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RECIPROCITY IN AN ISOTROPIC PLANE

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Abstract. The concept of reciprocity with respect to a triangle is introduced in an isotropic plane. A number of statements about the properties of this mapping is proved. The images of some well known elements of a triangle with respect to this mapping will be studied.

1. Introduction

The isotropic (or Galilean) plane is a projective–metric plane, where the absolute consists of one line, the absolute line ω and one point on that line, the absolute point Ω. The lines through the point Ω are isotropic lines, and the points on the line ω are isotropic points (the points at infinity). Two lines through the same isotropic point are parallel, and two points on the same isotropic line are parallel points. Therefore, an isotropic plane is in fact the affine plane with the pointed direction of isotropic lines where the principle of duality is valid.

The triangle in an isotropic plane is allowable if any two of its vertices are not parallel. Each allowable triangle in an isotropic plane can be set, by a suitable choice of coordinates, in the so called *standard position*, i.e. that its circumscribed circle has the equation $y = x^2$, and its vertices are of the form $A = (a, a^2)$, $B = (b, b^2)$, $C = (c, c^2)$ where $a + b + c = 0$. To prove the geometric facts for each allowable triangle it is sufficient to give a proof for the standard triangle (see [3]). With the labels $p = abc$, $q = bc + ca + ab$ it can be shown that some useful equalities are valid as for example $q = bc - a^2$. In the Euclidean geometry the mappings of isogonality and reciprocity with respect to a triangle are well known and have been thoroughly studied (see e.g. Vandeghen [6]). In an isotropic plane the concept of isogonality with respect to a triangle in the standard position have been considered in [4]. In this paper we are going to introduce and investigate the concept of reciprocity in an isotropic plane.

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2. Reciprocity in an isotropic plane

In this chapter we will define the concept of reciprocity with respect to a given triangle in an isotropic plane.

Theorem 2.1. *If the points $T = (x_o, y_o)$ and $T' = (x'_o, y'_o)$ do not lie on the lines BC, CA, AB, and if D, E, F and D', E', F' are the intersections of the lines AT, BT, CT and AT', BT', CT' with the lines BC, CA, AB then the pairs of the points D, D'; E, E'; F, F' have the midpoints at the midpoints of the sides BC, CA, AB if, and only if, the following equalities are valid*

$$qx_ox'_o + 3y_oy'_o + 3p(x_o + x'_o) + 4q(y_o + y'_o) + 4q^2 = 0, \tag{2.1}$$

$$3px_ox'_o + q(x_oy'_o + y_ox'_o) + 3p(y_o + y'_o) + 4pq = 0. \tag{2.2}$$

Proof. The line with the equation

$$(x_o - a)y = (y_o - a^2)x + a^2x_o - ay_o$$

obviously passes through the points $A = (a, a^2)$ and $T = (x_o, y_o)$, so this is the line AT. According to [3] $y = -ax - bc$ is the equation of the line BC. The last two equations imply the following one

$$(x_o - a)(ax + bc) + (y_o - a^2)x + a^2x_o - ay_o = 0,$$

whose solution for x is the abscissa d of the point $D = AT \cap BC$. This gives

$$d = \frac{qx_o - 2bcx_o + ay_o + p}{ax_o + y_o - 2a^2} \tag{2.3}$$

because of $-a^2 - bc = q - 2bc$. Analogously, the point $D' = AT' \cap BC$ has the abscissa

$$d' = \frac{qx'_o - 2bcx'_o + ay'_o + p}{ax'_o + y'_o - 2a^2}. \tag{2.4}$$

The midpoint of the points D and D' is at the midpoint of the side BC provided that $d + d' = b + c$, i.e., $d + d' + a = 0$. Owing to (2.3) and (2.4) this condition can be written in the form

$$\begin{aligned}(qx_o&-2bcx_o+ay_o+p)(ax'_o+y'_o-2a^2)\\ &+(qx'_o-2bcx'_o+ay'_o+p)(ax_o+y_o-2a^2)\\ &+a(ax_o+y_o-2a^2)(ax'_o+y'_o-2a^2)=0\end{aligned}$$

or, after arranging it, in the form

$$\begin{aligned}&(2aq - 4p + a^3)x_ox'_o + (q - 2bc + 2a^2)(x_oy'_o + y_ox'_o) + 3ay_oy'_o\\ &+ (5ap - 2a^2q - 2a^4)(x_o + x'_o) + (p - 4a^3)(y_o + y'_o) + 4a^5 - 4a^2p = 0.\end{aligned}$$

