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Computing semi-commuting differential operators in one and multiple variables

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Abstract. We discuss the concept of what we refer to as semi-commuting linear differential operators. Such operators hold commuting operators as a special case. In particular, we discuss a heuristic by which one may construct such operators. Restricting to the case in which one such operator is of degree two, we construct families of linear differential operators semi-commuting with some named operators governing special functions (with a focus on the hypergeometric case, as it holds many other cases as reductions); operators commuting with such special degree two operators will necessarily be contained in these families. In the partial differential operator case, we consider examples in the form of the wave equation with a variable wave speed, and then hypergeometric operators on two variables (such operators define Appell functions).

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1. Introduction

The theory of commuting differential operators has been an important area of study over the last hundred years. The basic results for ordinary differential operators were first laid out in the three classical papers [8] - [10] authored by Burchnall and Chaundy. The main theorem established by Burchnall and Chaundy is as follows: If P and Q are ordinary differential operators of orders m and n respectively, and $[P, Q] = 0$, then P and Q satisfy an algebraic identity $F(P, Q) = 0$ of degree n in P and m in Q. In the case when m and n are co-prime, then $F(P, Q) = 0$ implies that $[P, Q] = 0$.

With the development of algebraic geometry in the 1960's, the study of the geometric properties of such curves $F(P, Q)$ was furthered. Algebro-geometric results like those discussed in Mumford [21] generalize the results discussed in [8] - [10]. Let k be any field of characteristic zero, let $k[[t]]$ be the ring of formal power series over k, and let $k[[t]]\left[\frac{d}{dt}\right]$ be the ring of formal linear ordinary differential operators over k. Consider two sets of data:

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Data A [Algebraic Data] consists of:

(I) a complete curve X over field k;

(II) a smooth k-rational point $P \in X$ and an isomorphism $T_{x,p} \cong k$, and;

(III) a torsion-free rank one sheaf $\mathcal{F}$ on X such that $h^0(\mathcal{F}) = h^1(\mathcal{F}) = 0$.

Data B [Differential Operator Data] consists of commutative subrings $k \subset R \subset k[[t]]\left[\frac{d}{dt}\right]$ such that there exist elememts $A, B \in R$ of the form

$$A = \left(\frac{d}{dt}\right)^{\alpha} + a_1(t)\left(\frac{d}{dt}\right)^{\alpha-1} + \cdots + a_{\alpha}(t),$$
$$B = \left(\frac{d}{dt}\right)^{\beta} + b_1(t)\left(\frac{d}{dt}\right)^{\beta-1} + \cdots + b_{\beta}(t),$$

where α and β are coprime. Two subrings $R_1, R_2 \subset k[[t]]\left[\frac{d}{dt}\right]$ are identified if for some $u(t) \in k[[t]]$ with $u(0) \neq 0$ we have that $R_2 = u(t) \circ R_1 \circ u^{-1}(t)$.

Krichever shows that there is a natural bijection between the set Data A of algebraic data, and the set Data B of differential operator data. (See, for instance, [15, 16].) See also [20] for a discussion of the difference operator case.

Also, while a categorical presentation was not explicitly employed in Mumford [21], it may help one to "step-back" from the details and view the underlying themes present in the theory. Note that the spectrum of a commutative ring R denoted as $\mathrm{Spec}(R)$ is defined to be the set of all proper prime ideals of R. If P is a point in $\mathrm{Spec}(R)$ (that is, a prime ideal), then the stalk at P equals the localization of R at P, and this is a *local ring*. Hence, $\mathrm{Spec}(R)$ is a locally ringed space. In a sense, then one might consider **Spec** (as a functor) a mapping (equivalence, really) between our data sets: the functor **Spec** yields an equivalence between the category of commutative rings (or, the restriction to commutative rings of differential operators, as far as we are concerned) and the category of affine schemes (which provide, up to isomorphism, our algebraic data, as locally ringed spaces).

As can easily be seen from the literature in this area over the last 30 years, an understanding of commuting and non-commuting differential operators is useful. Most likely the best application explored recently is the construction of Lax pairs from certain non-commuting differential operators. Lax pairs allow for one to understand a potentially complicated nonlinear partial differential equation in terms of two potentially better behaved differential equations. As such, formulating a nonlinear model in terms of Lax pairs has become a frequent practice. Some examples of nonlinear equations which have been formulated via a Lax pair include the Sine-Gordon equation, the KdV hierarchy (and hence the standard Korteweg de Vries equation), and several instances of the nonlinear Schrödinger equation, to name a few. Excellent reviews and recent work in the area include [11, 28, 27, 29].

While a number of important differential operators in mathematical physics have been shown to commute with some class of operators, it is important to note that many do not. Indeed, perhaps the most famous of such relations is due to the Heisenberg uncertainty principle

$$[\hat{x}, \hat{p}] = i\hbar$$

from quantum mechanics. For some modern examples of non-commuting operators of physical relevance and their applicability, see [2, 18, 19, 4, 13, 14, 7, 26].

While non-commuting operators are of sufficent interest in and of themselves, in some cases there are certain symmetries or degeneracies which permit operators to "almost" commute, or, as we shall discuss, "semi-commute". In essence, suppose P is a differential operator of degree m and Q is an operator of degree n. Then the commutator $[P,Q]$ is, in general, an operator of degree $m+n-1$. Let us fix the choice of P. Then, how can one select Q so that the highest order terms in $[P,Q]$ vanish? This will be the focus of the present paper. Whereas P and Q might not commute, we show that Q may be selected so as to reduce the order of $[P,Q]$ to strictly less than $m+n-1$, under certain conditions. One may form the set of all such operators Q, and we say that such operators semi-commute with P. From this set of operators, one may then search for operators which commute with P; as such a set will be much smaller than the set of all degree n operators Q, searching within such a set for Q commutative with P is much more efficient.

In Section 2, we shall discuss the notion of semi-commuting ordinary differential operators. We provide theorems on the construction of semi-commuting operators for monic P in the case where P and Q have analytic coefficients. Most of the material in this section is known for the restricted case of commuting operators, and can be omitted by those familiar with this area. However, this material will be useful in the following sections, and is included for completeness. In Section 3, we consider the case of ordinary differential operators with singular coefficients, and show that operators Q which semi-commute with P preserve the singular structure of P. In Section 4, we make some remarks concerning semi-commuting partial differential operators in many variables. We then turn our attention to some applications in Section 5, considering Airy, Bessel and Hypergeometric operators to demonstrate the construction of semi-commuting operators for in the ordinary case. We also discuss cases for which ordinary differential operators locally commute. Meanwhile, in order to demonstrate the results for partial differential operators, we consider Appell operators (a generalization of the ${}_2F_1$ Hypergeometric operator to two variables). Finally, in Section 6 we provide some discussions on difficulties extending the results to infinite dimensional operators and nonlinear operators.

