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ENTROPY GENERATION IN STEADY HEAT CONDUCTION THROUGH A CYLINDER WALL WITH INTERNAL HEAT GENERATION

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Original scientific paper
Minimization of the entropy generation in steady heat conduction through a cylinder wall with an internal heat generation and isothermal boundary conditions is considered in a detail. The problem is solved analytically by introducing the relevant dimensionless variables: the dimensionless rate of internal heat generation, the ratio of the outside to the inside radius, and the ratio of the boundary temperatures. By means of these values and the dimensionless radius, the analytical expressions for the temperature distribution, and the local and total entropy generation are derived. A necessary condition for the existence of a minimum of the entropy generation is presented and the results of the analysis are explained. The field of the values of the ratio of the outside to the inside radius and the ratio of the boundary temperatures, for certain values of the internal heat generation, in which minimum of the entropy generation exists is presented.

Keywords: *entropy generation, cylinder wall, internal heat generation, isothermal boundary conditions*

Entropijska produkcija pri stacionarnom provođenju topline kroz stijenku cilindra s toplinskim izvorom

Izvorni znanstveni članak
U radu je detaljno analiziran problem minimizacije entropijske produkcije za model stacionarnog provođenja topline u stijenci cilindra s toplinskim izvorom i nametnutim izotermnim rubnim uvjetima. Problem je riješen uvođenjem relevantnih bezdimenzijskih varijabli: bezdimenzijske izdašnosti toplinskog izvora, omjera vanjskog i unutrašnjeg polumjera cilindra, kao i omjera rubnih temperatura. Pomoću tih veličina i bezdimenzijskog polumjera izvedeni su analitički izrazi za temperaturno polje, te lokalnu i ukupnu entropijsku produkciju te je postavljen kriterij za postojanje minimuma entropijske produkcije. Rezultati provedene analize su obrazloženi te je prikazano područje vrijednosti omjera polumjera cilindra, te omjera rubnih temperatura, za određene vrijednosti izdašnosti toplinskog izvora, u kojem postoji minimum entropijske produkcije.

Ključne riječi: *entropijska produkcija, stijenka cilindra, toplinski izvor, izotermni rubni uvjeti*

1 Introduction

In many engineering applications such as thermal insulation, heat transfer gauges and metal casting heat conduction with internal heat generation appears. Entropy generation and its minimization were investigated extensively by Bejan [1], who introduced the concept and optimization method of entropy generation minimization.
The entropy generation in solids, which is a consequence of the existence of temperature gradients, particularly with a goal to minimize the entropy generation, is considered in many papers. So, for example, in the papers [2, 3] the problem of entropy generation in a plane wall in a steady heat conduction is elaborated, and it is proved that in a steady state, without the existence of internal heat generation or heat sink, there is no minimum of the entropy generation. Ibanez et al. [4] and Bautista et al. [5] analyse the entropy generation in a plane wall with constant rate of heat generation with asymmetric boundary conditions. They assert that the minimum entropy generation can be obtained by combining the boundary conditions. In a very interesting and extended paper [6], Aziz and Makinde investigate a problem of minimization of entropy generation in a plane wall with asymmetric boundary conditions and temperature-dependent internal heat generation. In [7], Khan gives the model of minimization of entropy generation with linearly temperature-dependent internal heat generation and asymmetric boundary conditions. In recent paper [8], Torabi and Zhang investigate classical entropy generation in cooled slab of two types of materials: homogeneous material and functionally graded material using differential transformation method. They concluded that considering temperature- or coordinate-dependent thermal conductivity and radiation heat transfer at both sides of the slab have great effects on the entropy generation. From the above cases, it can be concluded that there is a possibility of minimization of entropy generation in a plane wall only in the cases of the existence of internal heat generation and imposed symmetric or asymmetric boundary conditions. It is evident that the entropy generation in heat conduction in a plane wall with internal heat generation and asymmetric boundary condition is a subject of interest of investigators. But, to the authors' knowledge, only few investigators consider entropy generation in heat conduction through a cylinder wall. Aziz and Khan [9] derived minimum entropy generation temperature profiles for steady conduction in three geometries, among other, in a hollow cylinder. Their study reveals that the effects of temperature or location dependent thermal conductivity, asymmetrical thermal boundary conditions, and internal heat generation have profound influence on the temperature profiles and the entropy generation rates. Entropy generation in a hollow cylinder which is cooled by convection on its inside surface and by convection and radiation on the outside surface is considered by Torabi and Aziz, [10].

