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On an inequality of I. Perić*

Sadia Khalid[1,†], Josip Pečarić[2] and Marjan Praljak[3]

[1] *Abdus Salam School of Mathematical Sciences, GC University, 68-B, New Muslim Town, Lahore 54 600, Pakistan*
[2] *Faculty of Textile Technology, University of Zagreb, Prilaz baruna Filipovića 28a, HR-10 000 Zagreb, Croatia*
[3] *Faculty of Food Technology and Biotechnology, University of Zagreb, Pierottijeva 6, HR-10 000 Zagreb, Croatia*

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Abstract. Generalizations of some results given in [1] and [5] are presented and it is shown that, after appropriate substitution, one of the results is equivalent to an inequality given in [10]. We also construct new families of exponentially convex functions and Cauchy-type means by looking at linear functionals associated with the obtained inequalities.
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1. Introduction and preliminaries

The following inequality is given by Hardy-Littlewood-Pólya (see [3, Theorem 134]).

Theorem 1. *If f is a convex and continuous function defined on $[0,\infty)$ and a_k, $k \in \mathbb{N}$ are non-negative and non-increasing, then*

$$f\left(\sum_{k=1}^{n} a_k\right) \geq f(0) + \sum_{k=1}^{n} \left[f(ka_k) - f((k-1)a_k)\right]. \tag{1}$$

If f' is a strictly increasing function, there is an equality only when a_k are equal up to a certain point and then zero.

In 1986, G. Bennett [1] proved the weighted version of (1) for power functions $f(x) = x^s$: if $a_k, k = 1, \ldots, n$ are non-negative and non-increasing and $p_k \geq 0$ for each $k = 1, \ldots, n$ with

$$P_k := \sum_{i=1}^{k} p_i, \quad k = 1, \ldots, n,$$

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†Corresponding author. *Email addresses:* `saadiakhalid176@gmail.com` (S. Khalid), `pecaric@element.hr` (J. Pečarić), `mpraljak@pbf.hr` (M. Praljak)

then for any real number $s > 1$,

$$\left(\sum_{k=1}^{n} p_k a_k\right)^s \geq \sum_{k=1}^{n} P_k^s \left[a_k^s - a_{k+1}^s\right] = (p_1 a_1)^s + \sum_{k=2}^{n} a_k^s \left[P_k^s - P_{k-1}^s\right] \tag{2}$$

holds. If $0 < s < 1$, then (2) holds in the reverse direction (see [1]).

Some generalizations of Hardy-Littlewood-Pólya inequality are presented by S. Khalid, J. Pečarić and M. Praljak in [6] and some generalizations of G. Bennett's result are proved by J. Pečarić, I. Perić and R. Roki in [7]. The aim of this paper is to present some new generalizations and refinements of inequalities (1) and (2). Our results generalize not only the G. Bennett's result but also some results recently proved by S. Khalid and J. Pečarić in [5]. In this paper, we also obtain some results which are related with the discrete weighted reversed Hardy-type inequality. In order to obtain our main results first, let us recall some definitions which are going to be used throughout this paper.

Definition 1. *A sequence $(a_k,\, k \in \mathbb{N}) \subset \mathbb{R}$ is non-increasing in weighted mean, if*

$$\frac{1}{P_n} \sum_{k=1}^{n} p_k a_k \geq \frac{1}{P_{n+1}} \sum_{k=1}^{n+1} p_k a_k, \quad n \in \mathbb{N}, \tag{3}$$

where a_k and $p_k, k \in \mathbb{N}$ are real numbers such that $p_i > 0, i = 1, \ldots, k$ with

$$P_k = \sum_{i=1}^{k} p_i, \quad k \in \mathbb{N}.$$

A sequence $(a_k,\, k \in \mathbb{N}) \subset \mathbb{R}$ is non-decreasing in weighted mean, if the opposite inequality holds in (3).
In a similar way, we can define when a finite sequence $(a_k,\, k = 1, \ldots, n) \subset \mathbb{R}$ is non-increasing or non-decreasing in weighted mean.

Remark 1. *It is easy to see that a sequence $(a_k,\, k \in \mathbb{N})$ is non-increasing in weighted mean (non-decreasing in weighted mean) if and only if*

$$\sum_{i=1}^{k-1} p_i a_i \geq P_{k-1} a_k \quad \left(\sum_{i=1}^{k-1} p_i a_i \leq P_{k-1} a_k\right)$$

for $k = 2, 3, \ldots$ holds.

The following property of a convex function will be used later (see [9, p.2]).

Definition 2. *A function $f : I \to \mathbb{R}$ is convex on I if*

$$(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geq 0 \tag{4}$$

holds for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$.

Another characterization of a convex function will be needed later (see [9, p.2]).

Proposition 1. *If f is a convex function on an interval I and if $x_1 \le y_1$, $x_2 \le y_2$, $x_1 \ne x_2$, $y_1 \ne y_2$, then the following inequality*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(y_2) - f(y_1)}{y_2 - y_1} \tag{5}$$

holds. If the function f is concave, the inequality reverses.

By letting $x_1 = x$, $x_2 = x + h$, $y_1 = y$ and $y_2 = y + h$, $x \le y, h \ge 0$ in (5), we have,

$$f(x + h) - f(x) \le f(y + h) - f(y). \tag{6}$$

The following definition of Wright-convex function is given in [9, p.7].

Definition 3. *A function $f : [a, b] \to \mathbb{R}$ is said to be Wright-convex if for all x, $y + h \in [a, b]$ such that $x \le y$, $h \ge 0$, (6) holds. The function f is said to be Wright-concave if the opposite inequality holds in (6).*

Remark 2. *If $K([a, b])$ and $W([a, b])$ denotes the class of all convex functions and the class of all Wright-convex functions, respectively, then $K([a, b]) \subsetneqq W([a, b])$. That is, a convex function must be a Wright-convex function but not conversely (see [9, p.7]).*

Wright-convex functions have an interesting and important generalization for functions of several variables (see [2]). Let $\mathbb{R}^m$ denote the m-dimensional vector lattice of points $\mathbf{x} = (x_1, \dots, x_m)$, $x_i \in \mathbb{R}$ for $i = 1, \dots, m$, with partial ordering

$$\mathbf{x} = (x_1, \dots, x_m) \le \mathbf{y} = (y_1, \dots, y_m)$$

if and only if $x_i \le y_i$ for $i = 1, \dots, m$ (see [9, p.13]).

For any two m-tuples $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, let us define component-wise multiplication and division as follows:

$$\mathbf{xy} = (x_1 y_1, \dots, x_m y_m),$$
$$\frac{\mathbf{x}}{\mathbf{y}} = \left(\frac{x_1}{y_1}, \dots, \frac{x_m}{y_m}\right), \quad \mathbf{y} \ne \mathbf{0}.$$

Definition 4. *A sequence $(\mathbf{a}_k, k \in \mathbb{N}) \subset \mathbb{R}^m$ is non-increasing in weighted mean if*

$$\frac{1}{\mathbf{P}_n} \sum_{k=1}^{n} \mathbf{p}_k \mathbf{a}_k \ge \frac{1}{\mathbf{P}_{n+1}} \sum_{k=1}^{n+1} \mathbf{p}_k \mathbf{a}_k, \quad n \in \mathbb{N}, \tag{7}$$

where

$$\mathbf{a}_k = \left(a_k^1, \dots, a_k^m\right), \mathbf{p}_k = \left(p_k^1, \dots, p_k^m\right) \text{ and } \mathbf{P}_k = \left(\sum_{i=1}^{k} p_i^1, \dots, \sum_{i=1}^{k} p_i^m\right) \in \mathbb{R}^m$$

such that $\mathbf{p}_k > \mathbf{0}, k \in \mathbb{N}$.

A sequence $(\mathbf{a}_k, k \in \mathbb{N}) \subset \mathbb{R}^m$ is non-decreasing in weighted mean if the opposite inequality holds in (7).