2. Semi-commutative operators: the ordinary case for sufficiently differentiable coefficients

Here we introduce the notion of ordinary differential operators which semi-commute. In general, for ordinary differential operators P and Q of degrees $m \geq 2$ and n, the commutator $[P,Q]$ is an ordinary differential operator of degree $m+n-1$. Fix P. Then, if Q is chosen such that the degree of $[P,Q]$ is less than or equal to $m-2$ (as we are free to select the $n+1$ variable coefficients of Q) we say that P and Q semi-commute. For arbitrary operator P, often the best we can do is to select an operator Q which semi-commutes with P. If additional conditions hold, we show that P and Q commute. Again, most of the material in this section is known for the restricted case of commuting operators, and can be omitted by those familiar with this area. However, this material will be useful in the following sections, and is included for completeness.

Let us introduce some notation. Let $\Lambda^{k,h}$ denote the set of degree k ordinary differential operators with variable coefficients which are all h times continuously differentiable, that is

$$\Lambda^{k,h} \stackrel{\text{def}}{=} \left\{ L \mid L = \sum_{i=0}^{k} a_i(x) D^i, \ a_i \in C^h(\mathbb{R}) \right\}. \tag{1}$$

We shall denote the set of degree k ordinary differential operators with variable coefficients which are all smooth ($C^\infty(\mathbb{R})$) by $\Lambda^{k,\infty}$.

Remark 1. *The following properties are apparent:*

1. $\Lambda^{k,0} \supset \Lambda^{k,1} \supset \Lambda^{k,2} \supset \cdots \quad \forall k \geq 0$,
2. $\Lambda^{k,\infty} \subset \Lambda^{k,h} \quad \forall k \geq 0 \quad \forall 0 \leq h < \infty$,
3. $\Lambda^{0,h} \subset \Lambda^{1,h} \subset \Lambda^{2,h} \subset \cdots \quad \forall k \geq 0$.

Let $\Gamma^k(P)$ denote the set

$$\Gamma^k(P) \stackrel{\text{def}}{=} \left\{ L \mid [P, L] \in \Lambda^{k,0} \right\}. \tag{2}$$

Meanwhile, define the set $\Gamma^*(P)$ by

$$\Gamma^*(P) \stackrel{\text{def}}{=} \{ L \mid [P, L] = 0 \}. \tag{3}$$

Thus, $\Gamma^k(P)$ denotes the set of all ordinary differential operators L such that $[P, L]$ is a k-jet of some function of D.

Remark 2. *The following properties are apparent:*

1. $\Gamma^0(P) \subset \Gamma^1(P) \subset \Gamma^2(P) \subset \cdots$,
2. $\Gamma^*(P) \subset \Gamma^k(P) \quad \forall_{k \geq 0}$.

Proposition 1. *Let $P \in \Lambda^{m,n}$, $Q \in \Lambda^{n,m}$. Then, $[P, Q] \in \Lambda^{m+n-1,0}$, $Q \in \Gamma^{m+n-1}(P)$, and $P \in \Gamma^{m+n-1}(Q)$.*

Proof. This follows from the above definitions. □

Definition 1. *Let $P \in \Lambda^{m,n}$ for $m \geq 2$, $n \geq 1$. For some $Q \in \Lambda^{n,m}$, if $[P, Q] \in \Lambda^{m-2,0}$, then we say that P and Q semi-commute.*

Then, we have

Theorem 1 (Semi-commutativity of P and Q)**.** *Given monic $P \in \Lambda^{m,n}$, we may, in principle, construct an $(m-1)(n+1)$ parameter family of (not necessarily monic) $Q \in \Lambda^{n,m}$ such that P and Q semi-commute.*

Proof. It is sufficient to consider

$$P = D^m + p_{m-1}(x)D^{m-1} + \cdots + p_1(x)D + p_0(x), \tag{4}$$

as P may always be made monic. By assumption, $p_j \in C^n(\mathbb{R})$. Then, consider

$$Q = q_n(x)D^n + \cdots + q_1(x)D + q_0(x), \tag{5}$$

where the q_j's are functions $q_j \in C^m(\mathbb{R})$ which are to be determined. Clearly, $[P, Q] \in \Lambda^{m+n-1,0}$, and in particular,

$$[P, Q] = \sum_{k=0}^{m+n-1} N_k[q_n, \ldots, q_1, q_0]D^k, \tag{6}$$

where the N_k's are linear functions of the q_j's and their derivatives up to order $m-1$. Explicitly, after some tedious computation, we find that

$$\begin{aligned} N_\chi = \sum_{\eta=0}^{\chi} & \left\{ p_\eta(x) \sum_{\xi=0}^{\eta} \binom{\eta}{\xi} D^{\eta-\xi} q_{\chi-\eta}(x) - q_{\chi-\eta}(x) \sum_{\xi=0}^{\chi-\eta} \binom{\chi-\eta}{\xi} D^{\chi-\eta-\xi} p_\eta(x) \right\} \\ & \times I_1(\eta) I_2(\chi - \eta), \end{aligned} \tag{7}$$

where $I_1(y) = 1$ if $y \leq m$ and $I_1(y) = 0$ for $y > m$, while $I_2(y) = 1$ if $y \leq n$ and $I_2(y) = 0$ for $y > n$ are indicator functions which prevent us from counting extraneous terms (we have summed an array along the diagonal).

