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# Biharmonic curves in $\widetilde{\mathbf{SL}(2,\mathbb{R})}$ space

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**Abstract.** In this paper, non-geodesic biharmonic curves in $\widetilde{SL(2,\mathbb{R})}$ space are characterized and the statement that only proper biharmonic curves are helices is proved. Also, the explicit parametric equations of proper biharmonic helices are obtained.

**AMS subject classifications**: 53A40

**Key words**: $\widetilde{SL(2,\mathbb{R})}$ geometry, biharmonic curves, helices

## 1. Introduction

A map $\phi : (M, g) \to (N, h)$ between two Riemannian manifolds is said to be *harmonic* if it is a critical point of the energy functional $E(\phi) := \frac{1}{2}\int_M |d\phi|^2 v_g$. A harmonic map is characterized by the vanishing of the first tension field $\tau(\phi) := \mathrm{trace}\nabla d\phi$.

A map $\phi : (M, g) \to (N, h)$ is said to be *biharmonic* if it is a critical point of the bienergy functional $E_2(\phi) := \frac{1}{2}\int_M |\tau(\phi)|^2 v_g$. A biharmonic map is characterized by the vanishing of the second tension field $\tau_2(\phi) := J(\tau(\phi)) = -\triangle^\phi \tau(\phi) - trace_g R^N(d\phi, \tau(\phi))d\phi = 0$.

Any harmonic map ($\tau(\phi) = 0$) is biharmonic ($\tau_2(\phi) = 0$), so the interesting are so called *proper biharmonic* maps, i.e. non-harmonic biharmonic maps.

In the last decade, several papers on *biharmonic curves* in 3-dimensional Riemannian manifolds have appeared (see [1, 2, 3, 8, 10, 12]). Generally speaking, in homogeneous spaces there either exists no proper biharmonic curve or all proper biharmonic curves are helices.

Here, biharmonic curves are examined using the hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ space. E. Molnár introduced the hyperboloid model as a part of the projective spherical model of the eight homogeneous Thurston 3-geometries ($E^3$, $S^3$, $H^3$, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\widetilde{SL(2,\mathbb{R})}$, $Nil$, $Sol$). One of the advantages of the hyperboloid model to the "standard" model (described in [11]) is the possibility of better visualization of curves and surfaces.

In this paper, we characterize proper biharmonic curves in $\widetilde{SL(2,\mathbb{R})}$ space (Proposition 3.1) and prove that the only proper biharmonic curves are helices (Theorem 3.3). We also give explicit parametric equations of proper biharmonic curves (Proposition 4.5).

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## 2. Hyperboliod model of $\widetilde{SL(2,\mathbb{R})}$ space

As we mentioned before, E. Molnar introduced the hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ space in [9]. The model is used in [5] and [6], where geodesics and minimal surfaces are considered, respectively.

The idea is to start with the collineation group which acts on projective 3-space $\mathcal{P}^3(R)$ and preserves a polarity, i.e. a scalar product of signature $(--++)$. Using the one-sheeted hyperboloid solid

$$\mathcal{H}:\ -x^0x^0 - x^1x^1 + x^2x^2 + x^3x^3 < 0 \quad \text{or} \quad -1 - x^2 + y^2 + z^2 < 0$$

with an appropriate choice of a subgroup of the collineation group of $\mathcal{H}$ as an isometry group, the universal covering space $\widetilde{\mathcal{H}}$ of hyperboloid $\mathcal{H}$ will give the so-called hyperboloid model of $\widetilde{SL(2,\mathbb{R})}$ geometry.

