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Perfect Matchings in Lattice Animals and Lattice Paths with Constraints*

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In the first part of this paper it is shown how to use ear decomposition techniques in proving existence and establishing lower bounds to the number of perfect matchings in lattice animals. A correspondence is then established between perfect matchings in certain classes of benzenoid graphs and paths in the rectangular lattice that satisfy certain diagonal constraints. This correspondence is used to give explicit formulas for the number of perfect matchings in hexagonal benzenoid graphs and to derive some identities involving Fibonacci numbers and binomial coefficients. Some of the results about benzenoid graphs are also translated into the context of polyominoes.

INTRODUCTION

A lattice animal is a 1-connected collection of congruent regular polygons arranged in a plane in such a way that two polygons are either completely disjoint or have a common edge. In other words, lattice animals are 1-connected subsets of regular tilings of the plane consisting of finite unions of basic polygonal tiles where any two tiles are either completely disjoint or share a whole edge. Obviously, the three different regular tilings of the plane give rise to three different classes of lattice animals. In this paper we are concerned with two of the three classes, namely with the animals consisting of squares and of regular hexagons. The square animals are also known as polyominoes, and the hexagonal ones, due to their obvious ressemblance to benzenoid hydrocarbons, as benzenoid systems.

To each lattice animal we can assign a corresponding animal graph taking the vertices of polygons as the vertices of the graph, and the sides of polygons as the edges of the graph. The resulting graph is simple, planar, and in the case of square and hexagonal animals, also bipartite. The non-bipartiteness of the graphs corresponding to triangular animals is the main reason for leaving them out of the scope of this paper, since the techniques used here do not work for non-bipartite graphs. The vertices of an animal graph lying on the border of the infinite face are called external; other vertices (if any) are

* Dedicated to Dr. Edward C. Kirby on the occasion of his 70^{th} birthday.

called internal. Borrowing a term from the chemistry of aromatic hydrocarbons, we call the animal graphs without internal vertices catacondensed. Otherwise, the graph is pericondensed. In the rest of this paper, when referring to lattice animals we will, in fact, be referring to the corresponding animal graphs.

In the following sections our attention will be (not always fairly) divided between the hexagonal and the square lattice animals. In section *Lattice Animals and Ear Decompositions* we use certain decomposition techniques to establish the existence of perfect matchings in certain classes of lattice animals and to derive simple lower bounds on their number. Then we proceed, in section *Benzenoid Parallelograms,* by demonstrating a correspondence between perfect matchings in some special classes of hexagonal animals and lattice paths in rectangular lattices. The correspondence is then used to obtain explicit formulas for the number of perfect matchings in some classes of hexagonal animals in terms of some well-known combinatorial sequences, and to establish certain identities involving Fibonacci numbers. We conclude with a section on polyominoes, using the correspondence with lattice paths to show how Catalan, Schröder and Delannoy numbers appear as the enumerating sequences of perfect matchings in some special polyominoes.

LATTICE ANIMALS AND EAR DECOMPOSITIONS

A perfect matching in a graph G is a collection M of edges of G such that every vertex of G is incident with exactly one edge from M. An edge e of G which appears in some perfect matching of G is called allowed; otherwise, it is forbidden. A graph G is elementary if its allowed edges form a connected subgraph of G. For bipartite graphs, elementarity is equivalent to the fact that all its edges are allowed.[10]

There are many criteria for deciding if a given lattice animal possesses a perfect matching. We present here some results that follow from the ear decomposition techniques. We refer the reader to Ref. 10 for a full description of these techniques.

Let G be a bipartite graph and G' subgraph of G. An ear of G relative to G' is any odd-length path in G having both endpoints – but no interior point – in G'. A bipartite ear decomposition of G starting with G' is a representation of G in the form $G = G'+P_1+\ldots+P_k$, where P_1 is an ear of $G'+P_1$ relative to G', and P_i is an ear of $G' + P_1+ \ldots +P_i$ relative to $G'+P_1+\ldots+P_{i-1}$ for $2 \le i \le k$.

An ear decomposition of a given graph is not unique.

Proposition 2.1

Every catacondensed lattice animal has a bipartite ear decomposition starting with any edge.