The purpose of this paper is to analyse, in detail, entropy generation in steady heat conduction through a cylinder wall with an internal heat generation and investigate the condition of the existence of the minimum local and total entropy generation in the cylinder wall.

2 Mathematical model

Differential equation of the steady state heat conduction through the cylinder wall with an internal heat generation is given by the following expression

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\vartheta}{\mathrm{d}r}\right)+\frac{\Phi_V}{\lambda}=0\,, \tag{1}$$

where r is radius, ϑ is temperature, Φ_V is the internal heat generation rate.

The first integration of the above expression gives the expression for the temperature gradient,

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}r}=-\frac{\Phi_V r}{2}+\frac{C_1}{r}, \tag{2}$$

where C_1 is the constant of integration. The second integration gives the following general solution of the temperature field through the cylinder wall

$$\vartheta(r)=-\frac{\Phi_V r^2}{4\lambda}+C_1\ln r+C_2\,, \tag{3}$$

where C_2 is the constant of integration, λ is the thermal conductivity.

The constants of integration are obtained from the boundary (isothermal) conditions

For $r = R_1$, $\vartheta = \vartheta_{s1}$; and for $r = R_2$, $\vartheta = \vartheta_{s2}$, (4)

where R_1 is the inside cylinder radius, R_2 is the outside cylinder radius, ϑ_{s1} is the temperature of the inside cylinder surface, ϑ_{s2} is the temperature of the outside cylinder surface.

Eq. (3) is transformed in the dimensionless form

$$\theta=1-\frac{1-\theta_s}{\ln R^*}\ln r^*+\Phi_V^*\frac{\left(R^*\right)^2-1}{\ln R^*}\ln r^*+\Phi_V^*\left(1-\left(r^*\right)^2\right), \tag{5}$$

where the dimensionless variables are defined as follows

$$\theta=\frac{\vartheta(r)}{\vartheta_{s1}};\ \theta_s=\frac{\vartheta_{s2}}{\vartheta_{s1}};\ \Phi_V^*=\frac{\Phi_V R_1^2}{4\lambda\vartheta_{s1}};\ R^*=\frac{R_2}{R_1};\ r^*=\frac{r}{R_1}\,. \tag{6}$$

The local entropy generation for a given problem is determined from the heat conduction equation,

$$S_{\mathrm{gen}}'''=\lambda\left(\frac{\frac{\partial\vartheta}{\partial r}}{\vartheta}\right)^2, \tag{7}$$

where S_{gen}''' is the local entropy generation.

Eq. (7) can be written, using Eq. (6) in the dimensionless form

$$\frac{S_{\mathrm{gen}}'''R_1^2}{\lambda}=\left(\frac{\frac{\partial\theta}{\partial r^*}}{\theta}\right)^2. \tag{8}$$

From Eq. (5), the dimensionless temperature gradient can be written in the following form:

$$\frac{\partial\theta}{\partial r^*}=\frac{1}{r^*\ln R^*}\left(\Phi_V^*\left(\left(R^*\right)^2-1\right)+\theta_s-1\right)-2\Phi_V^* r^*. \tag{9}$$