An invariant infinitesimal arc length square in $\widetilde{SL(2,\mathbb{R})}$, obtained by the standard pull back translation into the origin (see [9]), is given by

$$\begin{aligned}(ds)^2 = &\frac{1}{\Big(-(x^0)^2-(x^1)^2+(x^2)^2+(x^3)^2\Big)^2}\\ &\Bigg(\Big(-(dx^0)x^1+(dx^1)x^0-(dx^2)x^3+(dx^3)x^2\Big)^2\\ &+\Big(-(dx^0)x^2-(dx^1)x^3+(dx^2)x^0+(dx^3)x^1\Big)^2\\ &+\Big(-(dx^0)x^3+(dx^1)x^2-(dx^2)x^1+(dx^3)x^0\Big)^2\Bigg).\end{aligned}$$

After introducing new coordinates, the so-called hyperboloid ones

$$\begin{aligned} x^0 &= \cosh r\cos\varphi\\ x^1 &= \cosh r\sin\varphi\\ x^2 &= \sinh r\cos(\vartheta-\varphi)\\ x^3 &= \sinh r\sin(\vartheta-\varphi)), \end{aligned} \tag{1}$$

the following infinitesimal metric is obtained

$$(ds)^2 = (dr)^2 + \cosh^2 r\sinh^2 r(d\vartheta)^2 + \Big((d\varphi)+\sinh^2 r(d\vartheta)\Big)^2, \tag{2}$$

where $(r,\vartheta)$ are polar coordinates in the hyperbolic base plane and $\varphi$ is the fiber coordinate.

The Euclidean coordinates $(x=\frac{x^1}{x^0}, y=\frac{x^2}{x^0}, z=\frac{x^3}{x^0})$ corresponding to the hyperboloid coordinates $(r,\vartheta,\varphi)$ and respecting (1), are given as follows

$$\begin{aligned} x &= \tan\varphi,\\ y &= \tanh r\,\frac{\cos(\vartheta-\varphi)}{\cos\varphi},\\ z &= \tanh r\,\frac{\sin(\vartheta-\varphi)}{\cos\varphi}, \end{aligned} \tag{3}$$

where $r \in [0,\infty)$, $\vartheta \in [0,2\pi)$ and $\varphi \in (-\frac{\pi}{2},\frac{\pi}{2})$ with extension to $\mathbb{R}$ for the universal covering. These Euclidean coordinates are important for the visualization of surfaces in $E^3$,

If $\{\omega^1,\omega^2,\omega^3\}$ is an orthonormal coframe defined by

$$\omega^1 = dr, \quad \omega^2 = \frac{1}{2}\sinh 2r\ d\vartheta, \quad \omega^3 = \sinh^2 r\ d\vartheta + d\varphi, \tag{4}$$

then the dual orthonormal frame fields are given by

$$e_1 = \frac{\partial}{\partial r}, \quad e_2 = \frac{2}{\sinh 2r}\frac{\partial}{\partial\vartheta} - \tanh r\frac{\partial}{\partial\varphi}, \quad e_3 = \frac{\partial}{\partial\varphi}. \tag{5}$$

Notice that $d\omega^3 \wedge \omega^3 \neq 0$ which means that $\omega^3$ is a contact form.

In local coordinates $(r,\vartheta,\varphi)$ around an arbitrary point $p \in \widetilde{SL(2,\mathbb{R})}$ one has a natural local vector basis $\{\partial_r,\partial_\vartheta,\partial_\varphi\}$ (for the standard denotation see, e.g. [4]). The Levi-Civita connection $\nabla$ of the ambient space is defined by $\nabla_{\partial_i}\partial_j := \Gamma^k_{ij}\partial_k$, where the Cristoffel symbols $\Gamma^k_{ij}$ are given by

$$\begin{aligned}
\Gamma^1_{ij} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}(1-2\cosh 2r)\sinh 2r & -\cosh r\sinh r \\ 0 & -\cosh r\sinh r & 0 \end{pmatrix}, \\
\Gamma^2_{ij} &= \begin{pmatrix} 0 & \coth r + 2\tanh r & \frac{1}{\cosh r\sinh r} \\ \coth r + 2\tanh r & 0 & 0 \\ \frac{1}{\cosh r\sinh r} & 0 & 0 \end{pmatrix}, \\
\Gamma^3_{ij} &= \begin{pmatrix} 0 & -2\sinh^2 r\tanh r & -\tanh r \\ -2\sinh^2 r\tanh r & 0 & 0 \\ -\tanh r & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{6}$$

