{ for } n=12 \text{ and}$$
$$\hat{t}_M, \hat{t}_2, \hat{t}_m, \hat{t}_2, \hat{t}_M, \hat{t}_2, \hat{t}_m, \text{ for } n=4.$$

Here is one more example. If $n=30$ then $k=6$, $j=2$ and $k=10$, $j=1$.

Here are some other properties which may be interesting. First note the fact that obtained Fuss' relation for an even n in Theorem 1 is not only for rotation number 1 but also for all rotation numbers k such that $k \in \mathbb{S}$, where

$$\mathbb{S} = \{x : x \in \{1, 2, \dots, \frac{n-2}{2}\} \text{ and } \gcd(x, n) = 1\}.$$

To see this let $t_1, \dots, t_n$ be lengths such that $t_i t_{i+\frac{n}{2}} = t_M t_m$ and let $\hat{t}_1, \dots, \hat{t}_n$ be given by $\hat{t}_i = t_{1+(i-1)k}, i = 1, \dots, n$. It is easy to see that also $\hat{t}_i \hat{t}_{i+\frac{n}{2}} = t_M t_m, i = 1, \dots, n$. Especially, if $\hat{t}_1 = t_M$ then $\hat{t}_{1+\frac{n}{2}} = t_m$. Here is an example. Let (R_0, r_0, d_0) be a solution of Fuss' relation $F_4(R, r, d) = 0$, say, $R_0 = 7$, $r_0 = 4.8$, $d_0 = 1$ and let R_1, r_1, d_1 and R_2, r_2, d_2 be given by

$$R_1^2 = R_0(R_0 + r_0 + \sqrt{(R_0 + r_0)^2 - d_0^2}), \quad r_1^2 = (R_0 + r_0)^2 - d_0^2,$$
$$d_1^2 = R_0(R_0 + r_0 - \sqrt{(R_0 + r_0)^2 - d_0^2})$$

and

$$R_2^2 = R_0(R_0 - r_0 + \sqrt{(R_0 - r_0)^2 - d_0^2}), \quad r_2^2 = (R_0 - r_0)^2 - d_0^2,$$
$$d_2^2 = R_0(R_0 - r_0 - \sqrt{(R_0 - r_0)^2 - d_0^2}).$$

Using computer algebra it is not difficult to show that $C_8^{(1)}(R_1, r_1, d_1)$ is a class of bicentric octagons whose rotation number is 1, and $C_8^{(3)}(R_2, r_2, d_2)$ is a class of bicentric octagons whose rotation number is 3. Since $t_M = 6.4$ and $t_m = 3.6$ are the same for each of the classes $C_4(R_0, r_0, d_0)$, $C_8^{(1)}(R_1, r_1, d_1)$, $C_8^{(3)}(R_2, r_2, d_2)$ we can take for t_1 any length between t_M and t_m, say, $t_1 = 5$. Now, using formula (1.5) and starting from $t_1 = 5$ we get (writing only first nine decimal places):

$$t_1 = 5,\ t_2 = 4.043395991,\ t_3 = 3.610778912,\ t_4 = 3.814401072,$$
$$t_5 = 4.608000000,\ t_6 = 5.698180452,\ t_7 = 6.380894692,\ t_8 = 6.040266757,$$
$$\hat{t}_1 = 5,\ \hat{t}_2 = 3.814401072,\ \hat{t}_3 = 6.380894692,\ \hat{t}_4 = 4.043395991,$$
$$\hat{t}_5 = 4.608000000,\ \hat{t}_6 = 6.040266757,\ \hat{t}_7 = 3.610778912,\ \hat{t}_8 = 6.040266757.$$

As can be seen, it is valid that $\hat{t}_i = t_{1+(i-1)3}$. Also can be found that

$$2\sum_{i=1}^{8} \arctan \frac{t_i}{r_1} = 360°, \quad 2\sum_{i=1}^{8} \arctan \frac{\hat{t}_i}{r_2} = 3 \cdot 360°.$$

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Poboljšana metoda za nalaženje Fussovih relacija za bicentričke n-kute, gdje je $n \geq 4$ parni broj

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Sažetak. U članku [7] izložena je relativno jednostavna i praktična metoda za nalaženje Fussovih relacija za bicentričke n-kute, gdje je $n \geq 3$ neparan broj. U sadašnjem članku izložena je relativno jednostavna i praktična metoda za nalaženje Fussovih relacija za bicentričke n-kute, gdje je $n \geq 4$ paran broj. U članku [7] ključnu ulogu imaju rotacioni brojevi za bicentričke n-kute, a u sadašnjem članku ključnu ulogu imaju tangentne duljine bicentričkih n-kuta. Izložen je jedan algoritam za dobivanje Fussovih relacija za bicentičke n-kute, gdje je $n \geq 4$ paran broj. Navedeno je i nekoliko do sada nepoznatih Fussovih relacija.

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