In [2], H. D. Brunk explored an interesting class of multivariate real-valued functions defined as follows:

Definition 5. *A real-valued function f on an m-dimensional rectangle $I \subset \mathbb{R}^m$ is said to have non-decreasing increments if*

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) \leq f(\mathbf{y}+\mathbf{h}) - f(\mathbf{y}), \tag{8}$$

whenever $\mathbf{x}$, $\mathbf{y}+\mathbf{h} \in I$, $\mathbf{0} \leq \mathbf{h} \in \mathbb{R}^m$, $\mathbf{x} \leq \mathbf{y}$. The function f is said to have non-increasing increments if the opposite inequality holds in (8).

Remark 3. *It is easy to see that if a function f is defined on $[a,b] \subset \mathbb{R}$, then the functions having non-decreasing increments are Wright-convex functions.*

Our first main result is the direct generalization of the Hardy-Littlewood-Pólya inequality (1), a result of G. Bennett given in [1] and also a result of S. Khalid and J. Pečarić given in [5]. After appropriate substitutions, our first main result is equivalent to the following inequality given by I. Perić (see [10, Theorem 1.4]).

Theorem 2. *Let f be a Wright-concave function defined on $[0,\infty)$. Let $0 = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \cdots$, $C_n \geq 0$, $n \in \mathbb{N}$ and*

$$\sum_{k=1}^{n} C_k (x_k - x_{k-1}) \geq C_{n+1} x_n, \quad n \geq 1. \tag{9}$$

Then

$$f\left(\sum_{k=1}^{n} C_k (x_k - x_{k-1})\right) + \sum_{k=1}^{n-1} f(C_{k+1} x_k) \leq \sum_{k=1}^{n} f(C_k x_k), \quad n \in \mathbb{N}. \tag{10}$$

The organization of the paper is the following: in Section 2, we present generalizations of some results given in [1] and [5] and show that one of our result is equivalent to the inequality from Theorem 2. In Section 3, the objective is to study the functionals defined as the difference between the right-hand and the left-hand side of the generalized inequalities and also their properties, such as n-exponential and logarithmic convexity. Furthermore, we prove the monotonicity property of the generalized Cauchy means obtained via these functionals. Finally, in Section 4, we give several examples of the families of functions for which the results can be applied and also present a refinement of inequality (2).

2. Main results

Inequality (2) is already proved in [1], but we give a new proof in a more general setting. The following result is our first main result.

Theorem 3. *Let a_k and $p_k, k = 1,\ldots,n$ be real numbers such that $a_k \geq 0$ and $p_k > 0$. Let*

$$p_1 a_1, \ \sum_{k=1}^{n} p_k a_k, \ P_k a_k, \ P_{k-1} a_k \in [a,b]$$

for all $k = 2,\ldots,n$ and let $f : [a,b] \to \mathbb{R}$ be a Wright-convex function.

(*i*) *If the sequence* $(a_k,\ k=1,\ldots,n)$ *is non-increasing in weighted mean, then we have*

$$f\left(\sum_{k=1}^{n} p_k a_k\right) \geq f\left(p_1 a_1\right) + \sum_{k=2}^{n}\left[f\left(P_k a_k\right) - f\left(P_{k-1} a_k\right)\right]. \tag{11}$$

(*ii*) *If the sequence* $(a_k,\ k=1,\ldots,n)$ *is non-decreasing in weighted mean, then we have*

$$f\left(\sum_{k=1}^{n} p_k a_k\right) \leq f\left(p_1 a_1\right) + \sum_{k=2}^{n}\left[f\left(P_k a_k\right) - f\left(P_{k-1} a_k\right)\right]. \tag{12}$$

If the function f is Wright-concave, then the opposite inequalities hold in (11) *and* (12).

Proof. (i): Since the sequence $(a_k,\ k=1,\ldots,n) \subset \mathbb{R}$ is non-increasing in weighted mean, by definition we have

$$\sum_{i=1}^{k-1} p_i a_i \geq P_{k-1} a_k$$

for $k=2,\ldots,n$. As f is a Wright-convex function, by setting

$$x = P_{k-1} a_k,\ y = \sum_{i=1}^{k-1} p_i a_i \quad \text{and} \quad h = p_k a_k,\ k=2,\ldots,n$$

in (6), we have

$$f\left(\sum_{i=1}^{k} p_i a_i\right) - f\left(\sum_{i=1}^{k-1} p_i a_i\right) \geq f\left(P_k a_k\right) - f\left(P_{k-1} a_k\right).$$

Summing over k from 2 to n, we have

$$f\left(\sum_{i=1}^{n} p_i a_i\right) - f\left(p_1 a_1\right) \geq \sum_{k=2}^{n}\left[f\left(P_k a_k\right) - f\left(P_{k-1} a_k\right)\right],$$

and so (11) holds.

(ii): Since the sequence $(a_k,\ k=1,\ldots,n) \subset \mathbb{R}$ is non-decreasing in weighted mean, by definition we have

$$\sum_{i=1}^{k-1} p_i a_i \leq P_{k-1} a_k$$

for $k=2,\ldots,n$. As f is a Wright-convex function, by setting

$$x = \sum_{i=1}^{k-1} p_i a_i,\ y = P_{k-1} a_k \quad \text{and} \quad h = p_k a_k,\ k=2,\ldots,n$$

in (6), we have

$$f\left(\sum_{i=1}^{k} p_i a_i\right) - f\left(\sum_{i=1}^{k-1} p_i a_i\right) \leq f\left(P_k a_k\right) - f\left(P_{k-1} a_k\right).$$

Now summing over k from 2 to n and after simplification, we have (12).

If f is a Wright-concave function, then the opposite inequality holds in (6) and so the opposite inequalities hold in (11) and (12). □

Remark 4. *For a Wright-concave function f, Theorems 2 and 3(i) are equivalent. First of all, from the proof of Theorem 2 (see [10, p.6]) it is obvious that the interval $[0,+\infty)$ can be replaced by $[a,b]$. Furthermore, with the substitutions $a_k = C_k$ and $p_k = x_k - x_{k-1}, k = 1,\ldots,n$ condition (9) is equivalent to the condition that the sequence $(a_k,\ k = 1,\ldots,n)$ is non-increasing in weighted mean and inequality (10) is equivalent to the reverse of (11).*

Since the class of Wright-convex (Wright-concave) functions properly contains the class of convex (concave) functions (see for example [9, p.7]), the following result is valid:

Corollary 1. *Let a_k and $p_k, k = 1,\ldots,n$ be real numbers such that $a_k \geq 0$ and $p_k > 0$. Let*

$$p_1 a_1,\ \sum_{k=1}^{n} p_k a_k,\ P_k a_k,\ P_{k-1} a_k \in [a,b]$$

for all $k = 2,\ldots,n$ and let $f : [a,b] \to \mathbb{R}$ be a convex function.

- (i) *If the sequence $(a_k,\ k = 1,\ldots,n)$ is non-increasing in weighted mean, then (11) holds.*
- (ii) *If the sequence $(a_k,\ k = 1,\ldots,n)$ is non-decreasing in weighted mean, then (12) holds.*

If the function f is concave, then the opposite inequalities hold in (11) and (12).

The following corollary is an application of Corollary 1:

Corollary 2. *Let $f(x) = x^s$, where $x \in (0,\infty)$ and $s \in \mathbb{R}$.*

- (i) *If the sequence $(a_k > 0,\ k = 1,\ldots,n)$ is non-increasing in weighted mean, $p_k > 0, k = 1,\ldots,n$ and $s \in \mathbb{R}$ such that $s < 0$ or $s > 1$, then*

$$\left(\sum_{k=1}^{n} p_k a_k\right)^s \geq (p_1 a_1)^s + \sum_{k=2}^{n} a_k^s \left[P_k^s - P_{k-1}^s\right] \tag{13}$$

  holds. If $0 < s < 1$, then (13) holds in the reverse direction.