Incorporating Eq. (5) and Eq. (9) into Eq. (8), the Eq. (10) which describes the local entropy generation in the dimensionless form is obtained:

$$\frac{S_{\mathrm{gen}}'''R_1^2}{\lambda}=\left(\frac{\Phi_V^*\left(\left(R^*\right)^2-1\right)+\theta_s-1-\left(2\Phi_V^*\ln R^*\right)\left(r^*\right)^2}{r^*\ln R^*\left(1-\frac{1-\theta_s}{\ln R^*}\ln r^*+\Phi_V^*\frac{\left(R^*\right)^2-1}{\ln R^*}\ln r^*+\Phi_V^*\left(1-\left(r^*\right)^2\right)\right)}\right)^2 \tag{10}$$

The total entropy generation is defined by the following equation:

$$S_{\mathrm{gen}}=\lambda\int_{R_1}^{R_2}\left(\frac{\frac{\partial\vartheta}{\partial r}}{\vartheta}\right)^2 2\pi L r\mathrm{d}r, \tag{11}$$

where L is the cylinder length.

In the dimensionless form, the expression for the total entropy generation can be written as follows

$$\frac{S_{\mathrm{gen}}}{2\pi\lambda L}=\int_1^{R^*}\left(\frac{\Phi_V^*\left(\left(r^*\right)^2-1\right)+\theta_s-1-\left(2\Phi_V^*\ln R^*\right)\left(r^*\right)^2}{r^*\ln R^*\left(1-\frac{1-\theta_s}{\ln R^*}\ln r^*+\Phi_V^*\frac{\left(R^*\right)^2-1}{\ln R^*}\ln r^*+\Phi_V^*\left(1-\left(r^*\right)^2\right)\right)}\right)^2 r^*\mathrm{d}r^* \tag{12}$$

It can be seen from Eq. (10) and Eq. (12) that the total and local entropy generation depend on the dimensionless variables θ_s, Φ_V^*, R^* and r^*. The local entropy generation can be obtained directly from Eq. (10), while for the total entropy generation, the numerical integration has to be done, because there is no explicit solution to the integral contained in Eq. (12).

The minimum of the local entropy generation is determined from the condition in which the temperature gradient is zero. For the selected values of dimensionless variables θ_s, Φ_V^*, R^* using the zero temperature gradient condition $\partial\theta/\partial r^*=0$ and the Eq. (9), the following expression for the stationary point is acquired

$$r_{\mathrm{stat}}^*=+\sqrt{\frac{\Phi_V^*\left(\left(R^*\right)^2-1\right)+\theta_s-1}{2\Phi_V^*\ln R^*}}\,. \tag{13}$$

For θ_s = 1,0, the value of r^*_{stat} does not depend on the value of Φ^*_V. It depends only on the value of R^*, according to the following equation:

$$r^*_{stat}(\theta_s = 1{,}0) = +\sqrt{\frac{(R^*)^2 - 1}{2\cdot \ln R^*}}. \qquad (14)$$

Accordingly, the dimensionless temperature field is determined by Eq. (5), the dimensionless local entropy generation by Eq. (10), which has the position of the zero value (local extreme - minimum) determined by Eq. (13), while the total entropy generation is determined by Eq. (12). Eq. (13) also refers to a conclusion when the extreme does exist in the analysed problem.

The minimum of the local entropy generation will exist in the wall, within the interval $1 \le r^*_{stat} \le R^*$ if, for the selected values of the variables θ_s, Φ^*_V and R^*, the next condition is met

$$1 \le \sqrt{\frac{\Phi^*_V\left(\left(R^*\right)^2 - 1\right) + \theta_s - 1}{2\Phi^*_V \ln R^*}} \le R^*. \qquad (15)$$

That means that the minimum of the total entropy generation will exist in the cylinder wall if the ratio of the dimensionless boundary temperatures θ_s lies in the following interval

$$1 + \Phi^*_V\left(2\ln R^* - \left(R^*\right)^2 + 1\right) \le \theta_s \le 1 + \Phi^*_V\left(\left(R^*\right)^2\left(2\ln R^* - 1\right) + 1\right) \qquad (16)$$

3 Results and discussion

In this section the temperature field and the local and total entropy generation are presented. The examples are chosen in a way that the analysis is always carried out showing the total entropy generation as a function of the dimensionless variables Φ^*_V, R^* and θ_s.