The Levi-Civita connection $\nabla$ (in terms of the orthonormal frame) is given by

$$\begin{array}{lll}
\nabla_{e_1}e_1 = 0 & \nabla_{e_1}e_2 = -e_3 & \nabla_{e_1}e_3 = e_2 \\
\nabla_{e_2}e_1 = 2\coth 2r\ e_2 + e_3 & \nabla_{e_2}e_2 = -2\coth 2r\ e_1 & \nabla_{e_2}e_3 = -e_1 \\
\nabla_{e_3}e_1 = e_2 & \nabla_{e_3}e_2 = -e_1 & \nabla_{e_3}e_3 = 0.
\end{array} \tag{7}$$

The non-vanishing components of the Riemannien curvature tensor defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

are

$$\begin{array}{lll}
R(e_1,e_2)e_1 = 7e_2 & R(e_1,e_2)e_2 = -7e_1 & R(e_1,e_3)e_1 = -e_3 \\
R(e_1,e_3)e_3 = e_1 & R(e_3,e_2)e_2 = e_3 & R(e_3,e_2)e_3 = -e_2.
\end{array}$$

Moreover, if we put $R_{ijkl} = -g(R_{ijk},e_l)$, where $R_{ijk} = R(e_i,e_j)e_k$, we obtain

$$R_{1212} = -7, \qquad R_{1313} = 1 \qquad \text{and} \qquad R_{2323} = 1. \tag{8}$$

## 3. Biharmonic curves in $\widetilde{SL(2,\mathbb{R})}$ space

Let $\gamma : I \to \widetilde{SL(2,\mathbb{R})}$ be a differentiable curve parameterized by arc length and let $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ be the orthonormal frame field tangent to $\widetilde{SL(2,\mathbb{R})}$ along $\gamma$ and defined as follows: $\mathbf{T}$ is the unit vector field tangent to $\gamma$, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ normal to $\gamma$ and $\mathbf{B} = \mathbf{T}\times\mathbf{N}$.

With respect to the orthonormal basis $\{e_1, e_2, e_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1e_1 + T_2e_2 + T_3e_3 \\ \mathbf{N} &= N_1e_1 + N_2e_2 + N_3e_3 \\ \mathbf{B} &= B_1e_1 + B_2e_2 + B_3e_3. \end{aligned} \tag{9}$$

The following Frenet formulas hold

$$\begin{array}{lcrcr} \nabla_{\mathbf{T}}\mathbf{T} & = & & & \kappa\mathbf{N} \\ \nabla_{\mathbf{T}}\mathbf{N} & = & -\kappa\mathbf{T} & + & \tau\mathbf{B} \\ \nabla_{\mathbf{T}}\mathbf{B} & = & & & -\tau\mathbf{N}, \end{array} \tag{10}$$

where $\kappa = |\nabla_{\mathbf{T}}\mathbf{T}|$ is the geodesic curvature of $\gamma$ and $\tau$ is the geodesic torsion.

The biharmonic equation

$$\tau_2(\phi) = -\triangle^{\phi}\tau(\phi) - trace_g R^N(d\phi, \tau(\phi))d\phi = 0 \tag{11}$$

in the case of a curve $\gamma : I \to (N,g)$ from an open interval $I \subset R$ to a Riemannian manifold $(N,g)$ parameterized by arc length transforms ($\gamma = \phi, d\phi = \mathbf{T}, \tau(\phi) = \nabla_{\mathbf{T}}\mathbf{T}$) to the differential equation

$$\nabla^3_{\mathbf{T}}\mathbf{T} - R(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T})\mathbf{T} = 0. \tag{12}$$

