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Willmore spacelike submanifolds in a Lorentzian space form $N_p^{n+p}(c)$*

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Abstract. Let $N_p^{n+p}(c)$ be an $(n+p)$-dimensional connected Lorentzian space form of constant sectional curvature c and $\varphi : M \to N_p^{n+p}(c)$ an n-dimensional spacelike submanifold in $N_p^{n+p}(c)$. The immersion $\varphi : M \to N_p^{n+p}(c)$ is called a Willmore spacelike submanifold in $N_p^{n+p}(c)$ if it is a critical submanifold to the Willmore functional

$$W(\varphi) = \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv,$$

where S, H and ρ^2 denote the norm square of the second fundamental form, the mean curvature and the non-negative function $\rho^2 = S - nH^2$ of M. In this article, by calculating the first variation of $W(\varphi)$, we obtain the Euler-Lagrange equation of $W(\varphi)$ and prove some rigidity theorems for n-dimensional Willmore spacelike submanifolds in $N_p^{n+p}(c)$.
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1. Introduction

Let $N_p^{n+p}(c)$ be an $(n+p)$-dimensional connected Lorentzian space form of constant sectional curvature c. If $c > 0$, $c = 0$ or $c < 0$, we call $N_p^{n+p}(c)$ a Minkowski space R_p^{n+p}, a de Sitter space $S_p^{n+p}(c)$ or an anti-de Sitter space $H_p^{n+p}(c)$. A submanifold in $N_p^{n+p}(c)$ is said to be spacelike if the induced metric on the submanifold is positive definite. Let

$$\varphi : M \to N_p^{n+p}(c)$$

be an n-dimensional spacelike submanifold in $N_p^{n+p}(c)$. Denote by $h_{ij}^\alpha, S, \vec{H}$ and H the second fundamental form, the norm square of the second fundamental form, the mean curvature vector and the mean curvature of M and denote by ρ^2 the non-negative function $\rho^2 = S - nH^2$. We define the Willmore functional (see [4, 9, 16]):

$$W(\varphi) = \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv, \tag{1}$$

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which vanishes if and only if M is a totally umbilical submanifold, so the functional $W(\varphi)$ measures how far $\varphi(M)$ is from being a totally umbilical submanifold. If the critical points of the Willmore functional $W(\varphi)$ are submanifolds in $N_p^{n+p}(c)$, we call them Willmore spacelike submanifolds.

Due to their backgrounds in mathematics, we know that Willmore submanifolds in a unit sphere were extensively studied by many mathematicians. For example, the well-known Willmore conjecture, which says that $W(\varphi) \geq 4\pi^2$ holds for all immersed tori $\varphi : M \to S^3$, was investigated by Willmore [20, 21], Li and Yau [10] and many others; it was recently proved by Marques and Neves [12]. We should notice that the topic of Willmore submanifolds and their rigidity problem was also studied by Wang [19] (using conformal invariance), Li [8, 9] and the first author [18] (using metric invariants) and by Mondino-Riviere [13] (who established a divergence form of the Willmore equation in manifolds and exploited it to get rigidity results). On the other hand, we should see that the parallel problem in Lorentzian conformal geometry is also an important and interesting topic. As far as the authors know, the earliest work in this direction is L. Alias and B. Palmer's paper [2], in which they essentially used the conformal invariance. We notice that one of Alias and Palmer's main contributions is the generalization of Willmore surfaces to Lorentz geometry and a Bernstein type theorem for them, which implies that compact Willmore surfaces in 3-dimensional Lorentz space forms must be totally umbilic spheres. This research was motivated by Barros et al. [3], Li and Nie [11], Nie et al. [14, 15] and others. In this article, we consider the Willmore functional on spacelike submanifolds in Lorentzian space forms. By using the metric invariants, we compute the first variation of the Willmore functional $W(\varphi)$ and obtain the Euler-Lagrange equation and some rigidity results of n-dimensional Willmore spacelike submanifolds in $N_p^{n+p}(c)$.