- (ii) *If the sequence $(a_k > 0,\ k = 1,\ldots,n)$ is non-decreasing in weighted mean, $p_k > 0, k = 1,\ldots,n$ and $s \in \mathbb{R}$ such that $s < 0$ or $s > 1$, then*

$$\left(\sum_{k=1}^{n} p_k a_k\right)^s \leq (p_1 a_1)^s + \sum_{k=2}^{n} a_k^s \left[P_k^s - P_{k-1}^s\right] \tag{14}$$

  holds. If $0 < s < 1$, then (14) holds in the reverse direction.

Remark 5. *Theorem* 3 *is a generalization of inequality* (2) *in the sense that we can obtain* (2) *as a special case of our first main result.*

The following result is related with the discrete weighted reversed Hardy-type inequality (see [11]):

Theorem 4. *Let a_k and $p_k, k = 1, \dots, n$ be real numbers such that $a_k \geq 0$ and $p_k > 0$ and let $b_n \geq 0$, for $n = 2, \dots, m$. Let*

$$p_1 a_1,\ \sum_{k=1}^{n} p_k a_k,\ P_k a_k,\ P_{k-1} a_k \in [a, b]$$

for all $k = 2, \dots, n$ and let $f : [a, b] \to \mathbb{R}$ be a differentiable convex function.

(i) *If the sequence $(a_k,\ k = 1, \dots, n)$ is non-increasing in weighted mean, then we have*

$$\sum_{n=2}^{m} b_n f\left(\sum_{k=1}^{n} p_k a_k\right) \geq f\left(p_1 a_1\right) \sum_{n=2}^{m} b_n + \sum_{k=2}^{m} f^{'}\left(P_{k-1} a_k\right) p_k a_k \sum_{n=k}^{m} b_n. \tag{15}$$

(ii) *If the sequence $(a_k,\ k = 1, \dots, n)$ is non-decreasing in weighted mean, then we have*

$$\sum_{n=2}^{m} b_n f\left(\sum_{k=1}^{n} p_k a_k\right) \leq f\left(p_1 a_1\right) \sum_{n=2}^{m} b_n + \sum_{k=2}^{m} f^{'}\left(P_{k-1} a_k\right) p_k a_k \sum_{n=k}^{m} b_n. \tag{16}$$

If the function f is concave, then the opposite inequalities hold in (15) *and* (16)*.*

Proof. (i): By following the steps as given in the proof of Theorem 3 (i), we have (11). Multiplying (11) by $b_n \geq 0, n = 2, \dots, m$ and summing over n from 2 to m, we get

$$\sum_{n=2}^{m} b_n f\left(\sum_{k=1}^{n} p_k a_k\right) \geq f\left(p_1 a_1\right) \sum_{n=2}^{m} b_n + \sum_{k=2}^{m} \left[f\left(P_k a_k\right) - f\left(P_{k-1} a_k\right)\right] \sum_{n=k}^{m} b_n. \tag{17}$$

Due to convexity of f, we have

$$f\left(P_k a_k\right) - f\left(P_{k-1} a_k\right) \geq f'\left(P_{k-1} a_k\right)\left(P_k a_k - P_{k-1} a_k\right) = f'\left(P_{k-1} a_k\right) p_k a_k,$$

which together with (17) implies (15).

(ii): (12) can be obtained by using the same arguments as given in the proof of Theorem 3 (ii). Now multiplying (12) by $b_n \geq 0$, summing over n from 2 to m and also using convexity of f, we get (16).

If f is a concave function, then the opposite inequality holds in (11) and (12) and so the opposite inequalities hold in (15) and (16). □

Example 1. *Let $f(x) = x^s$, where $x \in (0,\infty)$ and $s \in \mathbb{R}$. If the sequence $(a_k > 0,\ k = 1,\dots,n)$ is non-increasing in weighted mean, $p_k > 0, k = 1,\dots,n$ and $s \in \mathbb{R}$ such that $s < 0$ or $s > 1$, then*

$$\sum_{n=2}^{m} b_n \left(\sum_{k=1}^{n} p_k a_k\right)^s \geq (p_1 a_1)^s \sum_{n=2}^{m} b_n + s \sum_{k=2}^{m} p_k a_k^s P_{k-1}^{s-1} \sum_{n=k}^{m} b_n \tag{18}$$

holds. If $0 < s < 1$, then (18) holds in the reverse direction.

Now, let $s > 1$ and take $b_n = p_n P_{n-1}^{-s}$. Using the fact that

$$\sum_{n=k}^{m} p_n P_{n-1}^{-s} \geq \int_{P_{k-1}}^{P_m} x^{-s} dx = \frac{P_{k-1}^{1-s} - P_m^{1-s}}{s-1},$$

from (18) we get

$$\sum_{n=2}^{m} p_n \left(\frac{1}{P_{n-1}} \sum_{k=1}^{n} p_k a_k\right)^s \geq \frac{p_1 a_1^s}{s-1}\left(1 - \left(\frac{p_1}{P_m}\right)^{s-1}\right) + \frac{s}{s-1} \sum_{k=2}^{m} p_k a_k^s \left(1 - \left(\frac{P_{k-1}}{P_m}\right)^{s-1}\right). \tag{19}$$

By adding $p_1 a_1^s$ to both sides of (19) and with the convention $P_0 = p_1$, if $m \to \infty$ and $P_m \to \infty$, inequality (19) becomes

$$\sum_{n=1}^{\infty} p_n \left(\frac{P_n}{P_{n-1}}\right)^s \left(\frac{1}{P_n} \sum_{k=1}^{n} p_k a_k\right)^s \geq \frac{s}{s-1} \sum_{k=1}^{\infty} p_k a_k^s$$

and represents a discrete weighted reversed Hardy-type inequality.

The multidimensional generalization is stated as follows:

Theorem 5. *Let*

$$\mathbf{a}_k,\ \mathbf{p}_k \text{ and } \mathbf{P}_k = \left(\sum_{i=1}^{k} p_i^1, \dots, \sum_{i=1}^{k} p_i^m\right) \in \mathbb{R}^m$$

be such that $\mathbf{a}_k \geq 0$ and $\mathbf{p}_k > 0$ for each $k = 1,\dots,n$. Let

$$\mathbf{p}_1\mathbf{a}_1,\ \sum_{k=1}^{n} \mathbf{p}_k\mathbf{a}_k,\ \mathbf{P}_k\mathbf{a}_k,\ \mathbf{P}_{k-1}\mathbf{a}_k \in \mathbf{I}$$

for all $k = 2,\dots,n$ and let $f : \mathbf{I} \to \mathbb{R}$ be a real valued function having non-decreasing increments on the rectangle $\mathbf{I} \subseteq \mathbb{R}^m$.

(*i*) *If the sequence $(\mathbf{a}_k,\ k = 1,\dots,n)$ is non-increasing in weighted mean, then we have*

$$f\left(\sum_{k=1}^{n} \mathbf{p}_k\mathbf{a}_k\right) \geq f(\mathbf{p}_1\mathbf{a}_1) + \sum_{k=2}^{n} \left[f(\mathbf{P}_k\mathbf{a}_k) - f(\mathbf{P}_{k-1}\mathbf{a}_k)\right]. \tag{20}$$

(*ii*) *If the sequence* $(\mathbf{a}_k,\ k=1,\ldots,n)$ *is non-decreasing in weighted mean, then we have*

$$f\left(\sum_{k=1}^{n}\mathbf{p}_k\mathbf{a}_k\right)\le f\left(\mathbf{p}_1\mathbf{a}_1\right)+\sum_{k=2}^{n}\left[f\left(\mathbf{P}_k\mathbf{a}_k\right)-f\left(\mathbf{P}_{k-1}\mathbf{a}_k\right)\right]. \tag{21}$$

If the function f *has non-increasing increments, then the opposite inequalities hold in* (20) *and* (21).

Proof. The idea of the proof is the same as in Theorem 3.

