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ANALYSIS OF THE STATE OF STRESS AND STRAIN OF A MEDIUM UNDER CONDITIONS OF INHOMOGENEOUS PLASTIC FLOW

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Under conditions of a complex state of stress of a medium a solution for a plane problem in the theory of plasticity was obtained for a complex medium being hardened in view of the temperature factor. Analytical relationships which have been obtained allow to predict mechanical characteristics of metal in zone of deformation.

Key words: *stress and strain, analysis inhomogeneous plastic flow*

Analiziranje naprezanja i deformacije medija u uvjetima nehomogenog plastičnog toka. U uvjetima složenog naprezanja medija dobiveno je teorijsko rješenje problema plastičnosti u ploštini za neki složeni medij koji je otvrdnut s obzirom na faktor temperature. Prethodno napravljene analitičke relacije omogućuju predviđanja mehaničkih svojstava metala u zoni deformacije.

Ključne riječi: *naprezanje i deformacija, analiza nehomogenog plastičnog toka*

INTRODUCTION

An analytical method is known for solution of problems in the theory of plasticity of a simple medium being hardened [1 - 2]. However the plastic medium responds not only to the rate of deformation, but also to the strain and temperature [3]. The problem proposed can be complicated and it is possible to obtain analytically a model of the medium taking into account the inflence of aforesaid parameters.

Initial equations [4]:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}=0;\ \frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0$$

$$(\sigma_x-\sigma_y)^2+4\cdot\tau_{xy}^2=4\cdot k^2$$

$$\frac{\sigma_x-\sigma_y}{2\cdot\tau_{xy}}=\frac{\xi_x-\xi_y}{\gamma_{xy}}=\frac{\varepsilon_x-\varepsilon_y}{\gamma_{xy}}=F$$

$$\xi_x+\xi_y=0;\ \varepsilon_x+\varepsilon_y=0 \tag{1}$$

$$\frac{\partial^2 \xi_x}{\partial y^2}+\frac{\partial^2 \xi_y}{\partial x^2}=\frac{\partial^2 \gamma_{xy}}{\partial y \partial x};\ \frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}=\frac{\partial^2 \gamma_{xy}}{\partial y \partial x}$$

$$\frac{\partial T}{\partial t}=a^2\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)$$

Model of a complex plastic medium

$$T_i=\alpha\cdot(H_i)^{m_1}\cdot(\Gamma_i)^{m_2}\cdot(T)^{-m_3} \tag{2}$$

In the system (1) ten equations and ten unknowns are represented. In comparison with the theory of the flow equations are included which estimate the influence of relative deformation as well as equation of heat conduction [5]. Combined equation of connection between stresses, rate of deformations and deformations conforms apparently to the assertion of L. M. Kachanov [6] that the two theories agree in the case of simple loading. To a certain degree this implicates that a general field "deformation - rate " being described by the common functional relationship to coordinates takes place. It is a real medium.

Boundary conditions for stresses [7]:

$$\tau_n=-T_i\cdot\sin\left[A\Phi-2\psi\right],\ \ T_i=k \tag{3}$$

or

$$\tau_n=-\left(\frac{\sigma_x-\sigma_y}{2}\cdot\sin\ 2\psi-\tau_{xy}\cdot\cos\ 2\psi\right)$$

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Additional conditions are given by the contact unit forces of friction (3) changing according to sinusoidal law with the strain and speed hardening.

THEORETICAL DEVELOPMENTS

To obtain the model (2) let us consider three equations of the second order in partial derivatives, inhomogeneous, of hyperbolic type.

$$
\begin{aligned}
&\frac{\partial^2 \tau_{xy}}{\partial x^2} - \frac{\partial^2 \tau_{xy}}{\partial y^2} = 2 \cdot \frac{\partial^2}{\partial x \partial y} \sqrt{k^2 - \tau_{xy}^2} \\
&\frac{\partial^2 \xi_x}{\partial y^2} - \frac{\partial^2 \xi_x}{\partial x^2} = 2 \cdot \frac{\partial^2}{\partial y \partial x} \frac{1}{F} \cdot \xi_x \\
&\frac{\partial^2 \varepsilon_x}{\partial y^2} - \frac{\partial^2 \varepsilon_x}{\partial x^2} = 2 \cdot \frac{\partial^2}{\partial y \partial x} \frac{1}{F} \cdot \varepsilon_x
\end{aligned}
\tag{4}
$$