Using the Frenet formulas (10), biharmonic equation (12) reduces to the system (see e.g. [3])

$$\begin{aligned} \kappa\kappa' &= 0 \\ \kappa'' - \kappa^3 - \kappa\tau^2 + \kappa R(\mathbf{T},\mathbf{N},\mathbf{T},\mathbf{N}) &= 0 \\ 2\kappa'\tau + \kappa\tau' + \kappa R(\mathbf{T},\mathbf{N},\mathbf{T},\mathbf{B}) &= 0 \end{aligned} \tag{13}$$

**Proposition 1.** *Let $\gamma : I \to \widetilde{SL(2,\mathbb{R})}$ be a differentiable curve parameterized by arc length. Then $\gamma$ is a proper non-geodesic biharmonic curve if and only if*

$$\begin{aligned} &\kappa = constant \neq 0 \\ &\kappa^2 + \tau^2 = 1 - 8B_3^2 \\ &\tau' = -8N_3B_3 \end{aligned} \tag{14}$$

**Proof**. By direct calculation, using (8), we obtain

$$\begin{aligned}
R(\mathbf{T},\mathbf{N},\mathbf{T},\mathbf{N}) &= \sum_{i,j,k,l=1}^{3} T_i N_j T_k N_l R_{ijkl} \\
&\vdots \\
&= -7(T_1N_2 - T_2N_1)^2 + (T_1N_3 - T_3N_1)^2 + (T_2N_3 - T_3N_2)^2 \\
&= -7B_3^2 + (-B_2)^2 + B_1^2 = -7B_3^2 + (1 - B_3^2) \\
&= 1 - 8B_3^2, \\
R(\mathbf{T},\mathbf{N},\mathbf{T},\mathbf{B}) &= \sum_{i,j,k,l=1}^{3} T_i N_j T_k B_l R_{ijkl} \\
&\vdots \\
&= (N_2T_3 - N_3T_2)(B_2T_3 - B_3T_2) + (N_1T_3 - N_3T_1)(B_1T_3 - B_3T_1) \\
&\quad -7(N_1T_2 - N_2T_1)(B_1T_2 - B_2T_1) = -B_1N_1 - B_2N_2 + 7B_3N_3 \\
&= N_3B_3 + 7N_3B_3 \\
&= 8N_3B_3.
\end{aligned}$$

If we insert the obtained result in the equation system (13), the theorem statement follows directly. □

**Definition 1.** *A differentiable curve in $\widetilde{SL(2,\mathbb{R})}$ space having constant both geodesic curvature and geodesic torsion is called a **helix**.*

**Theorem 1.** *Let $\gamma : I \to \widetilde{SL(2,\mathbb{R})}$ be a proper biharmonic curve parameterized by arc length. Then $\gamma$ is a helix.*

**Proof**. First we determine $\nabla_{\mathbf{T}}\mathbf{T}$, $\nabla_{\mathbf{T}}\mathbf{N}$ and $\nabla_{\mathbf{T}}\mathbf{B}$. Using (9) and (7), after long but straightforward computation we have

$$\begin{aligned}
\nabla_{\mathbf{T}} X^i = &\Big((X_1^i)' - 2T_2X_2^i \coth 2r - (T_2X_3^i + X_2^iT_3)\Big)e_1 \\
&+\Big((X_2^i)' + 2T_2X_1^i \coth 2r + (T_1X_3^i + X_1^iT_3)\Big)e_2 \\
&+\Big(T_2X_1^i - X_2^iT_1 + (X_3^i)'\Big)e_3, \quad i = 1,2,3,
\end{aligned} \tag{15}$$

where $X^1 = \mathbf{T}$, $X^2 = \mathbf{N}$ and $X^3 = \mathbf{B}$.