Consider the system of $n + 1$ linear differential equations

$$\begin{aligned} N_{m+n-1}[q_n, \ldots, q_1, q_0] &= 0 \\ N_{m+n-2}[q_n, \ldots, q_1, q_0] &= 0 \\ &\vdots \\ N_{m-2}[q_n, \ldots, q_1, q_0] &= 0, \end{aligned} \tag{8}$$

for $n + 1$ unknown functions $q_n, \ldots, q_0$. From the equations derived above, we see that $N_{m+n-1-\ell}$ is a $(m - 1)$th order linear ODE in q_{n-k}. In the general case, we may always express this as a system of $(m - 1)(n + 1)$ first order linear differential equations. By assumption, all coefficients are at worst $C^0(\mathbb{R})$, so a solution to the system in the form of an initial value problem exists and is unique. Such a system will then admit a solution $\tilde{q}_n(x; \beta_{n,1}, \ldots, \beta_{n,m-1})$,...,$\tilde{q}_0(x; \beta_{0,1}, \ldots, \beta_{0,m-1})$, where each q_k depends on $m - 1$ free parameters; let $\beta_{i,j} \in \mathcal{B} \subseteq \mathbb{R}^{(m-1)(n+1)}$. Such solutions identically satisfy (8). So, taking

$$Q = \tilde{q}_n(x)D^n + \cdots + \tilde{q}_1(x)D + \tilde{q}_0(x), \tag{9}$$

we see that

$$[P, Q] = \sum_{k=0}^{m-2} N_k[\tilde{q}_n, \ldots, \tilde{q}_1, \tilde{q}_0]D^k, \tag{10}$$

hence $[P, Q] \in \Lambda^{m-2,0}$. □

Corollary 1 (Commutativity of P and Q). *For Q taken in the proof of Theorem 2.5, if the additional conditions*

$$\begin{aligned} N_{m-2}[\tilde{q}_n,\dots,\tilde{q}_1,\tilde{q}_0] &= 0 \\ N_{m-3}[\tilde{q}_n,\dots,\tilde{q}_1,\tilde{q}_0] &= 0 \\ &\vdots \\ N_0[\tilde{q}_n,\dots,\tilde{q}_1,\tilde{q}_0] &= 0, \end{aligned} \tag{11}$$

hold, then $[P,Q]=0$; that is, P and Q commute.

Note that requiring $N_k = 0$ for all k results in an overdetermined system for the q_k's; the additional conditions listed here are not likely to hold in general. However, as $N_{m-3},\dots,N_0$ depend on the $m(n+1)$ parameters $\beta_{k,j}$, such parameters may be selected to permit conditions (11) in specific special cases.

For $P \in \Lambda^{1,n}$, semi-commutativity of P and $Q \in \Lambda^{n,1}$ is equivalent to commutativity of such P and Q. That is to say, all first order ordinary differential operators $P = D + p_0(x)$ commute with a differential operator Q if and only if P and Q semi-commute. In the case $P = D + p_0(x)$, from (7) we have the system of precisely $n+1$ linear ODEs

$$\begin{aligned} Dq_j(x) + p_0(x)q_j(x) = 0 \quad \text{for all} \quad j = 1,2,\dots,n, \\ Dq_0(x) = \sum_{k=1}^{n} q_k(x) D^k p_0(x). \end{aligned} \tag{12}$$

The solution to this system is given by

$$\begin{aligned} q_j(x) &= \beta_j \exp\left(-\int_0^x p_0(t)dt\right) \quad \text{for all} \quad j = 1,2,\dots,n, \\ q_0(x) &= \beta_0 + \int_0^x \exp\left(-\int_0^t p_0(\tau)d\tau\right)\left\{\sum_{k=1}^{n} \beta_k D^k p_0(x)|_{x=t}\right\} dt, \end{aligned} \tag{13}$$

an $n+1$ parameter family in the β's. Then, if solutions q_j to (12) are taken to be those given in (13), P and Q semi-commute. However, placing such a form of Q into $[P,Q]$, we find that $[P,Q]=0$ identically and hence P and Q commute.

However, if P is of degree greater than one, commutativity of course implies semi-commutativity while semi-commutativity does not imply commutativity. That said, computing the space of Q that semi-commute with P ($Q \in \Gamma^{m-2}(P)$ in the case of analytic coefficients) allows us to narrow the search for Q that are commuting with P (the space $\Gamma^*(P)$ in the case of analytic coefficients).

3. Semi-commutative operators: the singular ordinary case

We remark that in the above proofs and definitions, we have always taken the coefficients of P and Q to be sufficiently continuously differentiable. However, in some circumstances, this requirement will not hold. We would still like to say something about such cases, which include certain differential operators with solutions having singularities.

Consider the ordinary differential operator

$$P = D^m + p_{m-1}(x)D^{m-1} + \cdots + p_1(x)D + p_0(x), \tag{14}$$

and let $\{s_i\}_{i\in I}$ be discrete singularities of the coefficients (where I is an index set). In the case when P has only regular singular points close to the ith singularity we know that there exist analytic functions $\tilde{p}_{k;i}(x)$ such that

$$p_k(x) = \frac{\tilde{p}_{k;i}(x)}{(x-s_i)^{\gamma_{k;i}}}, \tag{15}$$

where $0 \le \gamma_{k;i} \le m-k$. Now, to determine all Q of degree n which semi-commute with P, let us assume

$$Q = q_n(x)D^n + q_{n-1}(x)D^{n-1} + \cdots + q_1(x)D + q_0(x). \tag{16}$$

To compute the commutator $[P,Q]$, we will need the formula

$$D^\ell\left(\frac{\tilde{p}_{k;i}(x)}{(x-s_i)^{\gamma_{k;i}}}\right) = \sum_{j=0}^{\ell} \frac{g_j(\gamma_{k;i})}{(x-s_i)^{\gamma_{k;i}+j}} \frac{d^{\ell-j}\tilde{p}_{k;i}(x)}{dx^j}, \tag{17}$$

where g_j is a degree j polynomial in $\gamma_{k;i}$ (i.e., a constant in x). From here, we find that near $x = s_i$

$$[P,Q] = \sum_{k=0}^{m+n-1} N_k[q_n,\dots,q_1,q_0]D^k, \tag{18}$$

where the N_k's are linear functions of the q_j's and their derivatives up to order $m-1$ with singular coefficients due to the singularity s_i. As in (7) of the previous section, after some tedious computation, we find that

$$\begin{aligned} N_\chi = \sum_{\eta=0}^{\chi}\{ &\frac{\tilde{p}_\eta(x)}{(x-s_i)^{\gamma_{\eta;i}}} \sum_{\xi=0}^{\eta}\binom{\eta}{\xi} D^{\eta-\xi} q_{\chi-\eta}(x) \\ &- q_{\chi-\eta}(x)\sum_{\xi=0}^{\chi-\eta}\binom{\chi-\eta}{\xi}\sum_{j=0}^{\chi-\eta-\xi}\frac{g_j(\gamma_\eta;i)}{(x-s_i)^{\gamma_{\eta;i}+j}} D^{\chi-\eta-\xi-j}p_\eta(x)\} \\ &\times\ I_1(\eta)I_2(\chi-\eta), \end{aligned} \tag{19}$$

where $I_1(y) = 1$ if $y \le m$ and $I_1(y) = 0$ for $y > m$, while $I_2(y) = 1$ if $y \le n$ and $I_2(y) = 0$ for $y > n$ are indicator functions which prevent us from counting extraneous terms (we have summed an array along the diagonal).