In order to satisfy boundary conditions of the form (3) it is necessary to assume that $\tau_{xy} = \kappa \cdot \sin A\Phi$. For a complex plastic medium the problem for stresses is solved in view of relationship

$$k = f(\Gamma_i,\ H_i,\ T,\ x,\ y)$$

In this case $\kappa = C_\sigma \cdot \exp \theta'$,

where:

$$\theta' = f(\Gamma_i,\ H_i,\ T,\ x,\ y)$$

It is necessary to take derivatives of shearing strength with respect to coordinates as of a complex function [8] and after substitution in the first equation (4) we shall have

$$
\begin{aligned}
&\Big\{(\theta'_{\text{н}} \cdot H_x + \theta'_{\text{г}} \cdot \Gamma_x + \theta'_t \cdot T_x)_x + \big[(\theta'_{\text{н}} \cdot H_x + \theta'_{\text{г}} \cdot \Gamma_x + \theta'_t \cdot T_x) + \\
&+ A\Phi_y\big]^2 - (\theta'_{\text{н}} \cdot H_y + \theta'_{\text{г}} \cdot \Gamma_y + \theta'_t \cdot T_y)_y + \\
&+ [(\theta'_{\text{н}} \cdot H_y + \theta'_{\text{г}} \cdot \Gamma_y + \theta'_t \cdot T_y) - A\Phi_x]^2 + 2A\Phi_{xy}\Big\} \cdot \sin(A\Phi) + \\
&+ \Big\{ 2 \cdot \big[(\theta'_{\text{н}} \cdot H_x + \theta'_{\text{г}} \cdot \Gamma_x + \theta'_t \cdot T_x) + A\Phi_y\big] \cdot \\
&\cdot \big[A\Phi_x - (\theta'_{\text{н}} \cdot H_y + \theta'_{\text{г}} \cdot \Gamma_y + \theta'_t \cdot T_y)\big] + A\Phi_{xx} - A\Phi_{yy} - \\
&- 2 \cdot (\theta'_{\text{нн}} \cdot H_x \cdot H_y + \theta'_{\text{н}} \cdot H_{xy} + \theta'_{\text{гг}} \cdot \Gamma_x \cdot \Gamma_y + \\
&+ \theta'_{\text{г}} \cdot \Gamma_{xy} + \theta'_{tt} \cdot T_x \cdot T_y + \theta'_t \cdot T_{xy})\Big\} \cdot \cos(A\Phi) = 0
\end{aligned}
\tag{5}
$$

Equation (5) is obtained on condition that the mixed derivatives of the function θ are equal to zero. This proves its value later on when constructing solution. In spite of the great quantity of partial derivatives and unwieldiness of conclusion it succeeded to simplify equation (4) in new variables. As for a simple plastic medium, there are variants allowing to order the non-linear terms of equations. Sums and differences of the squares of corresponding derivatives appeared in operators standing before trigonometric functions. Equation (5) is identically equal to zero; in fact,

$$
\begin{aligned}
&\theta'_{\text{н}} \cdot H_x + \theta'_{\text{г}} \cdot \Gamma_x + \theta'_t \cdot T_x = -A\Phi_y \\
&\theta'_{\text{н}} \cdot H_y + \theta'_{\text{г}} \cdot \Gamma_y + \theta'_t \cdot T_y = A\Phi_x
\end{aligned}
$$

Then, from the last relationships, we have

$$
\begin{aligned}
&(\theta'_{\text{н}} \cdot H_x + \theta'_{\text{г}} \cdot \Gamma_x + \theta'_t \cdot T_x)_x = -A\Phi_{yx} \\
&(\theta'_{\text{н}} \cdot H_y + \theta'_{\text{г}} \cdot \Gamma_y + \theta'_t \cdot T_y)_y = A\Phi_{xy}
\end{aligned}
$$

$$
\begin{aligned}
&A\Phi_{yy} = -(\theta'_{\text{нн}} \cdot H_x \cdot H_y + \theta'_{\text{н}} \cdot H_{xy} + \\
&+ \theta'_{\text{гг}} \cdot \Gamma_x \cdot \Gamma_y + \theta'_{\text{г}} \cdot \Gamma_{xy} + \theta'_{tt} \cdot T_x \cdot T_y + \theta'_t \cdot T_{xy}) \\
&A\Phi_{xx} = (\theta'_{\text{нн}} \cdot H_x \cdot H_y + \theta'_{\text{н}} \cdot H_{xy} + \\
&+ \theta'_{\text{гг}} \cdot \Gamma_x \cdot \Gamma_y + \theta'_{\text{г}} \cdot \Gamma_{xy} + \theta'_{tt} \cdot T_x \cdot T_y + \theta'_t \cdot T_{xy})
\end{aligned}
$$