Next, we find $g(\nabla_{\mathbf{T}} X^i, e_3)$, first using equations (15) (and having in mind $\mathbf{B} = \mathbf{T} \times \mathbf{N}$) and then using Frenet formulas (10). After comparing the corresponding scalar products, we obtain the following system

$$\begin{aligned}
T_3' &= \kappa N_3, \\
-B_3 + N_3' &= -\kappa T_3 + \tau B_3, \\
N_3 + B_3' &= -\tau N_3.
\end{aligned} \tag{16}$$

If we suppose that $N_3B_3 \neq 0$, then inserting $N_3$ from the third equation of (14) into the third equation of (16), after separating variables and integrating, we have

$$\tau^2 + 2\tau = 8B_3^2 + constant.$$

On the other hand, from the second equation of (14), we have
$8B_3^2 = 1 - \kappa^2 - \tau^2$. Thus, by comparing the last two equations we get

$$\tau^2 + \tau = constant,$$

which means that $\tau$ is a constant and this is in contradiction with our assumption $N_3B_3 \neq 0$.

So, the conclusion is that for any proper biharmonic curve there holds $N_3B_3 = 0$, i.e. $\tau$ is constant. Finally, the curve is a helix. □

**Corollary 1.** *Let $\gamma : I \to \widetilde{SL(2,\mathbb{R})}$ be a differentiable curve parameterized by arc length. Then $\gamma$ is a proper biharmonic curve if and only if*

$$\begin{aligned} &\kappa = constant \neq 0 \\ &\tau = constant \\ &\kappa^2 + \tau^2 = 1 - 8B_3^2 \\ &N_3B_3 = 0. \end{aligned} \tag{17}$$

**Remark 1.** *In [7], the authors have derived similar statements about biharmonic curves using the standard metric of $\widetilde{SL(2,\mathbb{R})}$ geometry.*

## 4. Biharmonic helices in $\widetilde{SL(2,\mathbb{R})}$

In this section, we determine helices in $\widetilde{SL(2,\mathbb{R})}$. From Corollary 1 it is clear that we have to study the coefficients $N_3$ and $B_3$, where from (17) we know that $B_3 = constant \in (\frac{-\sqrt{2}}{4}, \frac{\sqrt{2}}{4})$.

**Proposition 2.** *Let $\gamma : I \to \widetilde{SL(2,\mathbb{R})}$ be a differentiable curve parameterized by arc length. If the third component of the binormal vector field vanishes ($B_3 = 0$), then $\gamma$ is not a proper biharmonic curve.*

**Proof**. First, note that if a curve is proper biharmonic, then it is not a geodesic. So, by Corollary 1 we have two cases:

- $B_3 = 0$ and $N_3 \neq 0$,
- $B_3 = 0$ and $N_3 = 0$,

and we prove that both cases imply $\gamma$ is geodesic which further contradicts the fact that $\gamma$ is a proper biharmonic.

In the first case, from the third equation of (16) it follows $\tau = -1$. If we insert $\tau = -1$ in the second equation of (14), then we obtain $\kappa = 0$, which means $\gamma$ is a geodesic curve.

In the second case, we have $\mathbf{T} = \mathbf{N} \times \mathbf{B} = (0, 0, N_1B_2 - B_1N_2) = \pm e_3$. Thus $\nabla_{\mathbf{T}}\mathbf{T} = \nabla_{e_3}e_3$ vanishes by (7). So, $\kappa = 0$ and $\gamma$ is a geodesic curve. □

**Corollary 2.** *Let $\gamma : I \to \widetilde{SL(2,\mathbb{R})}$ be a differentiable curve parameterized by arc length. Then $\gamma$ is a proper biharmonic curve if and only if*

$$\begin{aligned} &\kappa = \mathit{constant} \neq 0 \\ &\tau = \mathit{constant} \\ &\kappa^2 + \tau^2 = 1 - 8B_3^2 \\ &B_3 = \mathit{constant} \neq 0 \\ &N_3 = 0. \end{aligned} \tag{18}$$

Further, using the obtained characterization of proper biharmonic curves we find explicit parametric equations of these curves.