The two cases are considered. In the first case the temperature on the outside cylinder surface is higher than the temperature of the inside cylinder surface (θ_s>1), and in the second case the temperature of the outside cylinder surface is lower than the temperature of the inside cylinder surface (θ_s<1). In both cases, the value of the internal heat generation is varied within the interval $0 \le \Phi^*_V \le 2{,}0$.

In the first case, the ratio of the boundary temperatures is taken θ_s = 2,0. The diagram in Fig. 1 represents the dimensionless temperature distribution, while the diagram in Fig. 2 shows the dimensionless local entropy generation in the cylinder wall. The ratio of the outside to the inside radius of the cylinder is taken R^*=2,0.

The line marked with crosses in Fig. 1 is a logarithmic function and it shows temperature distribution in the case without internal heat generation. It can be seen that the temperature in the cylinder wall monotonically increases with the increase of radius r^*, and there is no local extreme. It is also evident that the value of the local temperature extreme (maximum) increases with the increase of the internal heat generation and the position of the maximum of the temperature distribution (stationary point) is moved to the left with the increase of the value of the internal heat generation.

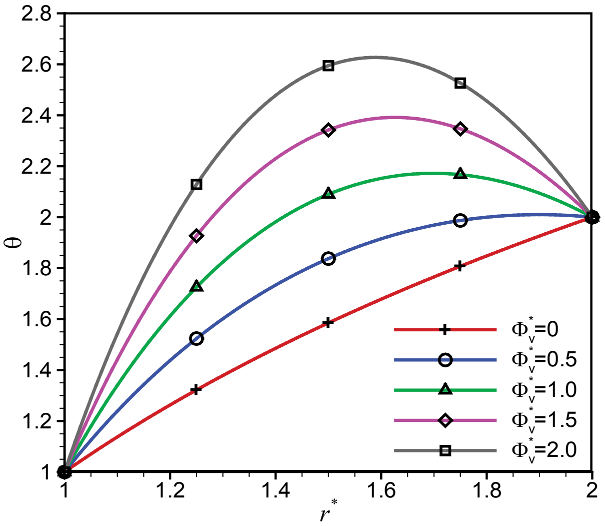

Figure 1 Temperature distribution in the cylinder wall as a function of dimensionless radius r^* and the dimensionless rate of internal heat generation Φ^*_V for θ_s= 2,0, R^*=2,0

The diagram in Fig. 2 shows the curves of the local entropy generation.

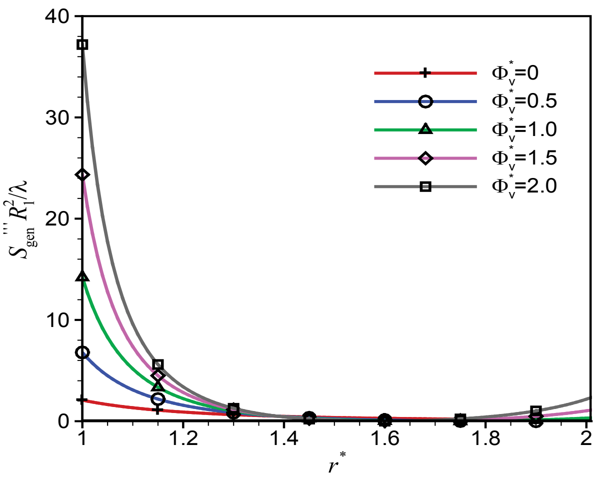

Figure 2 Dimensionless local entropy generation in the cylinder wall as a function of the dimensionless radius r^* and the dimensionless internal heat generation rate Φ^*_V for θ_s= 2,0, R^*=2,0