As the mixed derivatives are equal to zero, then

$$
\begin{aligned}
&d\theta'_1 = \theta'_{\text{н}} dH,\ \ d\theta'_2 = \theta'_{\text{г}} d\Gamma,\ \ d\theta'_3 = \theta'_t dT \\
&d\theta' = d\theta'_1 + d\theta'_2 + d\theta'_3 \\
&\theta'_1 = \theta'(H),\ \ \theta'_2 = \theta'(\Gamma),\ \ \theta'_3 = \theta'(T)
\end{aligned}
$$

The index of exponent is determinate as sum of three taking into account the influence of the rate of deformation, deformation and temperature, i.e.

$$\theta' = \theta'_1 + \theta'_2 + \theta'_3$$

Thus, the afore-cited mathematical computations show that solution of equation (10) admits the account of the influence of three noted factors upon the shearing strength. These factors are to be brought into accord with solution of the deformation and temperature problem.

Further relationships are:

$$
\begin{aligned}
&\theta' = -A\theta = \theta'_1 + \theta'_2 + \theta'_3 = -(A_1\theta + A_2\theta + A_3\theta) \\
&\theta'_1 = -A_1\theta,\ \ \theta'_2 = -A_2\theta,\ \ \theta'_3 = -A_3\theta
\end{aligned}
$$

hence $A = A_1 + A_2 + A_3$, where A_1, A_2, A_3 - are constants determining the influence of the rate and degree of deformation and temperature.

The shearing strength and components of the strain tension are:

$$\begin{aligned} &\kappa = C_\sigma \cdot \exp(-A_1\theta - A_2\theta - A_3\theta) \\ &\tau_{xy} = C_\sigma \cdot \exp(-A_1\theta - A_2\theta - A_3\theta) \cdot \sin(A\Phi) \\ &\sigma_x = C_\sigma \cdot \exp(-A_1\theta - A_2\theta - A_3\theta) \cdot \cos(A\Phi) + f(x) + C \\ &\sigma_y = -C_\sigma \cdot \exp(-A_1\theta - A_2\theta - A_3\theta) \cdot \cos(A\Phi) + f(x) + C \end{aligned} \tag{6}$$

Returning to the equations (4) it is possible to show that the two last equations can be brought into the form [1]. In this case

$$(\theta_1'')_y = B_1\Phi_x,\ (\theta_1'')_x = -B_1\Phi_y,$$
$$(\theta_2'')_y = B_2\Phi_x,\ (\theta_2'')_x = -B_2\Phi_y$$

where:

$\theta_1'' = -B_1\theta,\ \theta_2'' = -B_2\theta$ - indices of exponents, functions determining fields of rates of deformations and deformations;

$B_1\Phi, B_2\Phi$ - are arguments of trigonometrically functions for rates of deformations and deformations.

Expressions for determining rates of deformations and deformations are of the form

$$\begin{aligned} &\xi_x = -\xi_y = C_\xi \cdot \exp\theta_1'' \cdot \cos B_1\Phi = C_\xi \cdot \exp(-B_1\theta) \cdot \cos B_1\Phi \\ &\gamma_{xy} = C_\xi \cdot \exp(\theta_1'') \cdot \sin B_1\Phi = C_\xi \cdot \exp(-B_1\theta) \cdot \sin B_1\Phi \\ &H_i = 2 \cdot C_\xi \cdot \exp(\theta_1'') = 2 \cdot C_\xi \cdot \exp(-B_1\theta) \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\varepsilon_x = -\varepsilon_y = \\ &= C_\varepsilon \cdot \exp(\theta_2'') \cdot \cos B_2\Phi = C_\varepsilon \cdot \exp(-B_2\theta) \cdot \cos B_2\Phi \\ &\gamma_{xy} = C_\varepsilon \cdot \exp(\theta_2'') \cdot \sin B_2\Phi = C_\varepsilon \cdot \exp(-B_2\theta) \cdot \sin B_2\Phi \\ &\Gamma_i = 2 \cdot C_\varepsilon \cdot \exp(\theta_2'') = 2 \cdot C_\varepsilon \cdot \exp(-B_2\theta) \end{aligned} \tag{8}$$

at

$$(\theta_1'')_y = B_1\Phi_x,\ (\theta_1'')_x = -B_1\Phi_y,$$
$$(\theta_2'')_y = B_2\Phi_x,\ (\theta_2'')_x = -B_2\Phi_y$$

Comparing formulae (7), (8) with (6) we find out that the values θ and Φ (indices of exponents and arguments of trigonometric functions) are present in all expressions. Therefore fields of stresses, rates of deformations and deformations redeterminated by the same functions of coordinates.