Proof

Every catacondensed benzenoid graph with h hexagons can be constructed starting from an arbitrary edge, adding an ear of length five to obtain the first hexagon and then adding one hexagon at every further step. But adding a hexagon means adding only five new edges to the graph already constructed. It is obvious that these edges make an ear in the above sense. So, every catacondensed benzenoid has a bipartite ear decomposition starting from any of its edges. Similarly, every catacondensed polyomino can be constructed starting from a single edge and adding ears consisting of three edges, one at a time. Hence, the claim is also valid for polyominoes. ■

We refer the reader to the p. 124 of Ref. 10 for the proof of the fact that a graph is elementary bipartite if and only if it has a bipartite ear decomposition. Since each ear in a bipartite ear decomposition contributes at least one new perfect matching, we get a lower bound for the number of perfect matchings in a catacondensed lattice animal.

Corollary 2.2

There are at least $h + 1$ perfect matchings in a catacondensed lattice animal with h basic polygons. ■

The lower bound from the Corollary 2.2 is a sharp one, in the sense that there are catacondensed benzenoids for which this bound is attained. Namely, it is easy to see that a straight linear chain of h hexagons contains exactly h+1 perfect matchings. In the case of polyominoes, the lower bound is attained for the class of zig-zag polyominoes, like the one shown in Figure 1.

There is a vast literature on the subject of enumeration of perfect matchings in benzenoid graphs. For a review, the reader should consult Ref. 3. The polyominoes have so far attracted much less attention.

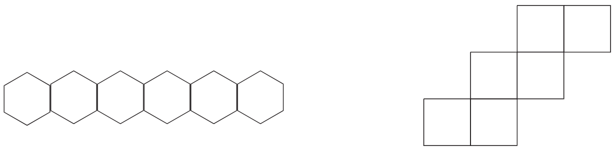

Figure 1. Extremal lattice animals with respect to the number of perfect matchings.

Elementary benzenoid graphs are also called normal in chemical literature. If we adopt the same terminology for the polyominoes, it is easy to show that the result and the lower bound of Corollary 2.2 remain valid also for the case of normal lattice animals. According to Theorem 7.6.2 of Ref. 10, any elementary bipartite graph on p vertices and q edges contains at least q–p+2 different perfect matchings. The claim now follows using Euler formula $q+1 = p+h$. Hence, we have the following result.

Corollary 2.3

There are at least h+1 different perfect matchings in a normal lattice animal with h polygons. ■

This result is a generalization of Theorem 13 of Ref. 3.

Non-elementary benzenoid graphs are, in chemical literature, called essentially disconnected. They contain edges that do not appear in any perfect matching. The linear lower bound of Corollary 2.3 is not valid for the case of non-normal benzenoids. In Figure 2 we show an essentially disconnected benzenoid with 11 hexagons and only 9 perfect matchings. From this example it is easy to see that there are essentially disconnected benzenoids with arbitrary many hexagons and no more than 9 perfect matchings.

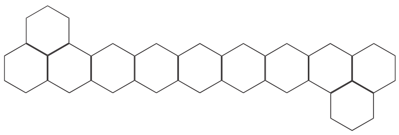

Figure 2. An essentially disconnected benzenoid with 11 hexagons and only 9 perfect matchings.

BENZENOID PARALLELOGRAMS

Let us now consider one simple class of benzenoid graphs, benzenoid parallelograms.

A benzenoid parallelogram $B_{m,n}$ consists of $m \times n$ hexagons, arranged in m rows, each row containing n hexagons, shifted for half a hexagon to the right from the row immediately below.

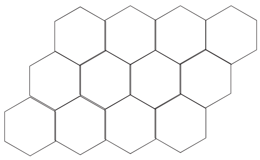

Figure 3. A benzenoid parallelogram.

Benzenoid parallelograms are not catacondensed, but we can still use the ear decomposition technique to show that they have perfect matchings.

Proposition 3.1

Every benzenoid parallelogram $B_{m,n}$ is an elementary graph, and it contains at least mn + 1 perfect matchings.

Proof

We shall display a bipartite ear decomposition of $B_{m,n}$ with exactly mn +1 ears. Starting from the rightmost hexagon in the lower-most row, add the hexagons to the left of it, one at a time. Each of them will contribute one 5-edge ear to the already constructed graph. After completing the lower-most row, start adding hexagons in the row immediately above it, again from the right to the left. The rightmost one will contribute a 5-edge ear, and every other will add a 3-edge ear to the already constructed graph. Repeating the same procedure for all the other rows, we get an ear decomposition of $B_{m,n}$ with exactly mn + 1 ears, counting the first hexagon as two ears. The claim now follows, since every ear in the ear decomposition brings at least one new perfect matching.[10] ■

An example of a perfect matching in $B_{3,4}$ is shown in Figure 4.