Since

$$2aq - 4p + a^3 = 2aq - 4p + a(bc - q) = aq - 3p,$$

$$q - 2bc + 2a^2 = -q,$$
$$5ap - 2a^2q - 2a^4 = 5ap - 2(bc - q)(q + a^2) = 3ap + 2q(q + a^2 - bc) = 3ap,$$
$$p - 4a^3 = p - 4a(bc - q) = 4aq - 3p,$$
$$4a^5 - 4a^2p = 4a^3(a^2 - bc) = -4a^3q = -4aq(bc - q) = 4aq^2 - 4pq,$$

the previous condition gives the following

$$(aq - 3p)x_o x'_o - q(x_o y'_o + y_o x'_o) + 3ay_o y'_o + 3ap(x_o + x'_o) + (4aq - 3p)(y_o + y'_o) + 4aq^2 - 4pq = 0, \tag{2.5}$$

i.e. with the labels

$$P = 3px_o x'_o + q(x_o y'_o + y_o x'_o) + 3p(y_o + y'_o) + 4pq,$$
$$Q = qx_o x'_o + 3y_o y'_o + 3p(x_o + x'_o) + 4q(y_o + y'_o) + 4q^2$$

the final form of the first equality of the three analogous equalities

$$aQ - P = 0, \qquad bQ - P = 0, \qquad cQ - P = 0, \tag{2.6}$$

where the other two equalities (2.6) are analogous conditions for the points E, E' and F, F'. The equalities (2.6) are evidently equivalent to the equalities $P = 0$, $Q = 0$, and they are the equalities (2.2) and (2.1). ∎

The points T and T' with the properties from Theorem 2.1 are called *reciprocal* with respect to the triangle ABC.

If we change the labels x_o, y_o, x'_o, y'_o to the labels x, y, x', y', then the equalities (2.1) and (2.2) can be written in x' and y' like this

$$(qx + 3p)x' + (3y + 4q)y' = -(3px + 4qy + 4q^2),$$
$$(3px + qy)x' + (qx + 3p)y' = -(3py + 4pq).$$

This system of the above equations has the solution

$$x' = -\frac{3pqx^2 + 4q^2xy - 9py^2 + (9p^2 + 4q^3)x - 12pqy - 4pq^2}{q^2x^2 - 9pxy - 3qy^2 - 6pqx - 4q^2y + 9p^2},$$
$$y' = \frac{9p^2x^2 + 12pqxy + 4q^2y^2 + 8pq^2x - (9p^2 - 4q^3)y - 12p^2q}{q^2x^2 - 9pxy - 3qy^2 - 6pqx - 4q^2y + 9p^2}. \tag{2.7}$$

The mapping $T \longmapsto T'$, where for $T = (x, y)$ the point $T' = (x', y')$ is given by the formula (2.7), will be called *reciprocity* with respect to the triangle ABC. Thus, we have proved:

Theorem 2.2. *Reciprocity with respect to the triangle ABC is given by the formulas (2.7).*

Reciprocity is an involutory mapping. Reciprocity defined by (2.7) maps the points on the conic with the equation

$$q^2x^2 - 9pxy - 3qy^2 - 6pqx - 4q^2y + 9p^2 = 0$$

to the points on the absolute line.

In [8] it is shown that the above equation is the equation of the circumscribed Steiner ellipse of the triangle ABC. Thus we get

THEOREM 2.3. *Reciprocity with respect to the triangle maps the points on its circumscribed Steiner ellipse to the points on the absolute line.*

If the point $T = (x, y)$ lies on the line $\mathcal{T}$ with the equation $y = kx + l$, then, from (2.7), we get for example

$$x' = -\frac{(3pq + 4q^2k - 9pk^2)x^2 + (4q^2l - 18pkl + 9p^2 + 4q^3 - 12pqk)x - 9pl^2 - 12pql - 4pq^2}{(q^2 - 9pk - 3qk^2)x^2 - (9pl + 6qkl + 6pq)x - 3ql^2 - 4q^2l + 9p^2}$$

and it is similar with y'. If the obtained equations are divided by x^2 and if we put $x \to \infty$, we get the equalities

$$x' = -\frac{3pq + 4q^2k - 9pk^2}{q^2 - 9pk - 3qk^2}, \qquad y' = \frac{9p^2 + 12pqk + 4q^2k^2}{q^2 - 9pk - 3qk^2}, \tag{2.8}$$

i.e., it leads to the following statement.