(i): Since the sequence $(\mathbf{a}_k,\ k=1,\ldots,n)\subset\mathbb{R}^m$ is non-increasing in weighted mean, by definition we have

$$\sum_{i=1}^{k-1}\mathbf{p}_i\mathbf{a}_i\ge\mathbf{P}_{k-1}\mathbf{a}_k,$$

for $k=2,\ldots,n$. By setting $\mathbf{x}=\mathbf{P}_{k-1}\mathbf{a}_k$, $\mathbf{y}=\sum_{i=1}^{k-1}\mathbf{p}_i\mathbf{a}_i$ and $\mathbf{h}=\mathbf{p}_k\mathbf{a}_k, k=2,\ldots,n$ in (8), where f has non-decreasing increments, we have

$$f\left(\sum_{i=1}^{k}\mathbf{p}_i\mathbf{a}_i\right)-f\left(\sum_{i=1}^{k-1}\mathbf{p}_i\mathbf{a}_i\right)\ge f\left(\mathbf{P}_k\mathbf{a}_k\right)-f\left(\mathbf{P}_{k-1}\mathbf{a}_k\right).$$

Summing over k from 2 to n, we have

$$f\left(\sum_{k=1}^{n}\mathbf{p}_k\mathbf{a}_k\right)-f\left(\mathbf{p}_1\mathbf{a}_1\right)\ge\sum_{k=2}^{n}\left[f\left(\mathbf{P}_k\mathbf{a}_k\right)-f\left(\mathbf{P}_{k-1}\mathbf{a}_k\right)\right].$$

and so (20) holds.

(ii): Since the sequence $(\mathbf{a}_k,\ k=1,\ldots,n)\subset\mathbb{R}^m$ is non-decreasing in weighted mean, by definition we have

$$\sum_{i=1}^{k-1}\mathbf{p}_i\mathbf{a}_i\le\mathbf{P}_{k-1}\mathbf{a}_k,$$

for $k=2,\ldots,n$. By setting $\mathbf{x}=\sum_{i=1}^{k-1}\mathbf{p}_i\mathbf{a}_i$, $\mathbf{y}=\mathbf{P}_{k-1}\mathbf{a}_k$ and $\mathbf{h}=\mathbf{p}_k\mathbf{a}_k, k=2,\ldots,n$ in (8), where f has non-decreasing increments, we have

$$f\left(\sum_{i=1}^{k}\mathbf{p}_i\mathbf{a}_i\right)-f\left(\sum_{i=1}^{k-1}\mathbf{p}_i\mathbf{a}_i\right)\le f\left(\mathbf{P}_k\mathbf{a}_k\right)-f\left(\mathbf{P}_{k-1}\mathbf{a}_k\right).$$

Now, summing over k from 2 to n and after simplification, we have (21).

If f has non-increasing increments, then the opposite inequality holds in (8) and so the opposite inequalities hold in (20) and (21). □

The following theorem is proven in the same way as Theorem 5 and it represents the m-dimensional generalization of Theorem 2.

Theorem 6. *Let $f : \mathbf{I} \to \mathbb{R}$ be a real valued function having non-decreasing increments on a rectangle $\mathbf{I} \subseteq \mathbb{R}^m$ and let $\mathbf{C}_k$, $\mathbf{x}_k \in \mathbb{R}^m$ be such that $0 = \mathbf{x}_0 \leq \mathbf{x}_1 \leq \mathbf{x}_2 \leq \cdots \leq \mathbf{x}_n$ and $\mathbf{C}_k \geq 0$ for $k = 1, \ldots, n$. Furthermore, let*

$$\mathbf{C}_1\mathbf{x}_1,\ \sum_{k=1}^{n} \mathbf{C}_k\left(\mathbf{x}_k - \mathbf{x}_{k-1}\right),\ \mathbf{C}_k\mathbf{x}_k \text{ and } \mathbf{C}_{k-1}\mathbf{x}_k \in \mathbf{I}$$

for all $k = 2, \ldots, n$.

(i) *If the inequalities*

$$\sum_{i=1}^{k-1} \mathbf{C}_i\left(\mathbf{x}_i - \mathbf{x}_{i-1}\right) \geq \mathbf{C}_k\mathbf{x}_{k-1},\ \text{for } k \geq 1 \tag{22}$$

hold, then

$$f\left(\sum_{k=1}^{n} \mathbf{C}_k\left(\mathbf{x}_k - \mathbf{x}_{k-1}\right)\right) + \sum_{k=1}^{n-1} f\left(\mathbf{C}_{k+1}\mathbf{x}_k\right) \geq \sum_{k=1}^{n} f\left(\mathbf{C}_k\mathbf{x}_k\right). \tag{23}$$

(ii) *If the inequalities in* (22) *are reversed, then*

$$f\left(\sum_{k=1}^{n} \mathbf{C}_k\left(\mathbf{x}_k - \mathbf{x}_{k-1}\right)\right) + \sum_{k=1}^{n-1} f\left(\mathbf{C}_{k+1}\mathbf{x}_k\right) \leq \sum_{k=1}^{n} f\left(\mathbf{C}_k\mathbf{x}_k\right) \tag{24}$$

holds. If the function f has non-increasing increments, then the opposite inequalities hold in (23) *and* (24)*.*

If we take $p_k^1 = p_k^2 = p_k^3 = \ldots = p_k^m = p_k, k = 1, \ldots, n$ in $\mathbf{p}_k = \left(p_k^1, \ldots, p_k^m\right)$ in Theorem 5, then we have the following result:

Corollary 3. *Let $\mathbf{a}_k \in [0, \infty)^m$ and $p_k, k = 1, \ldots, n$ be real numbers such that $p_k > 0$. Let*

$$p_1\mathbf{a}_1,\ \sum_{k=1}^{n} p_k\mathbf{a}_k,\ P_k\mathbf{a}_k,\ P_{k-1}\mathbf{a}_k \in \mathbf{I}$$

for all $k = 2, \ldots, n$ and let $f : \mathbf{I} \to \mathbb{R}$ be a real valued function defined on a rectangle $\mathbf{I} \subseteq \mathbb{R}^m$ having non-decreasing increments.

(i) *If the sequence $(\mathbf{a}_k,\ k = 1, \ldots, n)$ is non-increasing in weighted mean, then we have*

$$f\left(\sum_{k=1}^{n} p_k\mathbf{a}_k\right) \geq f\left(p_1\mathbf{a}_1\right) + \sum_{k=2}^{n} \left[f\left(P_k\mathbf{a}_k\right) - f\left(P_{k-1}\mathbf{a}_k\right)\right]. \tag{25}$$

(ii) *If the sequence $(\mathbf{a_k},\ k = 1, \ldots, n)$ is non-decreasing in weighted mean, then we have*

$$f\left(\sum_{k=1}^{n} p_k\mathbf{a}_k\right) \leq f\left(p_1\mathbf{a}_1\right) + \sum_{k=2}^{n} \left[f\left(P_k\mathbf{a}_k\right) - f\left(P_{k-1}\mathbf{a}_k\right)\right]. \tag{26}$$

If the function f has non-increasing increments, then the opposite inequalities hold in (25) *and* (26).

Remark 6. *If we make the substitutions $p_k \to 1, k = 1, \ldots, n$ in our results, then the results given in [5] are recaptured.*

Consider the inequalities (11) and (15) and define two functionals

$$\Phi_1(f) = f\left(\sum_{k=1}^{n} p_k a_k\right) - f(p_1 a_1) - \sum_{k=2}^{n} \left[f(P_k a_k) - f(P_{k-1} a_k)\right], \tag{27}$$

$$\Phi_2(f) = \sum_{n=2}^{m} b_n f\left(\sum_{k=1}^{n} p_k a_k\right) - f(p_1 a_1) \sum_{n=2}^{m} b_n - \sum_{k=2}^{m} f'(P_{k-1} a_k) p_k a_k \sum_{n=k}^{m} b_n, \tag{28}$$

where $a_k \geq 0$, $p_k > 0$;

$$p_1 a_1, \ \sum_{k=1}^{n} p_k a_k, \ P_k a_k, \ P_{k-1} a_k \in [a, b]$$

for all $k = 2, \ldots, n$ and $b_n \geq 0$, for $n = 2, \ldots, m$. If the function f is convex defined on $[a, b]$ and the sequence $(a_k,\, k = 1, \ldots, n) \subset \mathbb{R}$ is non-increasing in weighted mean, then Corollary 1 (i), implies that $\Phi_1(f) \geq 0$, and if in addition f is differentiable, then Theorem 4 (i), implies that $\Phi_2(f) \geq 0$.