We see that this matching contains exactly one vertical edge from each row. We can prove that this is always the case.

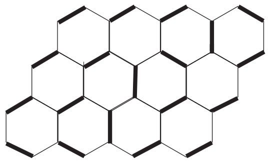

Figure 4. A perfect matching in $B_{3,4}$.

Lemma 3.2

Every perfect matching in a benzenoid parallelogram $B_{m,n}$ contains precisely one vertical edge of each row.

Proof

Let us consider a benzenoid parallelogram $B_{m,n}$ and a perfect matching M in $B_{m,n}$. The vertex set of $B_{m,n}$ is partitioned in two sets, W (for white) and B (for black) in such a way that the top vertices of all hexagons are white. Suppose that there is a row, say the i-th one, such that no vertical edge from it is contained in M. By removing all vertical edges of this row, we decompose the parallelogram $B_{m,n}$ into the components B^+ and B^-. An example of such procedure is shown in Figure 5.

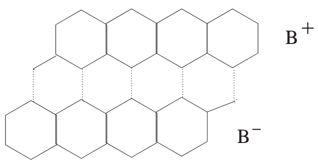

Figure 5. With the proof of Lemma 3.2.

Each of deleted edges connects a black vertex of B^+ with a white one of B^-. Further, in B^+ the number of black vertices exceeds the number of white ones by precisely one (and conversely in B^-). Hence, any perfect matching in $B_{m,n}$ must contain precisely one vertical edge from the considered row. ■

So, every perfect matching M of a benzenoid parallelogram $B_{m,n}$ contains exactly one vertical edge from each row of $B_{m,n}$. There are $n+1$ vertical edges in every row of $B_{m,n}$. Label them consecutively from the left to the right with integer labels $0,1,2, \ldots, n$. For a given perfect matching M, let i_p be the label of the vertical edge from the p-th row contained in M.

Lemma 3.3

The sequence $(i_1, \ldots, i_m)$ is non-decreasing for every perfect matching M of a benzenoid parallelogram $B_{m,n}$.

Proof

Consider a perfect matching M in $B_{m,n}$ and suppose that there is a $p \in [m-1]$ such that $i_p > i_{p+1}$. Remove all vertical edges from the p-th row which are to the left from i_p. (We count the rows from bottom to top.) The remaining graph, $B'_{m,n}$ as shown in Figure 6, will have a pendant vertex. Denote this vertex by a. Consider the shortest path connecting the vertex a with x, the lower endpoint of the vertical edge i_{p+1} from M, and denote it by P. No vertex of $P-\{x\}$ is covered by a vertical edge of M, and yet, as no edges from M were removed, all vertices of $p-\{x\}$ must be covered by some edge of M. But the cardinality of $V(P)-\{x\}$ is necessarily odd and we have arrived at a contradiction. ■

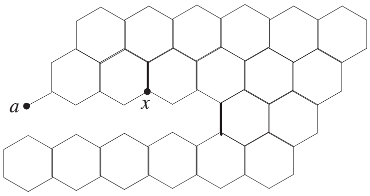

Figure 6. With the proof of Lemma 3.3.

Corollary 3.4

Let M be a perfect matching in $B_{m,n}$ containing the vertical edge i_p in the row p. Then the part of M lying in the rows $p+1,..., m$, left from their respective i_p-th vertical edges is uniquely determined. Similarly, the part of M lying in the rows $1,..., p-1$, right to their respective i_p-th rows is uniquely determined.

Proof

Let us first consider the part of $B_{m,n}$ above and left from the i_p-th vertical edge of the p-th row. No vertical edge from this part may be in M, and the conditions on the boundary force both non-vertical edges on the left side of every hexagon in this part of $B_{n.m}$ to be in M. A similar argument holds for the part of $B_{m,n}$ below and right of the considered vertical edge. ■

Proposition 3.5

There is a bijection between the set of all perfect matchings in $B_{m,n}$ and the set of all non-decreasing sequences of length m with elements from $\{0,1,... n\}$.