THEOREM 2.4. *Reciprocity with respect to the triangle ABC maps the point at infinity of the line with the slope k to the point $T' = (x', y')$ given by the the formula (2.8).*

According to [3] the orthic line of the standard triangle ABC, and owing to [2] and [7] its de Longchamps line and its Feuerbach line have the slope $k = 0$. With this value for k, owing to (2.8), $x' = -\frac{3p}{q}$ and $y' = \frac{9p^2}{q^2}$ follow, so the point T' is the Steiner point S of the triangle ABC (see [8]). According to [3] the Euler line of the triangle ABC has the slope $k = \infty$, and then from (2.8) $x' = -\frac{3p}{q}$, $y' = -\frac{4}{3}q$ follow. It can be easily proved that the point

$$\Gamma = \left(\frac{-3p}{q}, \frac{-4q}{3} \right) \tag{2.9}$$

is the intersection of the lines AA_i, BB_i, CC_i, where A_i, B_i, C_i are the points of contact of the sides of the triangle and its inscribed circle. By analogy to the Euclidean case the point Γ will be called the Gergonne point of the triangle ABC.

COROLLARY 2.5. *The Steiner point of the triangle is reciprocal to the point at infinity of its de Longcahamps line, its orthic line, its inertial axis and its Feuerbach line.*

COROLLARY 2.6. *Reciprocity with respect to the triangle associates the absolute point with its Gergonne point.*

The centroid G and the symmedian center K of the triangle ABC are given by the formulas (see [3] and [2])

$$G = \left(0, -\frac{2q}{3}\right), \; K = \left(\frac{3p}{2q}, -\frac{q}{3}\right). \tag{2.10}$$

It is easy to see that $3G = \Gamma + 2K$ is valid. It means that the point Γ lies on the line GK. Because of that the statement of Corollary 2.6 is in accordance to Vandeghen ([6]) in the Euclidean case which states that the centroid, the symmedian center of the triangle and the point reciprocal to its orthocenter are collinear.

Owing to [2] the Lemoine line of the standard triangle ABC has the slope $k = \frac{3p}{q}$. If we take this value for k we get

$$\begin{aligned}
q^2 - 9pk - 3qk^2 &= q^2 - \frac{27p^2}{q} - \frac{27p^2}{q} = -\frac{1}{q}(54p^2 - q^3),\\
3pq + 4q^2k - 9pk^2 &= 3pq + 12pq - 81\frac{p^3}{q^2} = -\frac{3p}{q^2}(27p^2 - 5q^3),\\
9p^2 + 12pqk + 4q^2k^2 &= 9p^2 + 36p^2 + 36p^2 = 81p^2,
\end{aligned}$$

and then

$$x' = -\frac{3p}{q} \cdot \frac{27p^2 - 5q^3}{54p^2 - q^3}, \qquad y' = -\frac{81p^2q}{54p^2 - q^3},$$

i.e. we have proved

Theorem 2.7. *Reciprocity with respect to the standard triangle ABC maps the point at infinity of its Lemoine line to the point*

$$\left(-\frac{3p}{q} \cdot \frac{27p^2 - 5q^3}{54p^2 - q^3}, -\frac{81p^2q}{54p^2 - q^3}\right).$$

Theorem 2.8. *Reciprocity with respect to the standard triangle ABC maps the point at infinity of the line with the slope $-\dfrac{3p}{q}$ to the point*

$$\left(\frac{81p^3}{q^4} + \frac{9p}{q}, \frac{9p^2}{q^2}\right),$$

whose joining line with the Steiner point of that triangle is parallel to its orthic line.