It can be seen that, for $\Phi^*_V = 0$, the local entropy generation continuously decreases with the increase of radius r^*. In the cases $\Phi^*_V > 0$, the local entropy generation continuously decreases in the interval r^* = 1,0 to r^*_{stat}. In all cases, at $r^* = r^*_{stat}$, the local entropy generation is equal to zero which is understandable because the temperature gradients are equal to zero, while the temperatures are greater than zero. Further increase of the value of r^* (to the value $r^*=R^*$= 2,0) leads to the slight increase of the local entropy generation because the increase of the absolute values of temperature gradients is greater than the decrease of the values of the temperatures.

By the integration of Eq. (12), the variable r^* disappears, and the total entropy generation is a function of the dimensionless variables θ_s, Φ_V^* and R^*. If we want to find out the necessary conditions of the existence of the minimum total entropy generation, we must certainly use the numerical method, since it is not possible to find the analytical solution of the integral in Eq. (12). Therefore, one of these variables is held constant, while the other two are changed, and based on the results of the calculations, it can be concluded for which values of the dimensionless variables θ_s, Φ_V^* and R^* the total entropy generation reaches a minimum.

In the case $\theta_s = 2{,}0$, when the ratio of the outside to the inside radius varies from 1,2 to 3,2 and the internal heat generation is in the interval from 0 to 2,0, the results of the calculations are shown in Fig. 3. In the calculation, to obtain the total entropy generation, the corresponding calculations of the local entropy generation, according to Eq. (10) should be done.

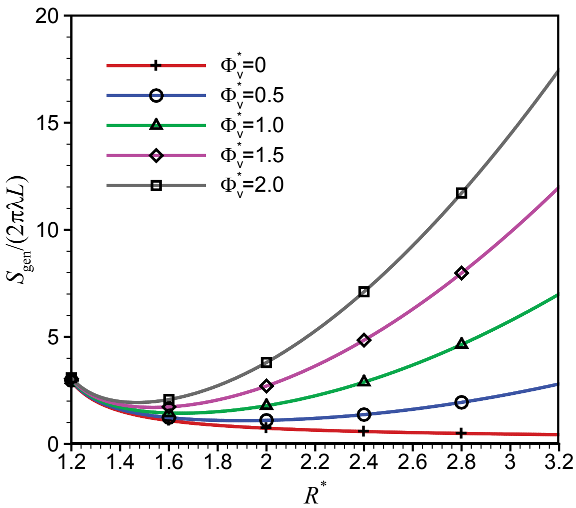

Figure 3 Dimensionless total entropy generation in the cylinder wall as a function of R^* and Φ_V^* for $\theta_s = 2{,}0$

It can be seen in Fig. 3 that the value of the ratio of the outside to the inside cylinder radius, in which the minimum of the total entropy generation is reached (stationary point R^*_{stat}), is moved to the lower values with the increase of the internal heat generation and the values of the minimum are also increased.

The following section describes the second case, when the heat transfer rate is directed from the inside to the outside cylinder surface, (θ_s<1). Fig. 4 shows the dimensionless temperature in the cylinder wall for $\theta_s = 0{,}5$ when the ratio of the outside to the inside cylinder radius is $R^* = 2$ and the internal heat generation varies from 0 to 2,0, as in the first case.

From the above diagram, it can be seen that the temperature is a logarithmic function of the dimensionless radius which is monotonically decreasing with the decrease of the temperature gradient in the case without internal heat generation in the cylinder wall. In presented cases with the internal heat generation ($\Phi_V^* > 0$), each curve of the temperature distribution has a maximum in the cylinder wall which is higher for higher values of the internal heat generation. Curves of the local entropy generation in the second case are shown in Fig 5.