Proof

It follows from Lemma 3.3 and Corollary 3.4 that the positions of vertical edges in a perfect matching uniquely define a non-decreasing sequence of length m with elements from $\{0,1,... n\}$. To prove the other part, take a nondecreasing sequence $(i_1, i_2,..., i_m)$ with elements from $\{0,1,...,n\}$ and construct a matching in $B_{m,n}$ by taking the vertical edge i_p in the row p. Denote this matching by V and suppose that there are two different perfect matchings, M' and M'', such that $V \subset M'$, $V \subset M''$. Consider their symmetric difference $M' \triangle M''$. Any edge from $M' \triangle M''$ must be non vertical. By Corollary 3.4, no edge of $M' \triangle M''$ may lie left and above of any edges of V. Similarly, no such edges can exist right and below the edges from V. The only remaining possibility is that they are on paths connecting the upper end of the vertical edge i_p with the lower end of the edge i_{p+1}. But all such paths must have an even number of inner vertices, and their perfect matchings are unique. So, we have $M' \triangle M'' = \emptyset$, and hence $M'=M''$. Therefore, each choice of m vertical edges with nondecreasing labels defines a unique perfect matching M of $B_{m,n}$. ■

CORRESPONDENCE WITH LATTICE PATHS

A lattice path of length n between the points P_0 and P_n in the (x,y) coordinate plane is any sequence P of n segments $(\overline{P_{j-1}P_j})_{j=1}^{n}$ both of whose endpoints have integer coordinates. The segment $\overline{P_{j-1}P_j}$ is called the j-th step of the path P. By imposing various restrictions on the size and orientation of steps, on the initial and final points, and on the areas of the lattice that must be visited or avoided by the path, we obtain different classes of lattice paths. We consider here lattice paths in a rectangular lattice with integer coordinates from (0,0) to (n,m) with steps of type (1,0) and (0,1). Denote the set of all such paths with $P_{n.m}$.

Proposition 4.1

There is a one-to-one correspondence between the set of all lattice paths from (0,0) to (n,m) with steps (1,0) and (0,1) and the set of all nondecreasing sequences of length m with elements from $\{0,...,n\}$.

Proof

Let us take a lattice path from $P_{n.m}$. It has $m+n$ steps, m of them vertical (*i.e.* of type (0,1)). By writing down their abscissas, we get a nondecreasing sequence of length

m with elements from $\{0, ..., n\}$, and the correspondence is obviously injective. On the other hand, take a nondecreasing sequence of length m with elements from $\{0, ..., n\}$ and construct a lattice path starting from (0,0) with vertical steps connecting the points $(i_j, j–1)$ and (i_j, j) for $j = 1, ..., m$. By inserting horizontal steps from (i_j, j) to (i_{j+1}, j), $j = 1, ..., m–1$ and the steps from (0,0) to $(i_1, 0)$ and (0,1) and from (i_m, m) to (n, m), if needed, we get a lattice path from (0,0) to (n, m) with steps (1,0) and (0,1), and the correspondence is again injective. ■

Theorem 4.2

There is a one-to-one correspondence between the set of all perfect matchings in a benzenoid parallelogram $B_{m,n}$ and the set of all lattice paths from (0,0) to (n, m) with steps (1,0) and (0,1).

Proof

Follows from combining Proposition 3.5. and Proposition 4.1. ■

An example of this correspondence is illustrated in Figure 7.

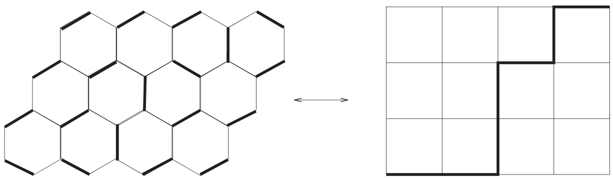

Figure 7. A perfect matching in $B_{n.m}$ and the corresponding lattice path.

CONSEQUENCES AND APPLICATIONS

The most obvious consequence of Theorem 4.2 is an exact formula for the number of perfect matchings in $B_{m,n}$. The result was obtained long time ago by Gordon and Davison, using essentially the same correspondence, but in a less formal manner.[3,7]

Proposition 5.1.

$$K(B_{m,n}) = \binom{m+n}{n} = \binom{m+n}{m}$$ ■

The Theorem 4.2. also enables us to get exact formulas for number of perfect matchings in various benzenoid graphs which may be obtained from benzenoid parallelograms by deleting some subgraphs.