Theorem 1. *Let $\varphi : M \to N_p^{n+p}(c)$ be an n-dimensional spacelike submanifold in $N_p^{n+p}(c)$. Then M is an n-dimensional Willmore spacelike submanifold if and only if for $n+1 \leq \alpha, \beta \leq n+p$*

$$\begin{aligned}\rho^{n-2}\Big\{SH^\alpha + \sum_{i,j,\beta} H^\beta h_{ij}^\beta h_{ij}^\alpha - \sum_{i,j,k,\beta} h_{ij}^\alpha h_{ik}^\beta h_{kj}^\beta - nH^2H^\alpha\Big\}&\\ + (n-1)\rho^{n-2}\Delta^\perp H^\alpha + 2(n-1)\sum_i(\rho^{n-2})_i H_{,i}^\alpha&\\ + (n-1)H^\alpha\Delta(\rho^{n-2}) - \Box^\alpha(\rho^{n-2}) = 0,&\end{aligned} \tag{2}$$

where

$$\begin{aligned}\Delta(\rho^{n-2}) &= \sum_i (\rho^{n-2})_{,ii},\\ \Delta^\perp H^\alpha &= \sum_i H_{,ii}^\alpha,\\ \Box^\alpha(\rho^{n-2}) &= \sum_{i,j}(\rho^{n-2})_{,ij}(nH^\alpha\delta_{ij} - h_{ij}^\alpha),\end{aligned}$$

and $(\rho^{n-2})_{,ij}$ is the Hessian of ρ^{n-2} with respect to the induced metric, $H_{,i}^\alpha$ and $H_{,ij}^\alpha$ are defined by (20) *and* (21).

Remark 1. *We should notice that in Theorem 1 (also in Proposition 3 - 5 and Corollary 2), when $n = 3$ and $n = 5$, we need to assume that M has no umbilical points to guarantee $(\rho^{n-2})_{,ij}$ is continuous on M. In fact, if we denote $x := S - nH^2$, then $\rho^{n-2} = x^{\frac{n-2}{2}}$. Thus, we have $(\rho^{n-2})_{,i} = \frac{n-2}{2} x^{\frac{n-4}{2}} x_{,i}$ and*

$$(\rho^{n-2})_{,ij} = \frac{n-2}{2}\Big(\frac{n-4}{2} x^{\frac{n-6}{2}} x_{,j} x_{,i} + x^{\frac{n-4}{2}} x_{,ij}\Big).$$

From the above equation, we see that when $n = 2, 4$ or $n \geq 6$, $(\rho^{n-2})_{,ij}$ is continuous on M and when $n = 3$ or 5, $(\rho^{n-2})_{,ij}$ is not continuous on the umbilical points of M. Therefore, the assumption $n \neq 3, 5$ is needed in Theorem 2 -Theorem 4 .

Remark 2. *We also notice that in conformal geometry of conformal spacelike submanifolds Nie and Wu [15] obtain the Willmore equation (Euler-Lagrange equation) in terms of conformal invariants.*

When $n = 2$, since $R_{ij} = \frac{R}{2}\delta_{ij}$ and $S = R - 2c + 4H^2$, from the Gauss equation (12) we see that

$$\begin{aligned} -\sum_{i,j,k,\beta} h^\alpha_{ij} h^\beta_{ik} h^\beta_{kj} &= -\sum_{i,j} h^\alpha_{ij}\big(R_{ij} - c\delta_{ij} + 2\sum_\beta H^\beta h^\beta_{ij}\big) \\ &= -SH^\alpha + 4H^2 H^\alpha - 2\sum_{i,j,\beta} H^\beta h^\beta_{ij} h^\alpha_{ij}. \end{aligned}$$

Thus, (2) reduces to

$$\Delta^\perp H^\alpha + 2H^2 H^\alpha - \sum_{i,j,\beta} H^\beta h^\beta_{ij} h^\alpha_{ij} = 0, \tag{3}$$

where $3 \leq \alpha, \beta \leq 2 + p$. From (3), we easily see

Proposition 1. *Every maximal spacelike surface $\varphi : M \to N_p^{2+p}(c)$ in a Lorentzian space form $N_p^{2+p}(c)$ is a Willmore spacelike surface.*

Proposition 2. *Every $n(n \geq 3)$-dimensional maximal and Einstein spacelike submanifold $\varphi : M \to N_p^{n+p}(c)$ in a Lorentzian space form $N_p^{n+p}(c)$ is a Willmore spacelike submanifold.*