Now, we give mean value theorems for the functional $\Phi_i, i = 1, 2$. These theorems enable us to define various classes of means that can be expressed in terms of linear functionals.

Theorem 7. *Let a_k and $p_k, k = 1, \ldots, n$ be real numbers such that $a_k \geq 0$ and $p_k > 0$ and let $b_n \geq 0$, for $n = 2, \ldots, m$. Let*

$$p_1 a_1, \ \sum_{k=1}^{n} p_k a_k, \ P_k a_k, \ P_{k-1} a_k \in [a, b]$$

for all $k = 2, \ldots, n$ and let the sequence $(a_k,\, k = 1, \ldots, n)$ be non-increasing in weighted mean. Suppose that Φ_1 and Φ_2 are linear functionals as defined in (27) *and* (28) *and $f \in C^2([a, b])$. Then there exists $\xi_1, \xi_2 \in [a, b]$ such that*

$$\Phi_i(f) = \frac{f''(\xi_i)}{2} \Phi_i(f_0), \quad i = 1, 2,$$

holds, where $f_0(x) = x^2$.

Proof. Analogous to the proof of Theorem 2.2 in [8]. $\square$

The following theorem is a new analogue to the classical Cauchy mean value theorem related to the functionals $\Phi_i,\ i = 1, 2$, and it can be proven by following the proof of Theorem 2.4 in [8].

Theorem 8. *Let all the assumptions of Theorem 7 be satisfied and let $f, g \in C^2([a,b])$. Then there exist $\xi_1, \xi_2 \in [a,b]$ such that*

$$\frac{\Phi_i(f)}{\Phi_i(g)} = \frac{f''(\xi_i)}{g''(\xi_i)}, \quad i = 1, 2, \tag{29}$$

holds, provided that the denominators are non-zero.

Remark 7.

(i) *By taking $f(x) = x^s$ and $g(x) = x^q$ in (29), where $s, q \in \mathbb{R} \setminus \{0,1\}$ are such that $s \neq q$, we have*

$$\xi_i^{s-q} = \frac{q(q-1)\Phi_i(x^s)}{s(s-1)\Phi_i(x^q)}, \quad i = 1, 2.$$

(ii) *If the inverse of the function f''/g'' exists, then (29) gives*

$$\xi_i = \left(\frac{f''}{g''}\right)^{-1}\left(\frac{\Phi_i(f)}{\Phi_i(g)}\right), \quad i = 1, 2.$$

3. n-Exponential convexity and log-convexity

We begin this section by recollecting definitions and properties which are going to be explored here and we also study some useful characterizations of these properties. In the sequel, let I be an open interval in $\mathbb{R}$.

Definition 6. *A function $h : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if*

$$\sum_{i,j=1}^{n} \alpha_i \alpha_j h\left(\frac{x_i + x_j}{2}\right) \geq 0$$

holds for every $\alpha_i \in \mathbb{R}$ and $x_i \in I$, $i = 1, \ldots, n$ (see [8]).

Definition 7. *A function $h : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.*

Remark 8. *From the above definition it is clear that 1-exponentially convex functions in the Jensen sense are non-negative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex functions in the Jensen sense for all $k \in \mathbb{N}$, $k \leq n$.*

By the definition of positive semi-definite matrices and some basic linear algebra, we have the following proposition:

Proposition 2. *If $h : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense, then the matrix*

$$\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k}$$

is a positive semi-definite matrix for all $k \in \mathbb{N},\ k \leq n$. Particularly,

$$\det \left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0 \quad \text{for every} \quad k \in \mathbb{N}, k \leq n, x_i \in I, i = 1, \dots, n.$$

Definition 8. *A function $h : I \to \mathbb{R}$ is exponentially convex in the Jensen sense if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.*

Definition 9. *A function $h : I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.*

Lemma 1. *A function $h : I \to (0, \infty)$ is* log*-convex in the Jensen sense, that is, for every $x, y \in I$,*

$$h^2\left(\frac{x+y}{2}\right) \leq h(x)\, h(y)$$

holds if and only if the relation

$$\alpha^2 h(x) + 2\alpha\beta\, h\left(\frac{x+y}{2}\right) + \beta^2 h(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$.

Remark 9. *It follows that a positive function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense. Also, by using basic convexity theory, a positive function is log-convex if and only if it is 2-exponentially convex.*

The following definition of divided difference is given in [9, p.14].

Definition 10. *The second order divided difference of a function $f : [a, b] \to \mathbb{R}$ at mutually distinct points $y_0, y_1, y_2 \in [a, b]$ is defined recursively by*

$$\begin{aligned} [y_i; f] &= f(y_i), \quad i = 0, 1, 2, \\ [y_i, y_{i+1}; f] &= \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i}, \quad i = 0, 1, \\ [y_0, y_1, y_2; f] &= \frac{[y_1, y_2; f] - [y_0, y_1; f]}{y_2 - y_0}. \end{aligned} \tag{30}$$

Remark 10. *The value $[y_0, y_1, y_2; f]$ is independent of the order of the points y_0, y_1 and y_2. This definition may be extended to include the case in which some or all the points coincide* (*see [9, p.16]*)*. Namely, taking the limit $y_1 \to y_0$ in* (30)*, we get*

$$\lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_2; f] = \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, \quad y_2 \neq y_0,$$

provided that f' exists; and furthermore, taking the limits $y_i \to y_0$, $i = 1, 2$, in (30)*, we get*

$$\lim_{y_2 \to y_0} \lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_0; f] = \frac{f''(y_0)}{2},$$

provided that f'' exists.

Remark 11. *Convex functions can be characterized by second order divided difference (see [9, p.16]): a function $f : [a,b] \to \mathbb{R}$ is convex if and only if for all choices of three distinct points $y_0, y_1, y_2, \in [a,b]$, $[y_0, y_1, y_2; f] \geq 0$.*

Next, we study n-exponential convexity and log-convexity of the functions associated with linear functionals $\Phi_i, i = 1,2$ defined in (27) and (28).

Theorem 9. *Let $\Phi_i, i = 1,2$ be linear functionals as defined in* (27) *and* (28). *Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a,b]$ such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is n-exponentially convex in the Jensen sense on I for every three mutually distinct points $y_0, y_1, y_2 \in [a,b]$ (for $i = 2$, the functions $f_s \in \Omega$ must be differentiable). Then the following statements hold:*

(i) *The function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex in the Jensen sense on I and the matrix*
$$\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m}$$
is a positive semi-definite matrix for all $m \in \mathbb{N}$, $m \leq n$ and $s_1, \dots, s_m \in I$. Particularly,
$$\det\left[\Phi_i\left(f_{\frac{s_j+s_k}{2}}\right)\right]_{j,k=1}^{m} \geq 0, \quad \forall\ m \in \mathbb{N},\ m \leq n.$$

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is n-exponentially convex on I.*

Proof. The idea of the proof is the same as that of Theorem 3.1 in [8].