As a first example, consider the benzenoid graph T_n consisting of n rows of hexagons, with the number of hexagons in a row decreasing by one from n in the lower-most row to one in the uppermost row, each row shifted for one and a half hexagon to the right from the row immediately below it. An example of such graph is shown in Figure 8.

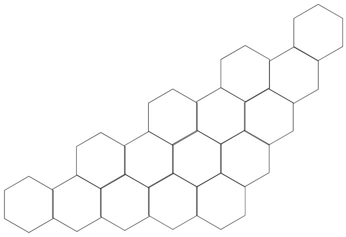

Figure 8. A triangular benzenoid.

Proposition 5.2

$$K(T_n) = C_{n+1},$$

where C_{n+1} is the $(n+1)$-st Catalan number.

Proof

Using Theorem 4.2. we can see that every perfect matching in T_n corresponds to a lattice path from (0,0) to $(n+1, n+1)$ with the steps (1,0) and (0,1) that always remains below the line $y=x$. Such paths are known as Dyck paths, and it is well known (see *e.g.* Ref. 15) that the number of Dyck paths on $2n$ steps is C_{n+1}. ■

The Catalan numbers, 1, 1, 2, 5, 14, 42, 132,... are one of the most ubiquitous sequences of enumerative combinatorics. They appeared already in the 18th century, in works of Euler, and got their name after a Belgian mathematician E. C. Catalan, who wrote a paper about them in 1838.[1] Since then, more than a hundred different families of objects counted by Catalan numbers have been found.[15] Certainly, this is not the first appearance of Catalan numbers in a chemical context. The best known example of their occurrence is as the number of ways to couple $2n$ electrons in $2n$ simply occupied orbitals to form a singlet wave-function. They also arise in the enumeration of Morgan trees[4] and in polymer statistics.

The result of Proposition 5.2 has been obtained in Ref. 16. Also, the triangular benzenoids T_n have not been overlooked by authors of Ref. 3. They give explicit formulas for $K(T_n)$, obtained by deriving recurrence relations for broader classes of benzenoids and then specializing certain parameters. Using the correspondence just established, all these results follow at a glance.

Let us now use some known results about enumeration of lattice paths with diagonal constraints.

Proposition 5.3

Let $W(n, m, r, s)$ denote the number of lattice paths from $(0, 0)$ to (n, m) not touching the lines $y = x - r$ and $y = x + s$, for some positive integers r, s. Then

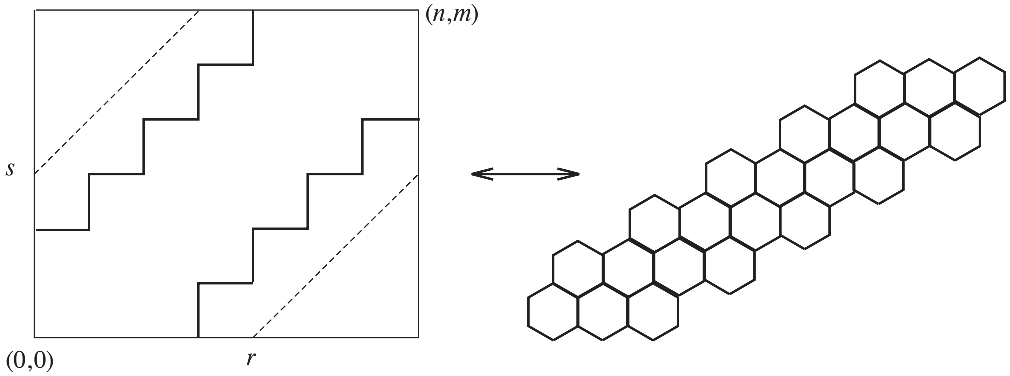

Figure 9. A part of lattice between diagonal boundaries and the corresponding benzenoid graph.

$$W(n,\ m,\ r,\ s) = \binom{m+n}{m} + \sum_{i\geq 1}\left[\binom{m+n}{m-it}+\binom{m+n}{n-it}\right] - \sum_{i\geq 0}\left[\binom{m+n}{n-r-it}+\binom{m+n}{m-s-it}\right]$$

where $t = r + s$. ■

For the proof of this proposition, one can see Ref. 8. Many other results about lattice paths between diagonal boundaries can be found in Ref. 11.