In fact, since M is maximal and Einstein, we have $H^\alpha = 0$ for all α and $R_{ij} = \frac{R}{n}\delta_{ij} = constant$. Thus, from (13), we see that $\rho^2 = S = R - n(n-1)c = constant$. From (2), we only need to prove $\sum_{i,j,k,\beta} h^\alpha_{ij} h^\beta_{ik} h^\beta_{kj} = 0$. From the Gauss equation (12), we have

$$\begin{aligned} \sum_{i,j,k,\beta} h^\alpha_{ij} h^\beta_{ik} h^\beta_{kj} &= \sum_{i,j} h^\alpha_{ij}\Big(\sum_{k,\beta} h^\beta_{ik} h^\beta_{kj}\Big) = \sum_{i,j}\big[R_{ij} - (n-1)c\delta_{ij}\big] h^\alpha_{ij} \\ &= \sum_{i,j}\Big[\frac{R}{n}\delta_{ij} - (n-1)c\delta_{ij}\Big] h^\alpha_{ij} = \Big[\frac{R}{n} - c(n-1)\Big] nH^\alpha = 0. \end{aligned}$$

We also have the example of Willmore spacelike hypersurfaces of $H_1^{n+1}(-1)$.

Example 1. *The hyperbolic cylinders*

$$H^k\big(\sqrt{\frac{n-k}{n}}\big)\times H^{n-k}\big(\sqrt{\frac{k}{n}}\big)\subset H_1^{n+1}(-1), 1\le k\le n-1,$$

have two distinct principal curvatures $\sqrt{k/(n-k)}$ *and* $-\sqrt{(n-k)/(k)}$ *with multiplicities* k *and* $n-k$*, respectively. We may easily check that they are Willmore spacelike hypersurfaces in* $H_1^{n+1}(-1)$ *(see [14]) and* $\rho^2 = S - nH^2 = n$.

Remark 3. *It is unknown whether there exist non-trivial examples of closed Willmore spacelike submanifolds whose normal bundle is timelike.*

Denote by K and Q the functions which assign to each point of M the infimum of the sectional curvature and the Ricci curvature at the point. We obtain the following integral inequalities of Simons' type and rigidity theorems in terms of ρ^2, K, Q and H.

Theorem 2. *Let* $\varphi : M \to N_p^{n+p}(c)$ *be an* $n(n\ge 2)$*-dimensional compact Willmore spacelike submanifold in a Lorentzian space form* $N_p^{n+p}(c)(c = 1, 0, -1)$*. If* $n \ne 3, 5$*, then*

(*1*) *for* $p = 1$*, we have*

(*i*) *if* $c = 1, 0$*, then* M *is totally umbilical;*

(*ii*) *if* $c = -1$ *and* $\rho^2 \ge n$*, then* M *is totally umbilical;*

(*2*) *for* $p \ge 2$*, we have*

$$\int_M \rho^n\left\{\frac{1}{p}\rho^2 + nc - nH^2\right\}dv \le 0. \tag{4}$$

In particular, if

$$\rho^2 \ge np(H^2 - c),$$

then M *is totally umbilical.*

Theorem 3. *Let* $\varphi : M \to N_p^{n+p}(c)$ *be an* $n(n\ge 2)$*-dimensional compact Willmore spacelike submanifold in a Lorentzian space form* $N_p^{n+p}(c)(c = 1, 0, -1)$*. If* $n \ne 3, 5$*, then the following integral inequality holds*

$$\int_M \rho^n\left\{K - \frac{n-2}{\sqrt{n(n-1)}}H\rho\right\}dv \le 0. \tag{5}$$

In particular, if

$$K \ge \frac{n-2}{\sqrt{n(n-1)}}H\rho,$$

then M *is totally umbilical.*

Theorem 4. *Let $\varphi : M \to N_p^{n+p}(c)$ be an $n(n \geq 2)$-dimensional compact Willmore spacelike submanifold in a Lorentzian space form $N_p^{n+p}(c)(c = 1, 0, -1)$. If $n \neq 3, 5$, then the following integral inequality holds*

$$\int_M \rho^n \left\{Q - (n-p-1)(c - H^2)\right\} dv \leq 0. \tag{6}$$

In particular, if

$$Q \geq (n-p-1)\left(c - H^2\right),$$

then M is totally umbilical.