**Proposition 3.** *The proper biharmonic curve $\gamma(t) = (r(t), \vartheta(t), \varphi(t))$ in $\widetilde{SL(2,\mathbb{R})}$ is a solution of the following system of differential equations.*

$$\begin{aligned} r'(t) &= \sin\alpha_0 \cos\beta(t) \\ \vartheta'(t) &= \frac{2}{\sinh 2r(t)} \sin\alpha_0 \sin\beta(t) \\ \varphi'(t) &= -\sinh^2 r(t)\vartheta'(t) + \cos\alpha_0 \\ \beta'(t) &= -2\sin\alpha_0 \sin\beta(t)\coth 2r(t) - 2\cos\alpha_0 + \omega, \end{aligned} \tag{19}$$

*where $\alpha_0 \in (0,\pi)$ and $\omega \in \mathbb{R}^+$.*

**Proof**. Suppose that $\gamma : I \to \widetilde{SL(2,\mathbb{R})}$ is a proper biharmonic curve parameterized by arc length. From (15) we have

$$\nabla_{\mathbf{T}}\mathbf{T} = (T_1' - 2T_2^2 \coth 2r - 2T_2T_3)e_1 + (T_2' + 2T_1T_2 \coth 2r + 2T_1T_3)e_2 + T_3'e_3. \tag{20}$$

Since $\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N}$, by Corollary 2 ($N_3 = 0$) it follows $T_3' = 0$, i.e. $T_3 = $ constant. Since $(e_1, e_2, e_3)$ is an orthonormal basis and $|\mathbf{T}| = 1$, it follows $T_3 \in \langle -1, 1\rangle$. If $T_3 = \pm 1$, then $T_1 = T_2 = 0$ and from (20) we get $\nabla_{\mathbf{T}}\mathbf{T} = 0$ which implies $\gamma$ is a geodesic curve. Therefore, there exists $\alpha_0 \in \langle 0, \pi\rangle$ and a smooth function $\beta$ such that

$$\mathbf{T}(t) = \sin\alpha_0 \cos\beta(t) \cdot e_1 + \sin\alpha_0 \sin\beta(t) \cdot e_2 + \cos\alpha_0 \cdot e_3. \tag{21}$$

If we put (21) on the right-hand side of (20), by direct computation we get

$$\nabla_{\mathbf{T}}\mathbf{T} = \omega \sin\alpha_0 \cdot (-\sin\beta \cdot e_1 + \cos\beta \cdot e_2), \tag{22}$$

where

$$\omega = \beta' + 2\sin\alpha_0 \sin\beta \coth 2r + 2\cos\alpha_0. \tag{23}$$

From (22) there follows

$$\kappa = \omega \sin\alpha_0 \qquad \text{and} \qquad N = -\sin\beta \cdot e_1 + \cos\beta \cdot e_2.$$

If the curve is given by $\gamma(t) = (r(t), \vartheta(t), \varphi(t))$, then we have

$$\begin{aligned}\gamma'(t) &= r'(t)\frac{\partial}{\partial r} + \vartheta'(t)\frac{\partial}{\partial\vartheta} + \varphi'(t)\frac{\partial}{\partial\varphi}\\ &= r'\cdot e_1 + \frac{1}{2}\sinh 2r\vartheta'\cdot e_2 + (\sinh^2 2r\vartheta' + \varphi')\cdot e_3.\end{aligned}$$

Using $\gamma'(t) = T(t)$, we obtain the system of differential equations (19). □

**Remark 2.** *Analogously to the relation $\kappa = \omega\sin\alpha_0$, using $\tau = \langle -\nabla_T B, N\rangle$ we have $\tau = \omega\cos\alpha_0 - 1$. Furthermore, from the third equation of (1) there follows*

$$\omega = \cos\alpha_0 \pm \sqrt{1 - 9\sin^2\alpha_0} = \textit{constant} > 0, \tag{24}$$