Prohibiting the lattice paths from touching the lines $y = x - r$ and $y = x + s$ effectively confines them to the hexagonal portion of the lattice, delimited by the said two lines and the lines $x = 0$, $x = n$, $y = 0$ and $y = m$. By replacing each unit square of the lattice which is whole contained in this area with a regular hexagon, we get a benzenoid graph whose every perfect matching corresponds to a lattice path confined between the lines $y = x - r$ and $y = x + s$. Let us denote such a graph, determined by the parameters n, m, r and s by $\mathrm{B}_{n,m,r,s}$. An example of such correspondence is shown in Figure 9.

Corollary 5.4

$$K(\mathrm{B}_{n,m,r,s}) = \binom{m+n}{m} + \sum_{i\geq 1}\left[\binom{m+n}{m-it}+\binom{m+n}{n-it}\right] - \sum_{i\geq 0}\left[\binom{m+n}{n-r-it}+\binom{m+n}{m-s-it}\right]$$

where $t = r + s$.

Proof

Follows by combining the Theorem 4.2 and Proposition 5.3. ■

The following results are obtained by specializing certain parameters in Corollary 5.4.

Corollary 5.5

$$K(\mathrm{B}_{n,m,r,s}) = \binom{m+n}{m} - \binom{m+n}{n+s}$$

for $m–n \leq s \leq m$, $r \geq n$. ■

Corollary 5.6

$$K(\mathrm{B}_{n,n,2,2}) = 2^n;$$
$$K(\mathrm{B}_{n,n,3,3}) = 2 \cdot 3^n;$$
$$K(\mathrm{B}_{n+1,n,3,3}) = 3^n;$$
$$K(\mathrm{B}_{k+1,k-1,3,3}) = 3^{k-1}.$$ ■

The correspondence can also be used in the opposite direction: By taking some known explicit formulas for number of perfect matchings in certain benzenoid graphs, we can derive some curious identities. Here are two examples involving Fibonacci numbers.

Corollary 5.7

$$F_{2k} = \binom{2k}{k+1} + \sum_{i\geq 1}\left[\binom{2k}{k+1-5i}+\binom{2k}{k-1-5i}\right] - \sum_{i\geq 0}\left[\binom{2k}{k-2-5i}+\binom{2k}{k-3-5i}\right]$$

Proof

It is well known[3] that the number of perfect matchings in a zig-zag chain with h hexagons is F_{h+2}. But the zig--zag chain with k–2 hexagons can be obtained as $B_{n,m,r,s}$, taking $n = k + 1$, $m = k - 1$, $r = 3$, $s = 2$. The claim now follows from Corollary 5.4. ■

By the same reasoning, we can prove the following result.

Corollary 5.8

$$F_{2k+1} = \binom{2k}{k} + 2\sum_{i\geq 1}\binom{2k}{k-5i} - \sum_{i\geq 0}\left[\binom{2k}{k-2-5i}+\binom{2k}{k-3-5i}\right]$$ ■

We conclude our review by presenting two explicit formulas derived from our correspondence, that are not in Ref. 3.

Corollary 5.9

$$K(B_{n+1,n,5,5}) = 5^{n/2}F_{n+1},$$

for n even. ■

Corollary 5.10

$$K(B_{n+3,n,5,5}) = 5^{(n+1)/2}F_{n+1},$$

for n odd. ■

In recent years, perfect matchings in benzenoids have been intensively studied for their connections with tilings of »Aztec diamonds« and plane partitions,[12] and many interesting results for special cases have been obtained (see, *e.g*. Refs. 2 and 5). Aztec diamonds also appear in our next section.

PERFECT MATCHINGS IN SOME CLASSES OF POLYOMINOES

We have already mentioned that the results expressing the number of perfect matchings in various special classes of polyominoes are far less abundant than those for hexagonal animals. The best known, and historically the most important, is the formula for the number of perfect matchings in a rectangular grid of $m \times n$ vertices (*i.e*., in a rectangular polyomino of $(m–1) \times (n–1)$ squares):

$$K(\mathrm{L}_{m-1,n-1}) = 2^{mn/2}\prod_{k=1}^{m}\prod_{l=1}^{n}\left[\cos^2\frac{k\pi}{m+1}+\cos^2\frac{l\pi}{n+1}\right]^{1/4}$$

For more information about this result we refer the reader to Ref. 10, p. 329. By setting one of the parameters, say n, to 2, we obtain the ladder graph with m rungs, L_m. It is easy to check that $K(\mathrm{L}_m)=F_{m+1}$, where F_{m+1} denotes the $(m+1)$-st Fibonacci number.