Remark 4. *If $p = 1$ and $c = 1, 0$, from Theorem 2 we know that M is totally umbilical. Thus, the conditions*

$$K \geq \frac{n-2}{\sqrt{n(n-1)}} H\rho \quad \textit{and} \quad Q \geq (n-p-1)\left(c - H^2\right)$$

can be omitted from Theorem 3 and Theorem 4 if $p = 1$ and $c = 1, 0$.

Remark 5. *For the Willmore spacelike surfaces, L. Alias and B. Palmer [2] proved that compact Willmore spacelike surfaces in 3-dimensional Lorentz space forms must be totally umbilical spheres. Thus, we notice that our results above generalize Alias and Palmer's uniqueness result to high dimension and high co-dimension Willmore spacelike submanifolds.*

If $\varphi : M \to N_p^{2+p}(c)$ is a maximal spacelike surface in a Lorentzian space form $N_p^{2+p}(c)$, from Proposition 1 and Theorem 2 - Theorem 4, we easily have the following result:

Corollary 1. *Let $\varphi : M \to N_p^{2+p}(c)$ be a compact maximal spacelike surface in a Lorentzian space form $N_p^{2+p}(c)(c = 1, 0, -1)$. Then*

(*1*) *if $c = 1, 0$, M is totally geodesic;*

(*2*) *if $c = -1$, $S \geq 2p$ or $K \geq 0$ or $Q \geq p - 1$, M is totally geodesic.*

Remark 6. *We notice that the result* (1) *of Corollary 1 was obtained by [7].*

2. Preliminaries

Let $N_p^{n+p}(c)$ be an $(n + p)$-dimensional Lorentzian space form with index p. Let M be an n-dimensional connected spacelike submanifold immersed in $N_p^{n+p}(c)$. We choose a local field of semi-Riemannian orthonormal frames $e_1, \dots, e_{n+p}$ in $N_p^{n+p}(c)$ so that at each point of M, $e_1, \dots, e_n$ span the tangent space of M and form an orthonormal frame there. We use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n + p, \quad 1 \leq i, j, k, \dots \leq n, \quad n + 1 \leq \alpha, \beta, \gamma, \dots \leq n + p.$$

Let $\omega_1, \ldots, \omega_{n+p}$ be its dual frame field so that the semi-Riemannian metric of $N_p^{n+p}(c)$ is given by

$$d\bar{s}^2 = \sum_i \omega_i^2 - \sum_\alpha \omega_\alpha^2 = \sum_A \varepsilon_A \omega_A^2,$$

where $\varepsilon_i = 1$ and $\varepsilon_\alpha = -1$. Then the structure equations of $N_p^{n+p}(c)$ are given by

$$d\omega_A = \sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{7}$$

$$d\omega_{AB} = \sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D K_{ABCD} \omega_C \wedge \omega_D, \tag{8}$$

$$K_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{9}$$

If we restrict these forms to M, then $\omega_\alpha = 0$, $n+1 \le \alpha \le n+p$ and

$$\omega_{\alpha i} = \sum_j h_{ij}^\alpha \omega_j, \quad h_{ij}^\alpha = h_{ji}^\alpha. \tag{10}$$

The Gauss equations are

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) - \sum_\alpha (h_{ik}^\alpha h_{jl}^\alpha - h_{il}^\alpha h_{jk}^\alpha), \tag{11}$$

$$R_{ik} = (n-1)c\delta_{ik} - \sum_\alpha (\sum_l h_{ll}^\alpha) h_{ik}^\alpha + \sum_{\alpha,j} h_{ij}^\alpha h_{jk}^\alpha, \tag{12}$$

$$R = n(n-1)c + S - n^2 H^2, \tag{13}$$

where

$$S = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2, \quad \vec{H} = \sum_\alpha H^\alpha e_\alpha, \quad H^\alpha = \frac{1}{n} \sum_k h_{kk}^\alpha, \quad H = |\vec{H}|$$

and R is the scalar curvature of M.