Proof. With $k = -\frac{3p}{q}$ we get

$$\begin{aligned}
q^2 - 9pk - 3qk^2 &= q^2 + \frac{27p^2}{q} - \frac{27p^2}{q} = q^2,\\
3pq + 4q^2k - 9pk^2 &= 3pq - 12pq - 81\frac{p^3}{q^2} = -\frac{9p}{q^2}(9p^2 + q^3),\\
9p^2 + 12pqk + 4q^2k^2 &= 9p^2 - 36p^2 + 36p^2 = 9p^2,
\end{aligned}$$

and because of (2.8) we obtain

$$x' = \frac{9p}{q^4}(9p^2 + q^3) = \frac{81p^3}{q^4} + \frac{9p}{q}, \qquad y' = \frac{9p^2}{q^2}.$$

□

THEOREM 2.9. *Reciprocity with respect to the allowable triangle maps the point at infinity of its Steiner axis to the point symmetrical to its Gergonne point with respect to its centroid.*

PROOF. Owing to [8] the Steiner axis has the slope $k = -\frac{3p}{2q}$. With this value for k we get

$$q^2 - 9pk - 3qk^2 = q^2 + \frac{27p^2}{2q} - \frac{27p^2}{4q} = \frac{1}{4q}(27p^2 + 4q^3),$$

$$3pq + 4q^2k - 9pk^2 = 3pq - 6pq - \frac{81p^3}{4q^2} = -\frac{3p}{4q^2}(27p^2 + 4q^3),$$

$$9p^2 + 12pqk + 4q^2k^2 = 9p^2 - 18p^2 + 9p^2 = 0,$$

and by (2.8) we get $x' = \frac{3p}{q}$, $y' = 0$. The point $(\frac{3p}{q}, 0)$ and the point Γ have the midpoint $(0, -\frac{2}{3}q)$, and owing to (2.10) it is the centroid G of the triangle ABC.

□

THEOREM 2.10. *Reciprocity with respect to the allowable triangle ABC maps any line to the circumscribed conic of the triangle ABC. Thus it maps the line with the equation $y = kx + l$ to the circumscribed conic of the triangle ABC which, with its circumcircle has, besides the points A, B, C, the fourth common point D with the abscissa*

$$d = \frac{4q^2k + 9pl + 12pq}{9pk - 3ql - 4q^2}. \tag{2.11}$$

PROOF. Due to (2.7) the line with the equation $y = kx + l$ is mapped to the conic with the equation

$$\begin{aligned} &9p^2x^2 + 12pqxy + 4q^2y^2 + 8pq^2x - 9p^2y + 4q^3y - 12p^2q \\ &+ k(3pqx^2 + 4q^2xy - 9py^2 + 9p^2x + 4q^3x - 12pqy - 4pq^2) \\ &- l(q^2x^2 - 9pxy - 3qy^2 - 6pqx - 4q^2y + 9p^2) = 0. \end{aligned} \tag{2.12}$$

Substitution $y = x^2$ in (2.12) gives the following equation in abscissa x

$$4q^2x^4 + 12pqx^3 + 4q^3x^2 + 8pq^2x - 12p^2q + k(-9px^4 + 4q^2x^3 - 9pqx^2 + 9p^2x + 4q^3x - 4pq^2)$$
$$-l(-3qx^4 - 9px^3 - 3q^2x^2 - 6pqx + 9p^2) = 0$$

for the intersection of the conic (2.12) and the circumscribed circle. This equation can also be written in the form

$$(4q^2x + 12pq)(x^3 + qx - p) + k(-9px + 4q^2)(x^3 + qx - p) - l(-3qx - 9p)(x^3 + qx - p) = 0,$$

i.e. it falls apart on the equations $x^3+qx-p=0$, i.e. $(x-a)(x-b)(x-c)=0$ and the equation

$$4q^2x+12pq+k(-9px+4q^2)+l(3qx+9p)=0$$

with the solution $x=d$ given by (2.11). Because of that these four intersections are the points A, B, C and the point $D=(d,d^2)$. □

Specially, for $k=0$ from (2.11) we get $d=-\frac{3p}{q}$ and the point D is the Steiner point S of the triangle ABC.

For $k=\infty$ the given line has the equation $x=0$ and it is, owing to [3], the Euler line of the triangle ABC, and by means of (2.11) $d=\frac{4q^2}{9p}$ follows. Because of that, we get

Corollary 2.11. *Reciprocity with respect to the allowable triangle ABC maps the Euler line of that triangle to its circumscribed conic with the equation*

$$3pqx^2+4q^2xy-9py^2+(9p^2+4q^3)x-12pqy-4pq^2=0, \tag{2.13}$$

which intersects the circumscribed circle of that triangle at the points A, B, C and $D=\left(\dfrac{4q^2}{9p},\dfrac{16q^4}{81p^2}\right)$.