Some other classes of polyominoes, most of them with the square symmetry, were considered by Hosoya in his 1986 paper.[9] Among them were the already mentioned Aztec diamonds. An Aztec diamond is a polyomino of the type shown in Figure 10. It has $2n–1$ rows of squares, the number of squares increasing by 2 from 1 in the uppermost row to $2n–1$ squares in the middle row, and then decreasing again to 1 in the lowermost row.

Hosoya stated, without proof, that the number of perfect matchings in an Aztec diamond A_n is given by $K(\mathrm{A}_n) = 2^{\binom{n+1}{2}}$. The first proof appeared in 1991, in a much cited paper.[6] By doubling the middle row in an Aztec diamond one obtains an augmented Aztec diamond; a graph of this type is shown in Figure 11. We denote such graph by AA_n.

It was shown by Sachs and Zernitz in Ref. 13 that there is a one-to-one correspondence between the set of all perfect matchings in an augmented Aztec diamond

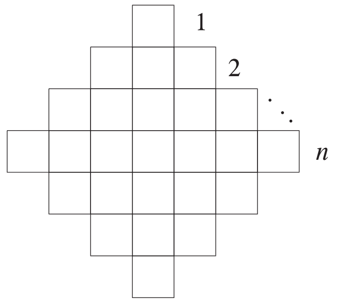

Figure 10. An Aztec diamond.

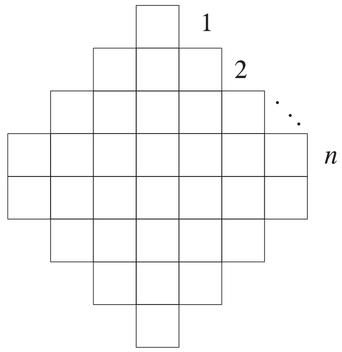

Figure 11. An augmented Aztec diamond.

AA_n and the set of all lattice paths from (0,0) to $(2n,0)$ using only the steps of the types (1,1), (1,–1), and (2,0). Such lattice paths are known in combinatorial literature as Delannoy paths, and their enumerating sequence D_n as the sequence of (central) Delannoy numbers. The sequence begins as 1,3,13,63,321,1683.... From its generating function $D(x) = \frac{1}{\sqrt{1-6x+x^2}}$ it can be deduced that $D_n = P_n(3)$, where $P_n(x)$ denotes the n-th Legendre polynomial, and that the sequence has the asymptotic behavior of $(3 + 2\sqrt{2})^n$.

By restricting the paths of Delannoy type to the upper half-plane, we get a class of paths known as Schröder paths. Their enumerating sequence, (large) Schröder numbers r_n, have a number of other combinatorial interpretations (see Ref. 15, Ex. 6.39), and their history goes all the way back to 1870.[14] It can be shown, following the approach of Sachs and Zernitz, that large Schröder numbers also enumerate the perfect matchings in the graph obtained from an augmented Aztec diamond by taking only its upper half. We denote such a polyomino by Z_n and call it a ziggurath of order n. A perfect matching in a ziggurath Z_n and the corresponding Schröder path are shown in Figure 12a). The correspondence becomes more clear if the images are superposed, as in Figure 12b).

The superposition makes particularly clear the relationship between the vertical edges in a perfect matching and the non-horizontal steps in the corresponding Schröder path.

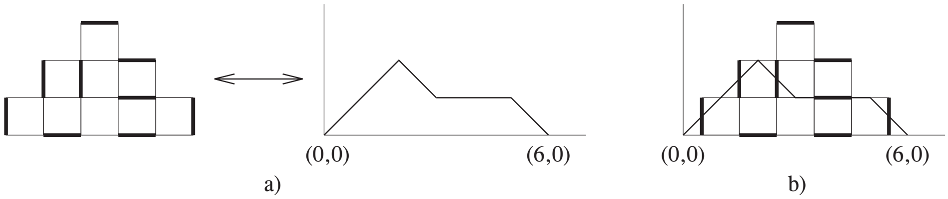

Figure 12. A perfect matching in a ziggurath and the corresponding Schröder path.