(i): Let $\alpha_j \in \mathbb{R}, j = 1, \dots, n$ and consider the function
$$\varphi(y) = \sum_{j,k=1}^{n} \alpha_j \alpha_k f_{\frac{s_j+s_k}{2}}(y),$$
where $s_j \in I$ and $f_{\frac{s_j+s_k}{2}} \in \Omega$. Then
$$[y_0, y_1, y_2; \varphi] = \sum_{j,k=1}^{n} \alpha_j \alpha_k \left[y_0, y_1, y_2; f_{\frac{s_j+s_k}{2}}\right]$$
and since
$$\left[y_0, y_1, y_2; f_{\frac{s_j+s_k}{2}}\right]$$
is n-exponentially convex in the Jensen sense on I by assumption, it follows that
$$[y_0, y_1, y_2; \varphi] = \sum_{j,k=1}^{n} \alpha_j \alpha_k \left[y_0, y_1, y_2; f_{\frac{s_j+s_k}{2}}\right] \geq 0.$$

And so by using Remark 11 we conclude that φ is a convex function. Hence
$$\Phi_i(\varphi) \geq 0, \quad i = 1,2,$$

which is equivalent to

$$\sum_{j,k=1}^{n} \alpha_j \alpha_k \Phi_i \left(f_{\frac{s_j+s_k}{2}} \right) \geq 0, \quad i = 1, 2,$$

and so we conclude that the function $s \mapsto \Phi_i(f_s)$ is n-exponentially convex in the Jensen sense on I. The remaining part follows from Proposition 2.

(ii): If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then from (i) and by Definition 7 it follows that it is n-exponentially convex on I. □

The following corollary is an immediate consequence of the above theorem.

Corollary 4. *Let $\Phi_i, i = 1, 2$ be linear functionals as defined in* (27) *and* (28)*. Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex in the Jensen sense on I for every three mutually distinct points $y_0, y_1, y_2 \in [a, b]$ (for $i = 2$, the functions $f_s \in \Omega$ must be differentiable). Then the following statements hold:*

(i) *The function $s \mapsto \Phi_i(f_s)$ is exponentially convex in the Jensen sense on I and the matrix*

$$\left[\Phi_i(f_{\frac{s_j+s_k}{2}}) \right]_{j,k=1}^{n}$$

is a positive semi-definite matrix for all $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in I$. Particularly,

$$\det \left[\Phi_i(f_{\frac{s_j+s_k}{2}}) \right]_{j,k=1}^{n} \geq 0, \quad \forall\ n \in \mathbb{N}.$$

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is exponentially convex on I.*

Corollary 5. *Let $\Phi_i, i = 1, 2$ be linear functionals as defined in* (27) *and* (28)*. Let $\Omega = \{f_s : s \in I \subseteq \mathbb{R}\}$ be a family of functions defined on $[a, b]$ such that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is 2-exponentially convex in the Jensen sense on I for every three mutually distinct points $y_0, y_1, y_2 \in [a, b]$ (for $i = 2$, the functions $f_s \in \Omega$ must be differentiable) and also assume that $\Phi_i(f_s), i = 1, 2$ are strictly positive for $f_s \in \Omega$. Then the following statements hold:*

(i) *If the function $s \mapsto \Phi_i(f_s)$ is continuous on I, then it is 2-exponentially convex on I and so it is log-convex on I and for $r, s, t \in I$ such that $r < s < t$, we have*

$$[\Phi_i(f_s)]^{t-r} \leq [\Phi_i(f_r)]^{t-s} [\Phi_i(f_t)]^{s-r}, \quad i = 1, 2, \tag{31}$$

known as Lyapunov's inequality. If $r < t < s$ or $s < r < t$, then the opposite inequality holds in (31)*.*

(ii) *If the function $s \mapsto \Phi_i(f_s)$ is differentiable on I, then for every $s, q, u, v \in I$ such that $s \leq u$ and $q \leq v$, we have*

$$\mu_{s,q}(\Phi_i, \Omega) \leq \mu_{u,v}(\Phi_i, \Omega), \quad i = 1, 2, \tag{32}$$

where

$$\mu_{s,q}\left(\Phi_i,\Omega\right)=\begin{cases}\left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(\frac{\frac{d}{ds}\Phi_i(f_s)}{\Phi_i(f_s)}\right), & s=q,\end{cases} \tag{33}$$

for $f_s, f_q \in \Omega$.

Proof. The idea of the proof is the same as that of Corollary 3.2 in [8].

(i): The claim that the function $s \mapsto \Phi_i(f_s)$ is log-convex on I, is an immediate consequence of Theorem 9 and Remark 9 and (31) can be obtained by replacing the convex function f by the convex function $f(z) = \log \Phi_i(f_z)$ for $z = r, s, t$ in (4), where $r, s, t \in I$ such that $r < s < t$.

(ii): Since by (i) the function $s \mapsto \Phi_i(f_s)$ is log-convex on I, that is, the function $s \mapsto \log \Phi_i(f_s)$ is convex on I. Applying Proposition 1 with setting $f(z) = \log \Phi_i(f_z)$, we get

$$\frac{\log \Phi_i(f_s) - \log \Phi_i(f_q)}{s-q} \le \frac{\log \Phi_i(f_u) - \log \Phi_i(f_v)}{u-v}, \tag{34}$$

for $s \le u$, $q \le v$, $s \ne q, u \ne v$; and therefore conclude that

$$\mu_{s,q}\left(\Phi_i,\Omega\right) \le \mu_{u,v}\left(\Phi_i,\Omega\right), \quad i=1,2.$$

If $s = q$, we consider the limit when $q \to s$ in (34) and conclude that

$$\mu_{s,s}\left(\Phi_i,\Omega\right) \le \mu_{u,v}\left(\Phi_i,\Omega\right), \quad i=1,2.$$

The case $u = v$ can be treated similarly. □

Remark 12. *Note that the results from Theorem 9, Corollary 4 and Corollary 5 still hold when two of the points* $y_0, y_1, y_2 \in [a,b]$ *coincide, say* $y_1 = y_0$*, for a family of differentiable functions* f_s *such that the function* $s \mapsto [y_0, y_1, y_2; f_s]$ *is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense on I); and furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs are obtained by recalling Remark 10 and using suitable characterizations of convexity.*

4. Examples

In this section, we present several families of functions which fulfil the conditions of Theorem 9, Corollary 4, Corollary 5 and Remark 12. This enables us to construct large families of functions which are exponentially convex.

Example 2. *Consider the family of functions*

$$\Omega_1 = \{g_s : \mathbb{R} \to [0,\ \infty) : s \in \mathbb{R}\}$$

defined by

$$g_s(x) = \begin{cases}\frac{1}{s^2}\, e^{sx}, & s \neq 0,\\ \frac{1}{2}\, x^2, & s = 0.\end{cases}$$

We have

$$\frac{d^2}{dx^2} g_s(x) = e^{sx} > 0,$$

which shows that g_s *is convex on* $\mathbb{R}$ *for every* $s \in \mathbb{R}$ *and* $s \mapsto g_s''(x)$ *is exponentially convex by definition* (*see also* [4]). *In order to prove that the function* $s \mapsto [y_0, y_1, y_2; g_s]$ *is exponentially convex, it is enough to show that*

$$\textstyle\sum_{j,k=1}^{n} \alpha_j \alpha_k \left[y_0, y_1, y_2; g_{\frac{s_j+s_k}{2}}\right] = \left[y_0, y_1, y_2; \sum_{j,k=1}^{n} \alpha_j \alpha_k g_{\frac{s_j+s_k}{2}}\right] \geq 0, \tag{35}$$