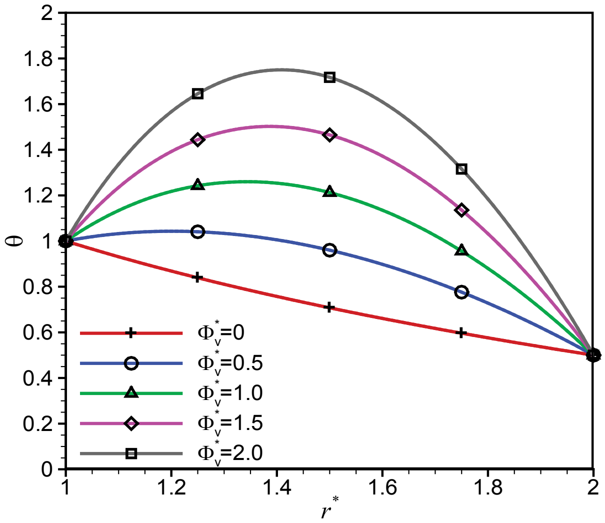

Figure 4 Temperature distribution in the cylinder wall as a function of radius r^* and the dimensionless rate of internal heat generation Φ_V^* for θ_s= 0,5, R^*=2,0

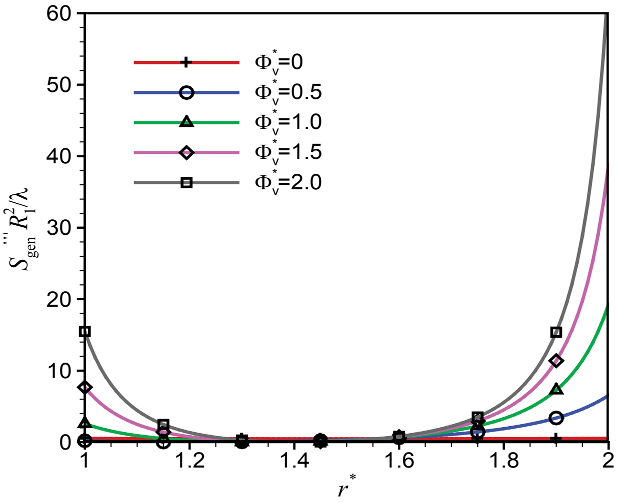

Figure 5 Dimensionless local entropy generation in the cylinder wall as a function of the dimensionless radius r^* and the dimensionless internal heat generation rate Φ_V^* ,for θ_s= 0,5, R^*=2,0

Comparison of the curves of the local entropy generation in the first and second case shows certain symmetry. In both cases, the curves have minima but the higher values of the local entropy generation appear on that edge of the cylinder wall on which the heat transfer rate leaves the cylinder.

The total entropy generation as a function of the internal heat generation and the ratio of the outside to the inside cylinder radius in the second case is shown in Fig. 6.

The curves of the total entropy generation in the second case are very similar to the curves in the first case, but the values of the total entropy generation are much higher. Also, in this case the curves have the minimum of the total entropy generation with the exception of the curve that represents the case without internal heat generation.

Fig. 7 shows dimensionless total entropy generation in the cylinder wall as a function of the ratio of boundary temperatures and the internal heat generation for R^*=2.

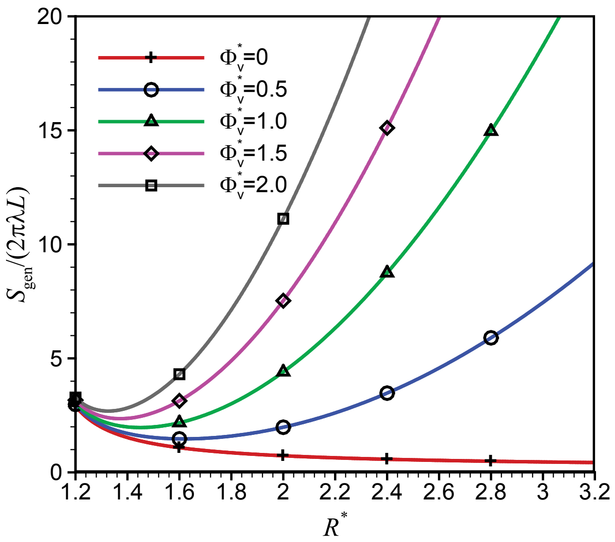

Figure 6 Dimensionless total entropy generation in the cylinder wall as a function of R^* and Φ_V^* for θ_s = 0,5