Theorem 2 (Regular singular points of P and Q)**.** *Given $P \in \Lambda^{m,n}$ with regular singular points $\{s_i\}_{i\in I}$, we may construct $Q \in \Lambda^{n,m}$ such that P and Q semi-commute and Q shares the regular singular points of P.*

Proof. From the analysis above, clearly the regular singular points of P are regular singular points of Q. Assume that Q has some singular point s_* that is not a singular point of P, yet P and Q semi-commute. Then, reversing the roles of P and Q, the singular points of P must be the singular points of Q. □

By a similar argument, irregular singular points are also transferred from P to Q. In other words,

Theorem 3. *Differential operators Q which semi-commute with P share the singular structure of P in the sense that regular (irregular) singular points of P are regular (irregular) singular points of Q.*

4. Semi-commutative operators: the partial differential operator case

In the setting of partial differential operators, we can recast the results discussed for the ordinary differential operators considered in the previous sections. To that end, we shall be required to consider partial differential operators in ν variables of the form

$$P = \sum_{n_1=0}^{N_1} \cdots \sum_{n_\nu=0}^{N_\nu} a_{n_1,\ldots,n_\nu}(x_1,\ldots,x_\nu)\partial_{x_1}^{n_1}\cdots\partial_{x_\nu}^{n_\nu}. \tag{20}$$

Here, $\partial_{x_k} = \partial/\partial x_k$. For brevity, we shall denote $\mathbf{n} = (n_1,\ldots,n_\nu)$, $\mathbf{x} = (x_1,\ldots,x_\nu) \in \mathbb{R}^\nu$, $\partial = (\partial_{x_1},\ldots,\partial_{x_\nu})$, and $\mathbf{N} = (N_1,\ldots,N_\nu)$. Making use of the multi-index notation, we may write

$$P = \sum_{\mathbf{0}\le\mathbf{n}\le\mathbf{N}} a_{\mathbf{n}}(\mathbf{x})\partial^{\mathbf{n}}. \tag{21}$$

For vectors $\mathbf{h} = (h_1,\ldots,h_\nu)$ and $\mathbf{n}$ (as above), let $\Pi^{\mathbf{k},\mathbf{h}}$ denote the set of partial differential operators of degree N_k in the kth variable, with variable coefficients which are all h_j times continuously differentiable in the variable x_j, that is

$$\Pi^{\mathbf{n},\mathbf{h}} \overset{\text{def}}{=} \left\{ L \mid L = \sum_{\mathbf{0}\le\mathbf{n}\le\mathbf{N}} a_{\mathbf{n}}(\mathbf{x})\partial^{\mathbf{n}},\ \forall_{\mathbf{n}}\ a_{\mathbf{n}} \ \text{ is } \ C^{h_j}(\mathbb{R}) \ \text{ for } \ x_j \right\}. \tag{22}$$

We denote the set of such operators with variable coefficients which are all smooth in all variables ($a_{\mathbf{n}}C^\infty(\mathbb{R}^\nu)$) by $\Pi^{\mathbf{n},\infty}$.

Remark 3. *The following properties are obvious:*

1. $\Pi^{\mathbf{n},(h_1,\ldots,h_{j-1},k_1,h_{j+1},\ldots,h_\nu)} \supset \Pi^{\mathbf{n},(h_1,\ldots,h_{j-1},k_2,h_{j+1},\ldots,h_\nu)}$ *for all* $0 \le k_1 \le k_2 \le \infty$ *and all* $|\mathbf{n}| \ge 0$,
2. $\Pi^{(n_1,\ldots,n_{j-1},k_1,n_{j+1},\ldots,n_\nu),\mathbf{h}} \subset \Pi^{(n_1,\ldots,n_{j-1},k_2,n_{j+1},\ldots,n_\nu),\mathbf{h}}$ *for all* $0 \le k_1 \le k_2 \le \infty$ *and all* $|\mathbf{h}| \ge 0$.

Let $\Omega^{\mathbf{n}}(P)$ denote the set

$$\Omega^{\mathbf{n}}(P) \stackrel{\text{def}}{=} \left\{L \mid [P, L] \in \Pi^{\mathbf{n},\mathbf{0}}\right\}, \tag{23}$$

and let $\Omega^*(P)$ denote the set

$$\Omega^*(P) \stackrel{\text{def}}{=} \{L \mid [P, L] = 0\}. \tag{24}$$

Thus, $\Omega^{\mathbf{n}}(P)$ denotes the set of all partial differential operators L such that $[P, L]$ is a k-jet of some function of ∂, whereas $\Omega^*(P)$ denotes the set of all partial differential operators L such that P and L commute.

Remark 4. *The following properties are obvious:*

1. $\Omega^{(n_1,\ldots,n_{j-1},k_1,n_{j+1},\ldots,n_\nu)}(P) \subset \Omega^{(n_1,\ldots,n_{j-1},k_2,n_{j+1},\ldots,n_\nu)}(P)$ *for all* $0 \le k_1 \le k_2 \le \infty$*,*
2. $\Omega^*(P) \subset \Omega^{\mathbf{0}}(P)$.

Proposition 2. *Let* $P \in \Pi^{\boldsymbol{m},\boldsymbol{n}}$*,* $Q \in \Pi^{\boldsymbol{n},\boldsymbol{m}}$*. Then,* $[P, Q] \in \Pi^{\boldsymbol{m}+\boldsymbol{n}-\boldsymbol{1},\boldsymbol{0}}$*,* $Q \in \Omega^{\boldsymbol{m}+\boldsymbol{n}-\boldsymbol{1}}(P)$*, and* $P \in \Omega^{\boldsymbol{m}+\boldsymbol{n}-\boldsymbol{1}}(Q)$*. Here,* $\boldsymbol{m} = (m_1, \ldots, m_\nu)$*,* $\boldsymbol{n} = (n_1, \ldots, m_\nu)$*,* $\boldsymbol{1} = (1, \ldots, 1) \in \mathbb{Z}^\nu$*,* $\boldsymbol{0} = (0, \ldots, 0) \in \mathbb{Z}^\nu$*.*