Let us consider the temperature problem. Solution of the latter can be determined by above-said variables too. The common method of solution for two-dimensional problems for Laplace's equation is the use of analytical functions of the form $w = u\,(x, y) + i \cdot v\,(x, y)$, components for them satisfying Cauchy - Rimann's conditions $u = v$, $u = -v$.

Analogous conditions were imposed on functions when solving the problem of the theory of plasticity (6) - (8). We seek solution of differential equation for the stationary temperature

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

in the form

$$T = C_t \cdot \exp(\theta_3'') \cdot (\sin B_3\Phi + \cos B_3\Phi) \tag{9}$$

at

$$(\theta_3'')_x = -B_3\Phi_y,\ (\theta_3'')_y = B_3\Phi_x$$

Let us show that expression (9) is a solution of the Laplace's equation. Substituting derivatives of (9) in the heat conduction equation we shall after simplifications obtain

$$\begin{aligned} &\Big\{(\theta_3'')_{xx} + \left[(\theta_3'')_x + B_3\Phi_y\right] \cdot \left[(\theta_3'')_x - B_3\Phi_y\right] + (\theta_3'')_{yy} + \\ &+ \left[(\theta_3'')_y + B_3\Phi_x\right] \cdot \left[(\theta_3'')_y - B_3\Phi_x\right]\Big\} \cdot \\ &\cdot (\sin B_3\Phi + \cos B_3\Phi) + \left[2 \cdot (\theta_3'')_x \cdot B_3\Phi_x + B_3\Phi_{xx} + \right. \\ &\left. + 2 \cdot (\theta_3'')_y \cdot B_3\Phi_y + B_3\Phi_{yy}\right] \cdot (\cos B_3\Phi - \sin B_3\Phi) = 0 \end{aligned} \tag{10}$$

In equation (10) ther appear brackets $\left[(\theta_3'')_x + B_3\Phi_y\right]$, $\left[(\theta_3'')_y - B_3\Phi_x\right]$ which are the result of transformations of operator's non-linear terms standing at trigonometric functions. Assuming the noted brackets equal to zero one obtains

$$(\theta_3'')_x = -B_3\Phi_y,\ (\theta_3'')_y = B_3\Phi_x$$

The last relationships correspond to Cauchy -Rimann's condition, which is imposed on the analytic functions when solving Laplace's equation. It follows from the latter that

$$(\theta_3'')_{xx} = -B_3\Phi_{yx},\ (\theta_3'')_{yy} = B_3\Phi_{xy},$$
$$(\theta_3'')_{yx} = B_3\Phi_{xx},\ (\theta_3'')_{xy} = -B_3\Phi_{yy}$$

Thus, coefficients at trigonometric functions in equation (10) are equal to zero at any values of arguments of trigonometric functions. Comparing solutions (6) - (9), with conditions imposed on functions we come to the conclusion that functions Φ and θ relate to the stress and strain states and to the field of temperatures, i.e.

$$\theta_3'' = B_3\theta$$

The common temperature - deformation - rate of deformation field tied up with the field of stresses and being described by the common relationships to coordinates is established in the process of plastic deformation. It follows hence that all the rates and temperature are given parametrically from afore-cited functions. In the fields of stresses, deformations, rates of deformations and temperatures we have the same function relationships to coordinates (parameters) which allows to express them analytically ones through others:

$$\exp(-\theta) = \left(\frac{H_i}{C_\xi}\right)^{\frac{1}{B_1}} = $$
$$= \left(\frac{\Gamma_i}{C_\varepsilon}\right)^{\frac{1}{B_2}} = \left(\frac{T}{C_t \cdot (\sin B_3\Phi + \cos B_3\Phi)}\right)^{-\frac{1}{B_3}}$$