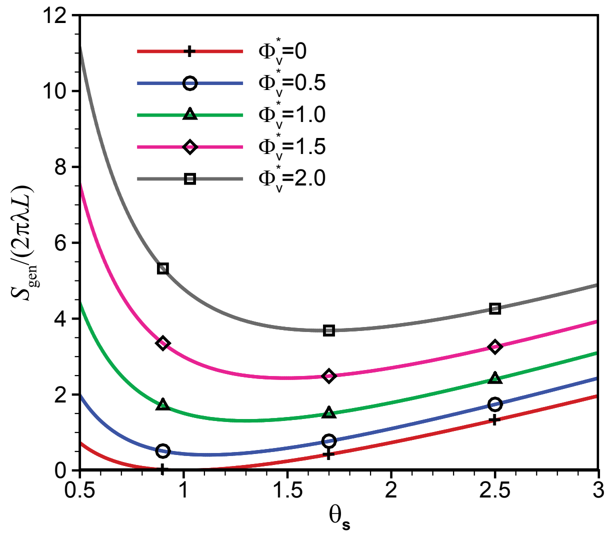

Figure 7 Dimensionless total entropy generation in the cylinder wall as a function of θ_s and Φ_V^* for R^*=2

It is clear from the above diagram that all curves reach the minimum of the total entropy generation including the curve that represents the case without internal heat generation. There is one value of the ratio of boundary temperatures for each value of the internal heat generation that gives the minimum total entropy generation, for the value of ratio of the outside to the inside cylinder radius R^*=2. Is it valid for all values of the ratio of outside to the inside radius in the interval 1,2 ≤R^*≤3,2? The answer is in the Eq. (16) which gives the field of the values of the ratio of boundary temperatures θ_s and the values of the ratio of the outside to the inside cylinder radius R^* in which the local entropy generation reaches minimum, for certain values of the internal heat generation. It is shown in Fig. 8a. This field is hatched for Φ_V^* =0,5.

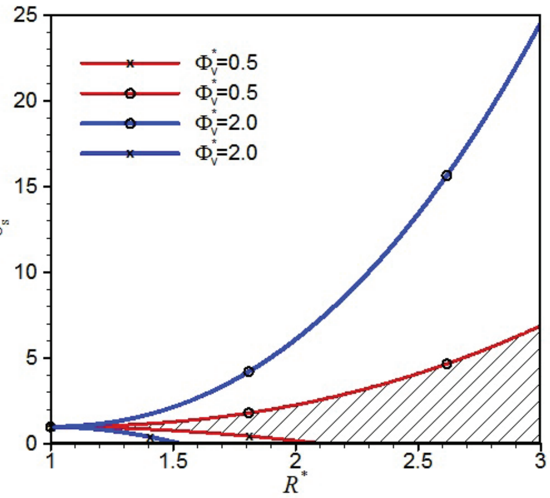

Figure 8a Field of the values of θ_s and R^*(Φ_V^* =0,5 and Φ_V^* =2,0), obtained from the condition of the existence of the dimensionless local entropy generation minimum, according to Eq. (16)

It can be seen that the higher value of the internal heat generation gives the larger field of the value of θ_s and R^* which meet the condition of the minimum of the local entropy generation.

Fig. 8b shows a detail from Fig. 8a, the interval 1,0≤R^*≤2,2 and the interval of the ratio of boundary temperatures 0<θ_s<3,0 for Φ_V^* =0,5 and for Φ_V^* =2,0.