$\forall\, n \in \mathbb{N}$, $\alpha_j, s_j \in \mathbb{R}$, $j = 1, \ldots, n$. *By Remark 11,* (35) *will hold if*

$$\Upsilon(x) := \sum_{j,k=1}^{n} \alpha_j \alpha_k g_{\frac{s_j+s_k}{2}}(x)$$

is convex. Since $s \mapsto g_s''(x)$ *is exponentially convex, i.e*

$$\sum_{j,k=1}^{n} \alpha_j \alpha_k g''_{\frac{s_j+s_k}{2}} \geq 0, \quad \forall n \in \mathbb{N}, \alpha_j, s_j \in \mathbb{R}, j = 1, \ldots, n,$$

which shows the convexity of $\Upsilon(x)$ *and so* (35) *holds. Now, as the function* $s \mapsto [y_0, y_1, y_2; g_s]$ *is exponentially convex,* $s \mapsto [y_0, y_1, y_2; g_s]$ *is exponentially convex in the Jensen sense and by using Corollary 4, we have that* $s \mapsto \Phi_i(g_s), i = 1, 2$ *is exponentially convex in the Jensen sense. Since this mapping is continuous (although the mapping* $s \mapsto g_s$ *is not continuous for* $s = 0$*),* $s \mapsto \Phi_i(g_s), i = 1, 2$ *is exponentially convex.*

For this family of functions, by taking $\Omega = \Omega_1$ *in* (33), $\mu_{s,q}(\Phi_i, \Omega_1), i = 1, 2$ *become*

$$\mu_{s,q}(\Phi_i, \Omega_1) = \begin{cases} \left(\frac{\Phi_i(g_s)}{\Phi_i(g_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(\frac{\Phi_i(id \cdot g_s)}{\Phi_i(g_s)} - \frac{2}{s}\right), & s = q \neq 0, \\ \exp\left(\frac{\Phi_i(id \cdot g_0)}{3\Phi_i(g_0)}\right), & s = q = 0. \end{cases}$$

By using Theorem 8, *it can be seen that*

$$M_{s,q}(\Phi_i, \Omega_1) = \log \mu_{s,q}(\Phi_i, \Omega_1), \quad i = 1, 2,$$

satisfy $a \leq M_{s,q}(\Phi_i, \Omega_1) \leq b$, *which shows that* $M_{s,q}(\Phi_i, \Omega_1)$ *is a family of mean.*

Example 3. *Consider the family of functions*

$$\Omega_2 = \{f_s : (0, \infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined by

$$f_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, \ s \neq 0, 1, \\ -\ln x, \ s = 0, \\ x \ln x, \ s = 1. \end{cases}$$

Here

$$\frac{d^2}{dx^2} f_s(x) = x^{s-2} = e^{(s-2)\ln x} > 0,$$

which shows that f_s is convex for $x > 0$ and $s \mapsto f_s''(x)$ is exponentially convex by definition (see also [4]). It is easy to prove that the function $s \mapsto [y_0, y_1, y_2; f_s]$ is exponentially convex. Arguing as in Example 2, we have that $s \mapsto \Phi_i(f_s), i = 1, 2$ is exponentially convex.
If $r, s, t \in \mathbb{R}$ are such that $r < s < t$, then from (31) we have

$$\Phi_i(f_s) \le [\Phi_i(f_r)]^{\frac{t-s}{t-r}} [\Phi_i(f_t)]^{\frac{s-r}{t-r}}, \quad i = 1, 2. \tag{36}$$

If $r < t < s$ or $s < r < t$, then the opposite inequality holds in (36).

Particularly, for $i = 1$ and $r, s, t \in \mathbb{R} \setminus \{0, 1\}$ such that $r < s < t$, we have

$$\begin{aligned}
&\frac{\left(\sum_{k=1}^n p_k a_k\right)^s - (p_1 a_1)^s - \sum_{k=2}^n a_k^s \left(P_k^s - P_{k-1}^s\right)}{s(s-1)} \\
&\quad \ge \left[\frac{\left(\sum_{k=1}^n p_k a_k\right)^r - (p_1 a_1)^r - \sum_{k=2}^n a_k^r \left(P_k^r - P_{k-1}^r\right)}{r(r-1)}\right]^{\frac{t-s}{t-r}} \\
&\quad\quad \times \left[\frac{\left(\sum_{k=1}^n p_k a_k\right)^t - (p_1 a_1)^t - \sum_{k=2}^n a_k^t \left(P_k^t - P_{k-1}^t\right)}{t(t-1)}\right]^{\frac{s-r}{t-r}},
\end{aligned}$$

where $a_k > 0$, $p_k > 0, k = 1, \ldots, n$ are such that $p_1 a_1$, $\sum_{k=1}^n p_k a_k$, $P_k a_k$, $P_{k-1} a_k$ $\in [a, b]$ for all $k = 2, \ldots, n$. In fact, for $s > 1$, (37) is the refinement of inequality (2) and for $0 < s < 1$, (37) holds in the reverse direction.

By taking $\Omega = \Omega_2$ *in* (33), $\Xi^i_{s,q} := \mu_{s,q}(\Phi_i, \Omega_2), i = 1,2$ *are of the form*

$$\Xi^1_{s,q} = \left(\frac{q(q-1)}{s(s-1)} \cdot \frac{\left(\sum_{k=1}^n p_k a_k\right)^s - (p_1 a_1)^s - \sum_{k=2}^n a_k^s \left(P_k^s - P_{k-1}^s\right)}{\left(\sum_{k=1}^n p_k a_k\right)^q - (p_1 a_1)^q - \sum_{k=2}^n a_k^q \left(P_k^q - P_{k-1}^q\right)} \right)^{\frac{1}{s-q}}, \quad s \neq q \neq 0,1,$$

$$\Xi^1_{s,0} = \left(\frac{1}{s(s-1)} \cdot \frac{\left(\sum_{k=1}^n p_k a_k\right)^s - (p_1 a_1)^s - \sum_{k=2}^n a_k^s \left(P_k^s - P_{k-1}^s\right)}{\ln(P_n a_1) - \ln\left(\sum_{k=1}^n p_k a_k\right)} \right)^{\frac{1}{s}}, \quad s \neq 0,1,$$

$$\Xi^1_{s,1} = \left(\frac{1}{s(s-1)} \cdot \frac{\left(\sum_{k=1}^n p_k a_k\right)^s - (p_1 a_1)^s - \sum_{k=2}^n a_k^s \left(P_k^s - P_{k-1}^s\right)}{\sum_{k=1}^n a_k \left(p_k \ln\left(\sum_{k=1}^n p_k a_k\right) - P_k \ln(P_k a_k)\right) + \sum_{k=2}^n a_k P_{k-1} \ln(P_{k-1} a_k)} \right)^{\frac{1}{s-1}},$$

$$s \neq 0,1,$$

$$\Xi^1_{0,1} = \frac{\sum_{k=1}^n a_k \left(p_k \ln\left(\sum_{k=1}^n p_k a_k\right) - P_k \ln(P_k a_k)\right) + \sum_{k=2}^n a_k P_{k-1} \ln(P_{k-1} a_k)}{\ln(P_n a_1) - \ln\left(\sum_{k=1}^n p_k a_k\right)},$$

$$\Xi^1_{s,s} = \exp\left(\frac{1-2s}{s(s-1)} + \frac{\left(\sum_{k=1}^n p_k a_k\right)^s \ln\left(\sum_{k=1}^n p_k a_k\right) - \sum_{k=1}^n a_k^s P_k^s \ln(P_k a_k)}{\left(\sum_{k=1}^n p_k a_k\right)^s - (p_1 a_1)^s - \sum_{k=2}^n a_k^s \left(P_k^s - P_{k-1}^s\right)} \right.$$
$$\left. + \frac{\sum_{k=2}^n a_k^s P_{k-1}^s \ln(P_{k-1} a_k)}{\left(\sum_{k=1}^n p_k a_k\right)^s - (p_1 a_1)^s - \sum_{k=2}^n a_k^s \left(P_k^s - P_{k-1}^s\right)} \right), s \neq 0,1,$$