Proof. This follows from the above definitions and properties. □

Definition 2. *Let* $P \in \Pi^{\boldsymbol{m},\boldsymbol{n}}$ *for* $\boldsymbol{m} \ge \boldsymbol{2}$*,* $\boldsymbol{n} \ge \boldsymbol{1}$*. For* $Q \in \Pi^{\boldsymbol{n},\boldsymbol{m}}$*, if* $[P, Q] \in \Pi^{\boldsymbol{m}-\boldsymbol{2},\boldsymbol{0}}$*, then we say that* P *and* Q *semi-commute. Note that by* $\boldsymbol{a} \ge \boldsymbol{b}$ *for* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^\nu$*, we mean that* $a_j \ge b_j$ *for all* $j = 1, 2, \ldots, \nu$*.*

The general relations for the partial differential equations governing the coefficients q of the partial differential operator Q semi-commuting with a given partial differential operator P are similar to, though in general far more complicated than, those given in the case of the ordinary differential operators (Sections 2 and 3). We omit the details here, as often for a given partial differential operator there will be certain symmetries to exploit so as to permit commutativity. In the definition given here, note that there are cases in which the reduction of order of the commutator $[P, Q]$ cannot be achieved strictly by selection of the coefficients of Q. For instance, consider operators P and Q which are second order in two variables, x_1 and x_2. Assuming no reductions or symmetries, each operator will in general have six terms. The commutator of P and Q will, however, have ten terms. Given P, in order for Q to semi-commute with P, we would need to pick Q so that the commutator is reduced to order one. Yet, of the ten terms, four are order three and three are order two. Hence, seven partial differential equations would need to be satisfied by the six free coefficients of Q, leading to an over determined system. This is the primary difference with semi-commuting *partial* differential operators: in general, we cannot select Q semi-commuting with general P via a restriction of the coefficients of Q. However, for particular forms of P, we are capable of finding semi-commuting operators Q. This is illustrated in Section 5.6, where we consider a Hypergeometric equation in two variables.

In analogy to the analytic and singular coefficient cases considered in Sections 2 and 3 for the ordinary differential operators, one may construct semi-commuting partial differential operators which preserve the singular structure of the original partial differential operator of interest in some instances. (Of course, this is impossible in full generality, as one is not always able to construct semi-commuting partial differential operators for a given operator, by the discussion in the above paragraph.)

5. Operators commutative and semi-commutative with well-known differential operators

We apply methods discussed above, along with basic knowledge of ordinary differential equations, in order to obtain operators commutative with several specific differential operators. For some of the cases, we are able to demonstrate the solutions to such operators.

5.1. The first order case

In Section 2, we have already shown that the operators Q semi-commuting with first order operators P are exactly those which commute with P. The first order ordinary differential equation $Py = (D + p_0(x))y = 0$ admits the solution $y = y(0)\exp\left(-\int_0^x p_0(t)dt\right)$. From the relations obtained in (12) and (13), we know precisely the form that differential operators Q satisfying $[P, Q]$ must take. So, let us consider solutions to the nth order ODE $Qz = 0$. It is clear that such an ODE equation takes the form

$$\frac{d^n z}{dx^n} + b_{n-1}\frac{d^{n-1}z}{dx^{n-1}} + \cdots + b_1\frac{dz}{dx} + f(x)z = 0, \tag{25}$$

where $b_k = \beta_k/\beta_n$ and

$$\begin{aligned} f(x) &= \exp\left(\int_0^x p_0(t)dt\right) \\ &\quad \times \left\{ b_0 + \int_0^x \exp\left(-\int_0^t p_0(\tau)d\tau\right)\left\{\sum_{k=1}^n b_k D^k p_0(x)|_{x=t}\right\} dt \right\}. \end{aligned} \tag{26}$$

From the Burchnall and Chaundy theory, P and Q must share a solution. Hence, whenever we encounter an ordinary differential equation of the form provided in (25), one may employ reduction of order (using the said shared solution of P and Q) in order to make the above ODE more amenable to analysis.

5.2. The general $\dim(P) = 2$ case and local commutativity

For general second order differential operators $P = D^2 + p_1(x)D + p_0(x)$, we now outline the explicit construction of operators Q of degree n commutative with P. The $\dim(P) = 2$ case is worthy of consideration, as (i) many interesting special functions are governed by second order linear differential operators and (ii) the dimension-two case is the first non-trivial instance in which semi-commutative operators may not be commutative. Interestingly, we find that even for some well-known special operators P there may exist a class of semi-commutative differential operators Q, none of which commute with P. We then discuss what we shall call *local commutativity* of operators.

We first give the explicit relations that the q_k's must satisfy in order for operators P and Q to be semi-commutative. Indeed, we find that if the q_k's satisfy

$$\begin{aligned} &Dq_n(x) = 0, \\ &Dq_k(x) = \frac{1}{2}\sum_{j=k+1}^{n}\binom{j}{k}q_j(x)D^{j-k}p_1(x) - \frac{1}{2}P[q_{k+1}(x)], \end{aligned} \tag{27}$$

for all $k = 0, 1, \ldots, n-1$, then P and Q semi-commute. One additional equation,

$$D^2q_0(x) + p_1(x)Dq_0(x) = \sum_{k=1}^{n} q_k(x)D^k p_0(x), \tag{28}$$

remains free. If this additional equation is satisfied, then P and Q commute provided that the q_k's satisfy (27). As discussed above, for sufficiently well-behaved p_1 and p_0, we can solve (27) to obtain the q_k's, which will depend on parameters β_k (there will be a total of $n+1$ arbitrary parameters β_k, as we have $n+1$ first order differential equations in (27)). Placing such solutions back into (28), we obtain a relation

$$r(x; \beta_0, \beta_1, \ldots, \beta_n) = 0. \tag{29}$$

In general, r will not be identically zero for all x. However, provided a solution $x_0 = x_0(\beta_0, \beta_1, \ldots, \beta_n)$ to relation (29) exists, all requirements for commutativity are satisfied at the point x_0. In such a case, we say that P and Q *locally commute* at $x = x_0$.

Let P and Q be semi-commuting differential operators with real valued variable coefficients over $\mathbb{R}$. Then, if $[P, Q] = 0$ at $x = x_\ell$, we say that P and Q locally commute at x_ℓ. In the case of $\dim(P) = 2$, the is equivalent to the 0-jet of $[P, Q]$ vanishing at x_ℓ. In the case of $\dim(P) = 1$, the notion is not needed as all semi-commuting operators trivially commute.