With $k=0$, $l=-\dfrac{4}{3}q$ the equation (2.12) after the multiplication by 3 gets the form

$$3(9p^2x^2+12pqxy+4q^2y^2+8pq^2x-9p^2y+4q^3y-12p^2q)$$
$$+4q(q^2x^2-9pxy-3qy^2-6pqx-4q^2y+9p^2)=0,$$

i.e. after arranging the form $(27p^2+4q^3)(x^2-y)=0$. As $27p^2+4q^3\neq 0$ (see [3]), we obtain the equation $y=x^2$ of the circumscribed circle of the triangle ABC, and the given line with the equation $y=-\frac{4}{3}q$ is, owing to [7], the de Longchamps line of the triangle ABC. We get

Corollary 2.12. *Reciprocity with respect to the allowable triangle mutually associates the points on its de Longchamps line to the points on its circumscribed circle.*

Corollary 2.5 follows easily from the Corollary 2.12 and the Theorem 2.3.

Now, we can prove a number of statements regarding to the Steiner point of the triangle.

Theorem 2.13. *The tangent line of the circumscribed circle of the allowable triangle at its Steiner point passes through the point reciprocal to the point at infinity of the Lemoine line of this triangle.* (In the Euclidean case the analogous statement can be found in [5]).

Proof. The circumscribed circle of the standard triangle ABC has the equation $y = x^2$ and the line $y = 2x_o x - x_o^2$ is the tangent line at the point (x_o, x_o^2), because the equation $x^2 - 2x_o x + x_o^2 = 0$ has a double solution x_o. Owing to [8] the Steiner point of the triangle ABC has the abscissa $x_o = -\frac{3p}{q}$, so the tangent line at that point has the equation

$$y = -\frac{6p}{q}x - \frac{9p^2}{q^2}.$$

The point from Theorem 2.7 lies on that line because of

$$-\tfrac{6p}{q}\left(-\tfrac{3p}{q}\right)\tfrac{27p^2-5q^3}{54p^3-q^3} - \tfrac{9p^2}{q^2} =$$
$$\tfrac{9p^2}{q^2(54p^3-q^3)}[2(27p^2 - 5q^3) - (54p^2 - q^3)] = -\tfrac{81p^2q}{54p^2-q^3}.$$

□

Theorem 2.14. *The isogonal image N' and the reciprocal image N'' of any point at infinity N for a given triangle are collinear with its Steiner point S.*

Proof. Let N be the point at infinity of the line with the slope k. Then, owing to [4] (Th. 3 and Th. 2.4) we get

$$N' = (-k, k^2), \qquad N'' = \left(-\frac{3pq + 4q^2k - 9pk^2}{q^2 - 9pk - 3qk^2}, \frac{9p^2 + 12pqk + 4q^2k^2}{q^2 - 9pk - 3qk^2}\right).$$

The line with the equation

$$qy = -(3p + qk)x - 3pk$$

obviously passes through the point N' and the Steiner point $S = \left(-\frac{3p}{q}, \frac{9p^2}{q^2}\right)$ of the triangle ABC. The point N'' lies on that line because of

$$q(9p^2+12pqk+4q^2k^2)-(3p+qk)(3pq+4q^2k-9pk^2)+3pk(q^2-9pk-3qk^2) = 0.$$

□

Theorem 2.15. *Let $\mathcal{T}$ and $\mathcal{T}'$ be the lines with the equations $y = kx + l$ and $y = k'x + l'$, which do not pass through the points A, B, C, and let D, E, F and D', E', F' be the intersections of these lines with the lines BC, CA, AB. The pairs of points D, D'; E, E'; F, F' have the midpoints at the midpoints of the sides BC, CA, AB if, and only if, these equalities*

$$kk' - l - l' - q = 0, \qquad kl' + lk' + q(k + k') + p = 0 \tag{2.14}$$

are valid.