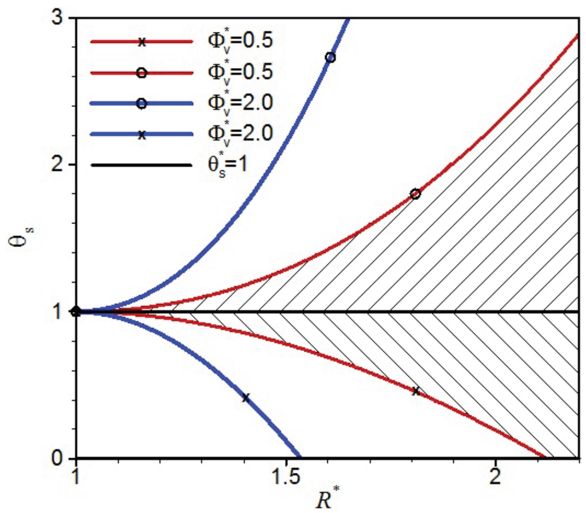

Figure 8b Detail of the diagram shown in Fig. 8a

For example, we consider the field of the values of θ_s and R^* in the case Φ_V^* = 0,5, (between thin curves) shown in Fig. 8b. The line θ_s=1 divides the field in two parts that represent two cases: the first in which the temperature of the outside cylinder surface is greater than the temperature of the inside cylinder surface, (θ_s>1) and the second in which the temperature of the outside cylinder surface is lower than the temperature on the inside cylinder surface, (θ_s<1). The minimum of the local entropy generation will exist within the cylinder wall if the ratio of the boundary temperatures lies in the interval $1 \le \theta_s \le 1 + 0{,}5\left((R^*)^2(2\ln R^* - 1) + 1\right)$ in the case $\theta_s > 1$ and $1 + 0{,}5\left(2\ln R^* - (R^*)^2 + 1\right) \le \theta_s \le 1$ in the case $\theta_s < 1$.

4 Conclusion

The analysis shows that the combination of the dimensionless variables Φ_V^*, R^* and θ_s in steady heat conduction through the cylinder wall with an internal heat generation can give the minimum of the total entropy generation. It is shown that the condition of the existence of the minimum total entropy generation is met, in the investigated cases, if the condition of the existence of the minimum local entropy generation is met in the cylinder wall.

The two cases are considered. In the case in which the temperature of the outside cylinder surface is greater than the temperature of the inside cylinder surface ($\theta_s > 1$), the minimum of the total entropy generation will exist within the cylinder wall, if the ratio of boundary temperatures θ_s lies in the interval $1 \le \theta_s \le 1 + \Phi_V^*\left((R^*)^2(2\ln R^* - 1) + 1\right)$, for each ratio of the outside to the inside cylinder radius $R^* > 1$.

In the case in which the temperature of the outside cylinder surface is lower than the temperature of the inside cylinder surface ($\theta_s < 1$), the minimum of the total entropy generation cannot be achieved for the pairs of values (R^*, θ_s) which gives the concave curve of the temperature distribution, for certain value of the internal heat generation. For these pairs, the value of the internal heat generation is not enough high to change the shape of the curve from concave to convex and reach the temperature maximum. The condition of the existence of the minimum of the entropy generation is that ratio of boundary temperatures θ_s should lie in the interval $1 + \Phi_V^*\left(2\ln R^* - (R^*)^2 + 1\right) \le \theta_s \le 1$, for each ratio of the outside to the inside cylinder radius $R^* > 1$, for certain value of the internal heat generation.

The presented analysis showed that in the case $\theta_s > 1$ the minimum of the total entropy generation exists for all combinations of the parameters (R^*, θ_s), but it exists in the cylinder wall if the given condition is met, while in the case $\theta_s < 1$ the minimum of the total entropy generation exists only if the given condition is met.

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