5.3. Airy's equation

The Airy function is named after the British astronomer George Biddell Airy, who encountered it in his study of optics [3]. The Airy differential equation is

$$y'' + xy = 0,$$

with solutions $\mathrm{Ai}(x)$ and $\mathrm{Bi}(x)$. Explicitly [1],

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt,$$

and

$$\mathrm{Bi}(x) = \frac{1}{\pi}\int_0^\infty \left\{ \exp\left(-\frac{t^3}{3} + xt\right) + \sin\left(\frac{t^3}{3} + xt\right)\right\} dt.$$

The Airy operator is the second order operator $P_{\mathrm{Airy}} = D^2 + x$, which falls into the set of second order operators with C^∞ coefficients, $\Lambda^{2,\infty}$, so the theory of Section 2 applies. We may thus apply the methods of Section 2 in order to construct nth order linear operators $Q = q_n D^n + \cdots q_1 D + q_0$ semi-commuting with the Airy operator. From (27) we see that the coefficients of Q must satisfy

$$\begin{aligned} &Dq_n = 0,\\ &Dq_k = -\frac{1}{2}P_{\mathrm{Airy}}[q_{k+1}] \quad \text{for} \quad k = 0, 1, \ldots, n-1. \end{aligned} \tag{30}$$

Such a system can be solved recursively starting with $q_n(x) = \beta_n$. In order for $\deg(Q) = n$ we need $\beta_n \neq 0$. Then, from relations (30) we see that q_k must be a polynomial of degree $2(n-k)$ in x. For such q_k's, Q semi-commutes with P_{Airy}. To illustrate this, let us constrict Q in the case $n = 3$. We find that all third order operators Q commuting with P take the form $Q = q_3(x)D^3 + q_2(x)D^2 + q_1(x)D + q_0(x)$, where $q_3(x) = \beta_3$ and

$$q_2(x) = -\frac{1}{4}x^2\beta_3 + \beta_2, \tag{31}$$

$$q_1(x) = \frac{1}{4}x\beta_3 + \frac{1}{32}x^4\beta_3 - \frac{1}{4}x^2\beta_2 + \beta_1, \tag{32}$$

$$q_0(x) = -\frac{5}{48}x^3\beta_3 + \frac{1}{4}x\beta_2 - \frac{1}{384}x^6\beta_3 + \frac{1}{32}x^4\beta_2 - \frac{1}{4}x^2\beta_1 + \beta_0. \tag{33}$$

Then, the condition (28) becomes

$$D^2 q_0 = q_1 \Rightarrow \frac{7\beta_3}{64}x^4 - \frac{5\beta_2}{8}x^2 + \frac{7\beta_3}{8}x + \frac{3\beta_1}{2} = 0, \tag{34}$$

which is not identically satisfied for any $\beta_3 \neq 0$. However, at the roots of this degree 4 polynomial we have that $[P, Q] = 0$ and hence P and Q locally commute at such roots.

We remark that, in general, the semi-commuting Q cannot be put into a form which commutes with P in the Airy case. In general, when $\dim(Q) = n$, we find that $r(x; \beta_0, \ldots, \beta_n) = 0$ has at least $2(n-1)$ complex roots (r being a degree $2(n-1)$ polynomial in x). There are too few coefficients to force $r \equiv 0$.

5.4. Bessel's equation

Bessel's differential equation is given by

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - a^2)y = 0.$$

The corresponding differential operator is

$$P_{\mathrm{Bessel}} = D^2 + \frac{1}{x}D + \frac{x^2 - a^2}{x^2}.$$

This operator falls into the category of differential operators with singular coefficients, so the theory of Section 3 applies. In order for a degree n differential operator Q to semi-commute with P_{Bessel}, the coefficients of $Q = q_n D^n + \cdots q_1 D + q_0$ must satisfy

$$\begin{aligned} Dq_n &= 0, \\ Dq_k &= \frac{1}{2} \sum_{j=k+1}^{n} \binom{j}{k} \frac{(-1)^{j-k}(j-k)!}{x^{j-k+1}} q_j - \frac{1}{2} P_{\text{Bessel}}[q_{k+1}] \end{aligned} \tag{35}$$

for $k = 0, 1, \ldots, n-1$. We find that the q_k's take the form

$$q_k = \frac{\tilde{q}_k}{x^{n-k}} \quad \text{for} \quad k = 0, 1, \ldots, n-1, \tag{36}$$

where $\tilde{q}_k$ is a polynomial in x. Hence, $x = 0$ is a regular singular point for the operator Q, as expected from the results in Section 3.

5.5. The hypergeometric equation

The hypergeometric differential equation is given by

$$x(1-x)\frac{d^2y}{dx^2} + [c - (a+b+1)x]\frac{dy}{dx} - aby = 0.$$

Solutions are given in terms of the hypergeometric function,

$${}_2F_1(a, b; c; x) = \sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n,$$

where

$$(q)_n = q(q+1)(q+2)\cdots(q+n-1) = \frac{\Gamma(q+n)}{\Gamma(q)} = \frac{(q+n-1)!}{(q-1)!}$$

is the rising factorial, or the *Pochhammer symbol.*

We define the corresponding differential operator as

$$P_{H1} = D^2 + \frac{c-(a+b+1)x}{x(1-x)} D - \frac{ab}{x(1-x)} 0.$$

In order for Q of degree n to semi-commute with P_{H1}, the coefficients of $Q = q_n D^n + \cdots q_1 D + q_0$ must satisfy

$$\begin{aligned} Dq_n &= 0, \\ Dq_k &= \frac{1}{2} \sum_{j=k+1}^{n} \binom{j}{k} D \left[\frac{c-(a+b+1)x}{x(1-x)} \right] q_j - \frac{1}{2} P_{\text{H1}}[q_{k+1}], \end{aligned} \tag{37}$$

for $k = 0, 1, \ldots, n-1$. The explicit expressions for the q_k's are complicated in general. However, we find that solutions will preserve the singular structure of P_{H1}.