Proof. The lines $\mathcal{T}$ and BC with the equations $y = kx + l$ and $y = -ax - bc$ meet at the point D with the abscissa

$$d = -\frac{l + bc}{k + a}. \tag{2.15}$$

Similarly the lines $\mathcal{T}'$ and BC meet at the point D' with the abscissa

$$d' = -\frac{l' + bc}{k' + a}. \tag{2.16}$$

The points D and D' have the midpoint at the midpoint of the side BC provided that $d + d' + a = 0$, which, owing to (2.15) and (2.16), gets the form

$$a(k + a)(k' + a) - (k + a)(l' + bc) - (k' + a)(l + bc) = 0,$$

i.e. because of $a^2 - bc = -q$ and

$$a^3 - 2abc = a(bc - q) - 2p = -aq - p$$

the form

$$akk' - (kl' + lk') - q(k + k') - a(l + l') - aq - p = 0.$$

With the labels

$$U = kk' - l - l' - q, \qquad V = kl' + lk' + q(k + k') + p$$

this condition is the first equality from the three analogous equalities

$$aU - V = 0, \qquad bU - V = 0, \qquad cU - V = 0, \tag{2.17}$$

where the remaining two equalities (2.17) are analogous conditions for the points E, E' and F, F'. The equalities (2.17) are obviously equivalent to the equalities $U = 0$, $V = 0$, and they are the equalities (2.14). □

The lines $\mathcal{T}$ and $\mathcal{T}'$ with the properties from Theorem 2.15 will be called *reciprocal* with respect to the triangle ABC.

The equalities (2.14) can be written in the form of the equations in k' and l'

$$kk' - l' = l + q, \qquad (l + q)k' + kl' = -(p + kq).$$

This system of equations has the solution

$$k' = \frac{kl - p}{k^2 + l + q}, \qquad l' = -\frac{k^2q + kp + l^2 + 2lq + q^2}{k^2 + l + q}. \tag{2.18}$$

The mapping $\mathcal{T} \longmapsto \mathcal{T}'$ defined by the formulas (2.18) will also be called *reciprocity* (on the set of lines) with respect to the triangle ABC. We have:

THEOREM 2.16. *Reciprocity of the lines with respect to the standard triangle ABC is given by the formulas (2.18), where $y = kx + l$ and $y = k'x + l'$ are the equations of the associated lines.*

Reciprocity of the lines is the involutory mapping too.
If $k = 0$, we have

COROLLARY 2.17. *The line with the equation*

$$y = -\frac{p}{l + q}x - l - q \tag{2.19}$$

is reciprocal to the line with the equation $y = l$.

With $l = -q$ from (2.19) it follows $x = 0$, i.e., reciprocity mutually associates the lines with the equations $y = -q$ and $x = 0$, and they are, according to [1] and [3], the Feuerbach and the Euler line of the triangle ABC. We have

Corollary 2.18. *The Euler and the Feuerbach line of the triangle are reciprocal with respect to this triangle.*

With $l = -\frac{4}{3}\,q$ from (2.19) the equation

$$y = \frac{3p}{q}x + \frac{q}{3}$$

follows, and it is according to [2] the equation of the Lemoine line of the triangle ABC. By analogy to the Euclidean case the line reciprocal to this line will be called the *de Longchamps line* of the triangle ABC. We obtain

Corollary 2.19. *The de Longchamps line of the triangle ABC at the standard position has the equation $y = -\frac{4}{3}\,q$.*

In [7] the de Longchamps line is obtained in a slightly different way.
Owing to (2.9) we have the following statement.

Corollary 2.20. *The de Longchamps line of the triangle passes through its Gergonne point.*

With $l = -\frac{q}{3}$ due to (2.19) we get

$$y = -\frac{3p}{2q}x - \frac{2}{3}q,$$

and this equation as well as the equation $y = -\frac{q}{3}$ are, owing to [8] and [3], the equations of the Steiner axis and the orthic line of the triangle ABC. Therefore, we get

Corollary 2.21. *The orthic line of the triangle is reciprocal to its Steiner axis.*

If the equation $y = kx + l$ is divided by k and if we take the value $lim\frac{l}{k} = -m$ for $k \to \infty$, then the line gets the equation $x = m$. By dividing the numerator and denominator in (2.18) by k^2 we get $k' = -m$, $l' = -(q + m^2)$ in the case $k^2 \to \infty$. Finally, it follows

Theorem 2.22. *Reciprocity with respect to the triangle ABC maps the line with the equation $x = m$ to the line with the equation $y = -mx - m^2 - q$.*

In [5] it is proved that the Brocard diameter of the standard triangle ABC has the equation $x = \frac{3p}{2q}$, so with $m = \frac{3p}{2q}$ from Theorem 2.22 we have

Corollary 2.23. *In the standard triangle ABC the Brocard diameter is mapped to the line with the equation $y = \frac{3p}{2q}x - \frac{9p^2}{4q^2} - q$.*

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Recipročnost u izotropnoj ravnini

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