$$\Xi^1_{0,0} = \exp\left(\frac{\left(\ln\left(\sum_{k=1}^n p_k a_k\right) - \ln(p_1 a_1)\right) \ln\left(e^2 p_1 a_1 \sum_{k=1}^n p_k a_k\right)}{2\left(\ln\left(\sum_{k=1}^n p_k a_k\right) - \ln(P_n a_1)\right)} \right.$$
$$\left. - \frac{\sum_{k=2}^n \left(\ln P_k - \ln P_{k-1}\right) \ln\left(e^2 P_k P_{k-1} a_k^2\right)}{2\left(\ln\left(\sum_{k=1}^n p_k a_k\right) - \ln(P_n a_1)\right)} \right),$$

$$\Xi^1_{1,1} = \exp\left(\frac{\sum\limits_{k=1}^n a_k \left(p_k \ln\left(\sum\limits_{k=1}^n p_k a_k\right) \ln\left(e^{-2} \sum\limits_{k=1}^n p_k a_k\right) - P_k \ln(P_k a_k) \ln\left(e^{-2} P_k a_k\right) \right)}{2\left(\sum_{k=1}^n a_k \left(p_k \ln\left(\sum_{k=1}^n p_k a_k\right) - P_k \ln(P_k a_k)\right) + \sum_{k=2}^n a_k P_{k-1} \ln(P_{k-1} a_k)\right)} \right)$$
$$\times \exp\left(\frac{\sum\limits_{k=2}^n a_k P_{k-1} \ln(P_{k-1} a_k) \ln\left(e^{-2} P_{k-1} a_k\right)}{2\left(\sum_{k=1}^n a_k \left(p_k \ln\left(\sum_{k=1}^n p_k a_k\right) - P_k \ln(P_k a_k)\right) + \sum_{k=2}^n a_k P_{k-1} \ln(P_{k-1} a_k)\right)} \right).$$

Similarly, we can obtain $\Xi^2_{s,q} =: \mu_{s,q}(\Phi_2, \Omega_2)$.

If $\Phi_i, i = 1,2$ *is positive, then Theorem 8 applied for* $f = f_s \in \Omega_2$ *and* $g = f_q \in \Omega_2$ *yields that there exists* $\xi_i \in [a,b]$ *such that*

$$\xi_i^{s-q} = \frac{\Phi_i(f_s)}{\Phi_i(f_q)}, \quad i = 1,2.$$

Since the function $\xi_i \mapsto \xi_i^{s-q}$ *is invertible for* $s \neq q$, *we have*

$$a \leq \left(\frac{\Phi_i(f_s)}{\Phi_i(f_q)} \right)^{\frac{1}{s-q}} \leq b, \quad i = 1,2, \tag{37}$$

which, together with the fact that $\mu_{s,q}(\Phi_i,\Omega_2)$ is continuous, symmetric and monotonous (by (32)), shows that $\mu_{s,q}(\Phi_i,\Omega_2)$ is a family of mean.

If $a=0$ and we consider functions defined on $[0,\infty)$, then we can obtain inequalities and means of the same form, but for parameters s and q restricted to $(0,\infty)$. More precisely, we consider the family of functions

$$\tilde{\Omega}_2=\{\tilde{f}_s:[0,\infty)\to\mathbb{R}:s\in(0,\infty)\}$$

defined by

$$\tilde{f}_s(x)=\begin{cases}\frac{x^s}{s(s-1)}, & s\neq 1,\\ x\ln x, & s=1,\end{cases}$$

with the convention that $0\ln 0=0$. For $s>0$ and $q>0$, by taking $\Omega=\tilde{\Omega}_2$ in (33),

$$\tilde{\Xi}^i_{s,q}=:\mu_{s,q}\left(\Phi_i,\tilde{\Omega}_2\right),\quad i=1,2$$

are of the same form as $\Xi^i_{s,q}$.

Remark 13. *If we make the substitutions $p_k\to 1, k=1,\dots,n$ in the means given in Example 3, then the results for the means given in [5] are recaptured.*

Example 4. *Consider the family of functions*

$$\Omega_3=\{h_s:(0,\infty)\to(0,\infty):s\in(0,\infty)\}$$

defined by

$$h_s(x)=\begin{cases}\frac{s^{-x}}{\ln^2 s}, & s\neq 1,\\ \frac{x^2}{2}, & s=1.\end{cases}$$

We have

$$\frac{d^2}{dx^2}h_s(x)=s^{-x}>0,$$

which shows that h_s is convex for all $s>0$. Since $s\mapsto h_s''(x)$ is the Laplace transform of a non-negative function (see [4, 12]), it is exponentially convex. It is easy to see that the function $s\mapsto[y_0,y_1,y_2;h_s]$ is also exponentially convex. Arguing as in Example 2, we have that $s\mapsto\Phi_i(h_s)$ is exponentially convex.
In this case, by taking $\Omega=\Omega_3$ in (33), $\mu_{s,q}(\Phi_i,\Omega_3)$, $i=1,2$ are of the form

$$\mu_{s,q}(\Phi_i,\Omega_3)=\begin{cases}\left(\frac{\Phi_i(h_s)}{\Phi_i(h_q)}\right)^{\frac{1}{s-q}}, & s\neq q,\\ \exp\left(-\frac{\Phi_i(id\cdot h_s)}{s\Phi_i(h_s)}-\frac{2}{s\ln s}\right), & s=q\neq 1,\\ \exp\left(-\frac{\Phi_i(id\cdot h_1)}{3\Phi_i(h_1)}\right), & s=q=1.\end{cases}$$

By using Theorem 8, it follows that

$$M_{s,q}(\Phi_i,\Omega_3)=-L(s,q)\log\mu_{s,q}(\Phi_i,\Omega_3),\quad i=1,2,$$

satisfy $a \leq M_{s,q}(\Phi_i, \Omega_3) \leq b$ *and so* $M_{s,q}(\Phi_i, \Omega_3)$ *is a family of mean, where* $L(s,q)$ *is a logarithmic mean defined by*

$$L(s,q) = \begin{cases} \frac{s-q}{\log s - \log q}, & s \neq q, \\ s, & s = q. \end{cases}$$

Example 5. *Consider the family of functions*

$$\Omega_4 = \{k_s : (0,\infty) \to (0,\infty) : s \in (0,\infty)\}$$

defined by

$$k_s(x) = \frac{e^{-x\sqrt{s}}}{s}.$$

Here,

$$\frac{d^2}{dx^2} k_s(x) = e^{-x\sqrt{s}} > 0,$$

which shows that k_s *is convex for all* $s > 0$. *Since* $s \mapsto k_s''(x)$ *is the Laplace transform of a non-negative function* (*see* [4, 12]), *it is exponentially convex. It is easy to prove that the function* $s \mapsto [y_0, y_1, y_2; k_s]$ *is also exponentially convex. Arguing as in Example* 2, *we have that* $s \mapsto \Phi_i(k_s)$ *is exponentially convex.*
In this case, by taking $\Omega = \Omega_4$ *in* (33), $\mu_{s,q}(\Phi_i, \Omega_4), i = 1,2$ *are of the form*

$$\mu_{s,q}(\Phi_i, \Omega_4) = \begin{cases} \left(\frac{\Phi_i(k_s)}{\Phi_i(k_q)}\right)^{\frac{1}{s-q}}, & s \neq q, \\ \exp\left(-\frac{\Phi_i(id \cdot k_s)}{2\sqrt{s}\Phi_i(k_s)} - \frac{1}{s}\right), & s = q. \end{cases}$$

By using Theorem 8, *it is easy to see that*

$$M_{s,q}(\Phi_i, \Omega_4) = -\left(\sqrt{s} + \sqrt{q}\right) \log \mu_{s,q}(\Phi_i, \Omega_4), \quad i = 1,2,$$

satisfy $a \leq M_{s,q}(\Phi_i, \Omega_4) \leq b$, *showing that* $M_{s,q}(\Phi_i, \Omega_4), i = 1,2$ *is a family of mean.*

Remark 14. *From* (33), *it is clear that* $\mu_{s,q}(\Phi_i, \Omega), i = 1,2$ *for* $\Omega = \Omega_1, \Omega_3$ *and* Ω_4 *are monotonous functions in parameters* s *and* q.

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