5.6. Hypergeometric equations in multiple variables

In order to demonstrate operators in multiple variables, we can consider the multivariate generalizations of the Hypergeometric function ${}_2F_1$. When there are two variables, we have the Appell functions (F_1, F_2, F_3, F_4) [5] - [12]. Meanwhile, in the case of three or more variables, we have the Lauricella functions [17], [6], [25]. Here we shall take the Appell operators corresponding to F_1. It has been shown [5] that F_1 satisfies the two partial differential equations

$$\left(x(1-x)\frac{\partial^2}{\partial x^2} + y(1-x)\frac{\partial^2}{\partial x \partial y} + [c-(a+b_1+1)x]\frac{\partial}{\partial x} - b_1 y\frac{\partial}{\partial y} - ab_1\right)F_1 = 0, \tag{38}$$

$$\left(y(1-y)\frac{\partial^2}{\partial y^2} + x(1-y)\frac{\partial^2}{\partial x \partial y} + [c-(a+b_2+1)y]\frac{\partial}{\partial y} - b_2 x\frac{\partial}{\partial x} - ab_2\right)F_1 = 0. \tag{39}$$

From here, we define the corresponding partial differential operators P_{H2a} and P_{H2b} by

$$P_{\mathrm{H2a}} = x(1-x)\partial_x^2 + y(1-x)\partial_x\partial_y + [c-(a+b_1+1)x]\partial_x - b_1 y\partial_y - ab_1, \tag{40}$$

$$P_{\mathrm{H2b}} = y(1-y)\partial_y^2 + x(1-y)\partial_x\partial_y + [c-(a+b_2+1)y]\partial_y - b_2 x\partial_y - ab_2. \tag{41}$$

Consider a general operator Q of the form

$$Q = \sum_{0\le i+j\le 2,\ i,j>0} q_{ij}\partial_x^i\partial_y^j, \tag{42}$$

so that there are 6 coefficient q_{ij}'s. Now, the commutator of P_{H2a} and Q takes the form

$$[P_{\mathrm{H2a}}, Q] = \sum_{0\le i+j\le 3,\ i,j>0} \kappa_{ij}\partial_x^i\partial_y^j. \tag{43}$$

Here κ_{ij} will depend on the q_{ij}'s through somewhat lengthy expressions (of which we omit the explicit forms). Without any assumptions on symmetry, the commutator will have 10 such κ's. However, for the case of P_{H2a}, we observe that $\kappa_{03} \equiv \kappa_{02} \equiv 0$ for all Q. In order for the operators P_{H2a} and Q to semi-commute, we need all higher-order terms (specifically, degree 2 and 3 terms) in commutator (43) to vanish. As there are five such non-zero κ_{ij}'s which depend linearly on some of the six q_{ij}'s we can, in principle, ensure that the operator Q can be taken so that P_{H2a} and Q do semi-commute. However, we can actually do better in the present case. Due to the structure of commutator (43) we can actually select q_{ij}'s such that P_{H2a} and Q commute.

We find that $q_{02} \equiv 0$ implies $\kappa_{12} \equiv \kappa_{21} \equiv 0$. We then find that $\kappa_{11} \equiv 0$ if and only if $q_{01}(x,y) = f_{01}(y)$ (i.e., q_{01} is constant in x). The condition for $\kappa_{01} \equiv 0$ then becomes equivalent to $yf_{01}'(y) = f_{01}(y)$, i.e., $f_{01}(y) = \mu_{01}y$, where μ_{01} is a constant of integration. From here we are left with four equations and four unknown q_{ij}'s. The remaining commutativity conditions and dependences are (in order of degree)

$$\begin{aligned} &\kappa_{30} = \kappa_{30}(q_{11}, q_{20}) \equiv 0, \kappa_{20} = \kappa_{20}(q_{11}, q_{20}, q_{10}) \equiv 0,\\ &\kappa_{10} = \kappa_{10}(q_{10}, q_{00}) \equiv 0, \kappa_{00} = \kappa_{00}(q_{00}) \equiv 0, \end{aligned} \tag{44}$$

a system of coupled linear partial differential equations. The simplest of these equations is $\kappa_{00}(q_{00}) \equiv 0$, which is the parabolic equation

$$(1-x)(x+y)\partial_x^2 q_{00} + [c - (a+b+1)x]\partial_x q_{00} - b_1 y \partial_y q_{00} = 0. \tag{45}$$

One then substitutes the solution q_{00} to this partial differential equation into the equation $\kappa_{10}(q_{10}, q_{00}) \equiv 0$ to solve for q_{10}. Then, one places the solution into κ_{20} to arrive at a system of two partial differential equations for the remaining unknown functions q_{11} and q_{20}.

5.7. The wave equation over one spatial dimension with variable wave speed

Consider the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{46}$$

and associated linear partial differential operator

$$P_{\text{Wave}} = \partial_t^2 - c^2 \partial_x^2. \tag{47}$$

Note that we may write P_{Wave} as the product $P_{\text{Wave}} = P_{W1}P_{W2} = P_{W2}P_{W1}$ where

$$P_{W1} = \partial_t - c\partial_x \quad \text{and} \quad P_{W2} = \partial_t + c\partial_x \tag{48}$$

are first order operators which clearly commute. One may wonder if such a factorization is possible in the case of a non-constant wave speed, say $c = c(x,t)$. Such a factorization was possible for constant c because P_{W1} and P_{W2} commute. Consider the operator

$$\tilde{P}_{\text{Wave}} = \partial_t^2 - \partial_x \left\{ (c(x,t))^2 \partial_x \right\}, \tag{49}$$

which permits variable wave speed. Let us define first order operators

$$\tilde{P}_{W1} = \partial_t - g_-(x,t)\partial_x \quad \text{and} \quad \tilde{P}_{W2} = \partial_t + g_+(x,t)\partial_x, \tag{50}$$

and note that an unknown function $g(x,t)$, rather than $c(x,t)$, is taken. It is clear that in general these operators do not commute. Let us see if they can be made to commute or semi-commute with an appropriate choice of $g(x,t)$. Calculating the commutator, we find

$$[\tilde{P}_{W1}, \tilde{P}_{W2}] = \{\partial_t(g_- + g_+) + g_-\partial_x g_+ - g_+\partial_x g_-\}\,\partial_x. \tag{51}$$

Thus, the operators do not automatically semi-commute. Note that if we set $g_- = -g_+$, then the operators identically commute; however, this is not useful as in such a case $\tilde{P}_{W1} \equiv \tilde{P}_{W2}$ and $\tilde{P}_{\text{Wave}} \neq \tilde{P}_{W1}\tilde{P}_{W2}$. Let us consider the case where $g_- = g_+$. Then commutator (51) becomes

$$[\tilde{P}_{W1}, \tilde{P}_{W2}] = \{2\partial_t g_+\}\,\partial_x. \tag{52}$$

In this case, it is clear that the operators $\tilde{P}_{W1}$ and $\tilde{P}_{W2}$ commute if $g_+ = g_+(x)$ (i.e., g_+ does not depend on t). This stands to reason, as the substitution $\partial_x \to \tilde{\partial}_x \equiv g_+(x)\partial_x$ yields operators of the form (48). Setting $g_+(x,t) = G(x)$ and $g_-(x,t) = G(x)$ for some function $G(x) \in C^1(\mathbb{R})$ we have that

$$\begin{aligned} \tilde{P}_{W1}\tilde{P}_{W2} &= \partial_t^2 - G(x)G'(x)\partial_x - (G(x))^2\partial_x^2 \\ &= \partial_t^2 - \partial_x\left\{(G(x))^2\partial_x\right\}. \end{aligned} \tag{53}$$

This equation exactly matches (49) when $c(x,t) \equiv G(x)$. Hence, for $c(x,t)$ depending only on x, we obtain a factorization $\tilde{P}_{\text{Wave}} = \tilde{P}_{W1}\tilde{P}_{W2}$.