*where $\alpha \in \langle 0, \arcsin\frac{1}{3}\rangle \cup \langle \pi - \arcsin\frac{1}{3}, \pi\rangle$.*

Now, we try to solve the system (19). Observe that the most important part is to solve the subsystem consisting of the first and the fourth equation. Therefore, from the fourth equation we have

$$r = \frac{1}{2}\mathrm{Arcth}\left(\frac{\omega - 2\cos\alpha_0 - \beta'}{2\sin\alpha_0\sin\beta}\right). \tag{25}$$

After differentiating (25) and comparing with the first equation of the system (19) we have the following equation for $\beta(t)$

$$\begin{aligned}2\sin\alpha_0\Big(&\sin\beta\cdot\beta'' - 2\cos\beta\cdot\beta'^2 + 3\cos\beta\cdot(\omega - 2\cos\alpha_0)\beta'\\ &+ 4\cos\beta\cdot\sin^2\alpha_0\sin^2\beta - \cos\beta\cdot(\omega - 2\cos\alpha_0)^2\Big) = 0\end{aligned} \tag{26}$$

The obtained second order differential equation is generally not solvable, but it is obvious that $\beta = \frac{\pi}{2} + k\pi$ is a solution of this equation.

In this case, from the first equation of the system (19), we have $r(t) = R_0 = \mathit{const}$, where since (24)

$$R_0 = \frac{1}{2}\mathrm{Arcth}\left(\frac{-\cos\alpha_0 \pm \sqrt{1 - 9\sin^2\alpha_0}}{2\sin\alpha_0}\right). \tag{27}$$

If we insert the obtained function in the second and the third equation of the system (19), we have the following proposition

**Proposition 4.** *The proper biharmonic curve $\gamma(t) = (r(t), \vartheta(t), \varphi(t))$ in $\widetilde{SL(2,\mathbb{R})}$ is given by following parametric equations*

$$\begin{aligned}r(t) &= R_0\\ \vartheta(t) &= \frac{2\sin\alpha_0\sin\beta_0}{\sinh(2R_0)}\cdot t + c_1\\ \varphi(t) &= (\cos\alpha_0 - \tanh R_0\sin\alpha_0\sin\beta_0)\cdot t + c_2,\end{aligned} \tag{28}$$

*where $\alpha_0 \in \langle \pi - \arcsin\frac{1}{3}, \pi\rangle, \beta_0 = \frac{\pi}{2} + k\pi$, $R_0$ is given by (27), $c_1, c_2 \in \mathbb{R}$.*

Figure 1 shows the obtained proper biharmonic curves for $\alpha_0 = 3, c_1 = c_2 = 0, \beta_0 = \frac{\pi}{2}$ (left curve) and $\beta_0 = \frac{3\pi}{2}$ (right curve).

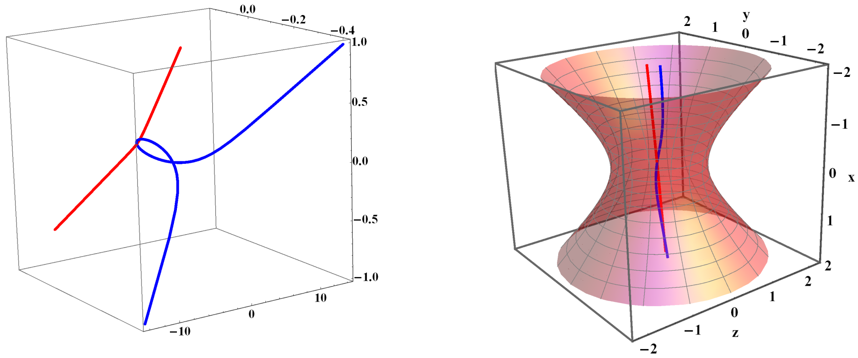


Figure 1: *Proper biharmonic curves in $\widetilde{SL(2,R)}$ space*

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