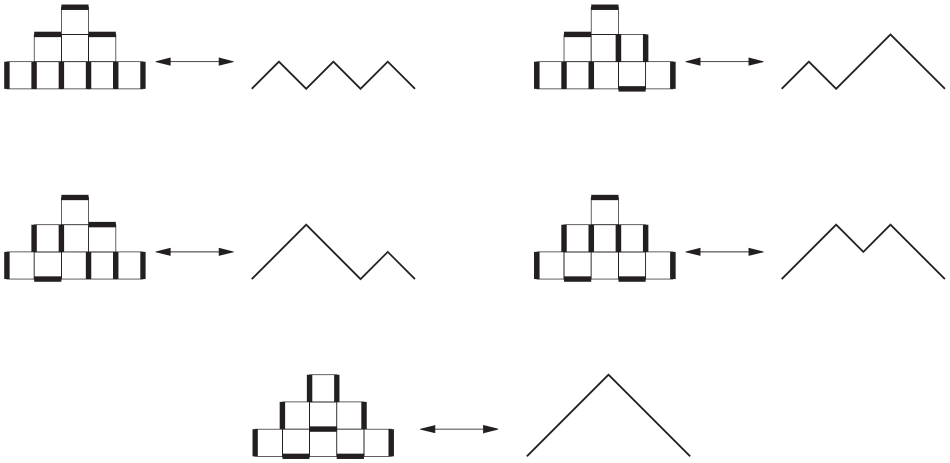

Figure 13. All-vertical perfect matchings in Z_3 and the corresponding Dyck paths.

As our last task, we consider the perfect matchings in a ziggurath Z_n that have exactly $2n$ vertical edges. (It is clear from the above correspondence, and it can be also shown using parity arguments, that no perfect matching in Z_n can contain more than $2n$ vertical edges.) We call such perfect matchings all-vertical. Such perfect matchings correspond to the Schröder paths without horizontal steps, and these, in turn, are nothing else but the Dyck paths on $2n$ steps, reflected across the line $y = x$ and rotated 45 degrees clockwise. Hence, we have the following result:

Proposition 6.1

The number of all-vertical perfect matchings in a ziggurath Z_n is the n-th Catalan number C_n. ■

The five all-vertical perfect matchings of Z_3 and the corresponding Schröder paths are shown in Figure 13.

In a similar way it can be shown that the number of all-vertical perfect matchings in an augmented Aztec diamond AA_n is equal to $\binom{2n}{2}$. Hence, perfect matchings in zigguraths and in augmented Aztec diamonds provide combinatorial interpretations for the well-known pair of formulas

$$r_n = \sum_{k=0}^{n} \binom{n+k}{n-k} C_k$$

$$D_n = \sum_{k=0}^{n} \binom{n+k}{n-k} \binom{2k}{k}.$$

From Propositions 6.1 and 5.2 it follows that there must exist a bijection between perfect matchings in a triangular benzenoid T_n and perfect matchings in a ziggurath Z_{n+1}. We invite the reader to work it out explicitly. Taking this bijection as the starting point, it is possible to derive the results about the number of perfect matchings in zigguraths and augmented Aztec diamonds in an alternative way.

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SAŽETAK

Savršeno sparivanje kod rešetkastih životinja i rešetkastih putova uz ograničenja

Tomislav Došlić

U prvom je dijelu članka pokazana uporaba tehnika ušnog rastava u dokazivanju postojanja i izvođenju donjih ocjena broja savršenih sparivanja u benzenoidnim grafovima i poliominima. Nakon toga je uspostavljena korespondencija između savršenih sparivanja u nekim klasama benzenoidnih grafova i putova u pravokutnim rešetkama koji zadovoljavaju određena ograničenja zadana dijagonalama. Korespondencija je zatim rabljena za dobivanje eksplicitnih formula za broj savršenih sparivanja u benzenoidnim grafovima i za izvođenje identiteta koji uključuju Fibonaccijeve brojeve i binomne koeficijente. Neki od rezultata za benzenoidne grafove su zatim prevedeni u kontekst poliomina.