6. Conclusions

While we were able to obtain results for finite linear operators with variable coefficients, it is worthwhile to note that we omit two large classes of differential operators. First, we omit infinite dimensional linear operators. Our constructive proofs relied on the fact that for a given finite order P we were constructing finite order Q such that the variable coefficients of Q satisfy a system of deg$Q+1$ linear differential equations for which a solution exists (locally). One could, for finite order P, construct infinite order Q such that Q semi-commutes with P. However, additional requirements would be needed to ensure that the infinite system of differential equations governing the coefficients of Q admits a solution; such conditions would be on the variable coefficients of P. The case in which both P and Q are infinite becomes more complicated.

We have also omitted any discussion of commuting nonlinear operators. Even in the case of constant coefficients, there appears to be no good way to go about constructing commuting or semi-commuting nonlinear differential operators, at least in the general case. However, for some nonlinear operators which depend on small parameters, one may expand the operator in a perturbation about the small parameter (see below). Let us consider the example of a nonlinear operator and a linear operator which share a solution yet do not commute. Let

$$M[y] = Dy + y^2 \quad \text{and} \quad L[y] = D^2y - \frac{2}{(x+1)^2}y \tag{54}$$

denote the nonlinear and linear differential operators, respectively. Observe that both operators admit a solution

$$y(x) = \frac{1}{x+1} \tag{55}$$

for initial conditions $y(0) = 1$, $y'(0) = -1$. Although these operators share a solution, observe that the resulting operator formed by the commutator is non zero:

$$\begin{aligned} [M,L]\,[y] &= \left(D^2y\right)^2 - 2\,(Dy) - \left(2 + \frac{4}{(x+1)^2}\right)y(D^2y) \\ &\quad + \left(\frac{2}{(x+1)^2} + \frac{4}{(x+1)^4}\right)y^2 + \frac{4}{(x+1)^3}y. \end{aligned} \tag{56}$$

In many cases, two given operators may not commute. However, they may semi-commute or even locally commute for some x. In analogy to local commutativity for some x in the domain, note that an operator Q with parameter ϵ may commute with a given operator P for some fixed values of ϵ. Indeed, we have illustrated this through some of the examples given in Section 5: in some cases, the arbitrary constants of integration obtained in finding the coefficients of Q can be fixed so as to permit commutativity. Let us take a different look at this approach. Suppose Q and P commute at $\epsilon_0 = 0$ (we can take $\epsilon_0 = 0$ through linear scaling) and semi-commute for all other values of ϵ. Then, for ϵ near $\epsilon_0 = 0$ we consider the expansion of Q in terms of ϵ:

$$Q = Q_0 + \epsilon Q_1 + \epsilon^2 Q_2 + \cdots . \tag{57}$$

Then,

$$[P, Q] = [P, Q_0] + \epsilon[P, Q_1] + \epsilon^2[P, Q_2] + \cdots . \tag{58}$$

Yet, $Q_0 = Q|_{\epsilon=0}$ and $[P, Q] = 0$ at $\epsilon = 0$, so $[P, Q_0] = 0$ and hence

$$[P, Q] = \epsilon[P, Q_1] + \epsilon^2[P, Q_2] + \cdots . \tag{59}$$

Consider a solution $y(x; \epsilon)$ to the homogeneous differential equation $Qy = 0$. As $[P, Q_0] = 0$ we have from the Burchnall and Chaundy theory that P and Q_0 share a solution. Let $y_0(x) = y(x; 0)$ be this solution. Now, P and Q commute up to zeroth order, and the higher order corrections in (59) are deformations due to the non-commutativity. For small ϵ, the corrections are minor, and as $|\epsilon| \to 0$, the semi-commutativity collapses to commutativity. Such decompositions (59) may be useful in the study of semi-commuting operators, as the sequence $Q_1, Q_2, Q_3 \ldots$ is often times simpler than the original operator Q. Such a perturbation approach might also prove useful for nonlinear operators. If Q is nonlinear, and P is linear, then in some cases we can expand Q as given in (57) in the sense that $Q_1, Q_2, Q_3 \ldots$ are all linear operators. Observe that, for such a case, if Q is finite then the $Q_1, Q_2, Q_3 \ldots$ are finite. Hence this gives us a way to study the commutativity of nonlinear operators Q with a linear operator P over certain parameter regimes by studying the relation between the linear operator P and the linear $Q_1, Q_2, Q_3 \ldots$.

As another way to view the relation between commuting and semi-commuting operators, recall that for a given operator P of degree $m \geq 2$, Q_1 of degree n semi-commutes with P provided $[P, Q_1]$ has degree $m-2$. Let $P_1 \equiv [P, Q_1]$. Now acting on P_1, let us select another operator, Q_2 of degree n, so that P_1 and Q_2 semi-commute. Then, $[P_1, Q_2]$ is of degree $m - 4$. After a finite number of reductions ($\frac{m}{2} + 1$ for even m, $\frac{m+1}{2} + 1$ for odd m) the process will terminate, yielding operators P_κ and $Q_{\kappa+1}$ which commute. The sequence of Q_j's thus satisfies the nested commutator relation

$$[\cdots[[[P, Q_1], Q_2], Q_3] \cdots, Q_{\kappa+1}] = 0. \tag{60}$$

In some cases, this may help in finding an operator $\mathcal{Q}$ which commutes with P; such a $\mathcal{Q}$ should satisfy the relation

$$[P, \mathcal{Q}] = [\cdots[[[P, Q_1], Q_2], Q_3] \cdots, Q_{\kappa+1}]. \tag{61}$$

Semi-commuting operators can be useful, as they comprise a large set of non-commuting operators, of which commuting operators are a subset. Hence, computing the large set of operators, we can impose additional conditions to ensure that the operators commute. In this manner, one may constructively obtain operators Q commuting with some fixed operator P, by considering a restriction of the set of operators P and Q which semi-commute.

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