Substituting in the intensivity of tangential stresses, we shall obtain

$$T_i = \alpha \cdot H_i^{m_1} \cdot \Gamma_i^{m_2} \cdot T^{-m_3} \qquad (11)$$

We seek solution for the non-stationary heat conduction equation in the form

$$T = C_t \cdot \exp(Q'') \cdot \left(\sin B_3\Phi + \cos B_3\Phi\right) +$$
$$+ \sum_{k=1}^{\infty}\left[C_1' \cdot \exp(-a^2 \cdot \lambda_k \cdot t) \cdot \cos\left(\sqrt{\frac{\lambda_k}{2}} \cdot (y-x) + C_{1n}\right)\right] +$$
$$+ \sum_{k=1}^{\infty}\left[C_2'' \cdot \exp\left(-a^2 \cdot \lambda_k \cdot t\right) \cdot \sin\left(\sqrt{\frac{\lambda_k}{2}} \cdot (y-x) + C_{2m}\right)\right]$$

Differentiating the proposed solution with respect to time we shall obtain zero from the first term: two following sums satisfy identically the heat conduction equation at the combined differentiation. While differentiating with respect to coordinates the first term, as it is obvious from solution for the stationary heat conduction equation, reduces to zero. As differential equation is linear, the common solution can be represented as superposition of solutions. It is obvious from analysis that solutions connected with effect of initial temperatures and temperatures on the boundary of the zone of deformation are superposed on the stationary solution corresponding as to functions to fields of stresses and deformations. One may write in general form

$$T = C_t \cdot \exp(\theta_3'') \cdot (\sin B_3\Phi + \cos B_3\Phi) + T'$$

where:

T' - is the temperature falling at the non-stationary part.

Then

$$T'' = T - T' = C_t \cdot \exp(\theta_3'') \cdot (\sin B_3\Phi + \cos B_3\Phi)$$

where:

T'' - is the temperature determining the stationary component.

When rearranging we obtain

$$\exp(-\theta) = \left(\frac{H_i}{C_\xi}\right)^{\frac{1}{B_1}} = $$
$$= \left(\frac{\Gamma_i}{C_\varepsilon}\right)^{\frac{1}{B_2}} = \left(\frac{T''}{C_t \cdot (\sin B_3\Phi + \cos B_3\Phi)}\right)^{-\frac{1}{B_3}}$$

If substituting $k = T_i$, we obtain

$$k = C_\sigma \cdot \frac{(H_i)^{\frac{A_1}{B_1}} (\Gamma_i)^{\frac{A_2}{B_2}} (T'')^{-\frac{A_3}{B_3}}}{\left(\frac{1}{C_\xi}\right)^{\frac{1}{B_1}} \cdot \left(\frac{1}{C_\xi}\right)^{\frac{1}{B_2}} \cdot \left[\frac{1}{C_t \cdot (\sin B_3\Phi + \cos B_3\Phi)}\right]^{-\frac{1}{B_3}}}$$

After performed simplifications, we obtain expression analogous to (11) where:

$$m_1 = \frac{A_1}{B_1},\ m_2 = \frac{A_2}{B_2},\ m_3 = \frac{A_3}{B_3},$$

In conditions of a real zone of deformation, when the state of stress and strain of medium is complex and inhomogeneous $[T_i = T_i\ (x, y),\ H_i = H_i\ (x, y),\ \Gamma_i = \Gamma_i\ (x, y)]$

obeying conditions stated while defining the problem, it is possible to have a complex plastic medium. Symbols used are presented in the list of references.

EXPERIMENTAL CONFIRMATION

Expression (11) corresponds as to its form to the relationship of the flow stress to the rate and degree of deformation and temperature obtained on the basis of treatment of a great number of experimental data presented in [9]. It follows from this statement that such relationship in the field of metal forming is basically established by the way of experiments and it was more than once confirmed by the leading scientists and has really the form

$$\sigma_u = S \cdot \sigma_0 \cdot u^a (10 \cdot \varepsilon)^b \cdot (0{,}001 \cdot t)^c \tag{12}$$

where:

S, σ_0, a, b, c - are the constant numbers being determined for different grades of steel,

u, ε - are rate and degree of deformation.

Expression (12) establishes the real existence of a complex medium were elements of simple loading (power relationship as to each component are present.

CONCLUSION

1. Closed solution of a plane problem of the theory of plasticity for media of the complex load shows that fields of stresses, rates of deformations and temperatures are described with the same functional relationships to coordinates.
2. Presented analytic solution of the closed system of equations for a complex plastic medium allows to realize prediction of the mode of the plastic medium.
3. Expression (11) allows to determinate in every point of zone of deformate mechanical characteristics of metals depending on parameters of metal forming: deformation, rate of deformation and temperature.

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