Define the first and the second covariant derivatives of h_{ij}^α, say h_{ijk}^α and h_{ijkl}^α, by

$$\sum_k h_{ijk}^\alpha \omega_k = dh_{ij}^\alpha + \sum_k h_{ik}^\alpha \omega_{kj} + \sum_k h_{jk}^\alpha \omega_{ki} - \sum_\beta h_{ij}^\beta \omega_{\beta\alpha}, \tag{14}$$

$$\sum_l h_{ijkl}^\alpha \omega_l = dh_{ijk}^\alpha + \sum_m h_{mjk}^\alpha \omega_{mi} + \sum_m h_{imk}^\alpha \omega_{mj} + \sum_m h_{ijm}^\alpha \omega_{mk} - \sum_\beta h_{ijk}^\beta \omega_{\beta\alpha}. \tag{15}$$

The Codazzi equations and the Ricci identities are

$$h_{ijk}^\alpha = h_{ikj}^\alpha, \tag{16}$$

$$h_{ijkl}^\alpha - h_{ijlk}^\alpha = \sum_m h_{im}^\alpha R_{mjkl} + \sum_m h_{jm}^\alpha R_{mikl} + \sum_\beta h_{ij}^\beta R_{\alpha\beta kl}. \tag{17}$$

The Ricci equations are

$$R_{\alpha\beta kl} = \sum_m (h_{km}^\alpha h_{ml}^\beta - h_{lm}^\alpha h_{mk}^\beta). \tag{18}$$

The Laplacian of h_{ij}^α is defined by $\Delta h_{ij}^\alpha = \sum\limits_k h_{ijkk}^\alpha$. From (17), for any $\alpha, n+1 \leq \alpha \leq n+p$, we obtain

$$\Delta h_{ij}^\alpha = \sum_k h_{kkij}^\alpha + \sum_{k,m} h_{km}^\alpha R_{mijk} + \sum_{k,m} h_{im}^\alpha R_{mkjk} + \sum_{k,\beta} h_{ik}^\beta R_{\alpha\beta jk}. \tag{19}$$

Define the first, second covariant derivatives and Laplacian of the mean curvature vector field $\vec{H} = \sum\limits_\alpha H^\alpha e_\alpha$ in the normal bundle $N(M)$ as follows

$$\sum_i H_{,i}^\alpha \theta_i = dH^\alpha + \sum_\beta H^\beta \theta_{\beta\alpha}, \tag{20}$$

$$\sum_j H_{,ij}^\alpha \theta_j = dH_{,i}^\alpha + \sum_j H_{,j}^\alpha \theta_{ji} + \sum_\beta H_{,i}^\beta \theta_{\beta\alpha}, \tag{21}$$

$$\Delta^\perp H^\alpha = \sum_i H_{,ii}^\alpha, \quad H^\alpha = \frac{1}{n}\sum_k h_{kk}^\alpha. \tag{22}$$

Let f be a smooth function on M. The first, second covariant derivatives $f_i, f_{,ij}$ and Laplacian of f are defined by

$$df = \sum_i f_i \theta_i, \quad \sum_j f_{,ij}\theta_j = df_i + \sum_j f_j \theta_{ji}, \quad \Delta f = \sum_i f_{,ii}. \tag{23}$$

For the fix index $\alpha (n+1 \leq \alpha \leq n+p)$, we introduce an operator $\Box^\alpha$ due to Cheng-Yau [5] by

$$\Box^\alpha f = \sum_{i,j} (nH^\alpha \delta_{ij} - h_{ij}^\alpha) f_{,ij}. \tag{24}$$

Since M is compact, the operator $\Box^\alpha$ is self-adjoint (see [5]) if and only if

$$\int_M (\Box^\alpha f) g dv = \int_M f(\Box^\alpha g) dv, \tag{25}$$

where f and g are smooth functions on M. We need the following:

Lemma 1 (See [17]). *Let A, B be symmetric $n \times n$ matrices satisfying $AB = BA$ and* $\mathrm{tr}A = \mathrm{tr}B = 0$. *Then*

$$|\mathrm{tr}A^2 B| \leq \frac{n-2}{\sqrt{n(n-1)}} (\mathrm{tr}A^2)(\mathrm{tr}B^2)^{1/2}, \tag{26}$$

and the equality holds if and only if $(n-1)$ of the eigenvalues x_i of B and the corresponding eigenvalues y_i of A satisfy

$$|x_i| = (\mathrm{tr}B^2)^{1/2}/\sqrt{n(n-1)}, \quad x_i x_j \geq 0,$$
$$y_i = (\mathrm{tr}A^2)^{1/2}/\sqrt{n(n-1)}.$$

By the same method as in the proof of Lemma 4.2 in [9], we also have the following:

Lemma 2. *Let $\varphi : M \to N_p^{n+p}(c)$ be an n-dimensional $(n \geq 2)$ spacelike submanifold in $N_p^{n+p}(c)$. Then we have*

$$|\nabla h|^2 \geq \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2, \tag{27}$$

where $|\nabla h|^2 = \sum\limits_{i,j,k,\alpha} (h_{ijk}^\alpha)^2, \quad |\nabla^\perp \vec{H}|^2 = \sum\limits_{i,\alpha}(H_{,i}^\alpha)^2.$

3. First variation and Euler-Lagrange equation

In this section, we shall calculate the first variation of the Willmore functional $W(\varphi_0)$ and obtain the Euler-Lagrange equation (2).

Let $\varphi_0 : M \to N_p^{n+p}(c)$ be an n-dimensional compact spacelike submanifold in $N_p^{n+p}(c)$ with (possibly empty) boundary ∂M. If otherwise, we will consider the variation with compact support. Let $\varphi : M \times R \to N_p^{n+p}(c)$ be a smooth variation of φ_0 such that $\varphi(\cdot, t) = \varphi_0$ on the boundary. Along $\varphi : M \times R \to N_p^{n+p}(c)$, we choose a local orthonormal basis $\{e_A\}$ for $TN_p^{n+p}(c)$ with dual basis $\{\omega_A\}$, so that $\{e_i(\cdot, t)\}$ forms a local orthonormal basis for $\varphi_t : M \times \{t\} \to N_p^{n+p}(c)$. Since $T^*(M \times R) = T^*M \oplus T^*R$, the pullback of $\{\omega_A\}$ and $\{\omega_{AB}\}$ on $N_p^{n+p}(c)$ through $\varphi : M \times R \to N_p^{n+p}(c)$ have the decomposition

$$\varphi^*\omega_\alpha = V_\alpha dt, \quad \varphi^*\omega_i = \theta_i + V_i dt, \tag{28}$$

$$\varphi^*\omega_{ij} = \theta_{ij} + L_{ij}dt, \quad \varphi^*\omega_{i\alpha} = \theta_{i\alpha} + M_{i\alpha}dt, \quad \varphi^*\omega_{\alpha\beta} = \theta_{\alpha\beta} + N_{\alpha\beta}dt, \tag{29}$$

where $\{V_i, V_\alpha, L_{ij}, M_{i\alpha}, N_{\alpha\beta}\}$ are local functions on $M \times R$ with $L_{ij} = -L_{ji}$, $N_{\alpha\beta} = -N_{\beta\alpha}$ and

$$V = \frac{d}{dt}|_{t=0}\varphi_t = \sum_i V_i d\varphi_0(e_i) + \sum_\alpha V_\alpha e_\alpha, \tag{30}$$

is the variation vector field of $\varphi_t : M \to N_p^{n+p}(c)$. We note that forms $\{\theta_i, \theta_{ij}, \theta_{i\alpha}, \theta_{\alpha\beta}\}$ are defined on $M \times \{t\}$, for $t = 0$, they reduce to the forms with the same notation on M. We denote by d_M the differential operator on T^*M; then $d = d_M + dt\frac{\partial}{\partial t}$ on $T^*(M \times R)$.

Let K_{ABCD} be the components of the Riemannian curvature tensor of $N_p^{n+p}(c)$. On $M \times \{t\}$, if we assume that h_{ij}^α and the covariant derivatives $V_{i,j}, V_{\alpha,i}$ and $M_{i\alpha,j}$ are defined similarly to [6] (see (3.7) - (3.10) in [6]), by the proof similar to Lemma 3.1 and Lemma 3.2 in [6], we have the following lemmas:

Lemma 3. *Under the above notations, we have*

$$\frac{\partial \theta_i}{\partial t} = \sum_j (V_{i,j} + L_{ij})\theta_j + \sum_{j,\alpha} h_{ij}^\alpha V_\alpha \theta_j, \tag{31}$$

$$M_{i\alpha} = V_{\alpha,i} + \sum_j h_{ij}^\alpha V_j, \tag{32}$$

$$\begin{aligned}\frac{\partial \theta_{i\alpha}}{\partial t} = \sum_j \Big(& M_{i\alpha,j} + \sum_k L_{ik} h_{jk}^\alpha - \sum_\beta N_{\beta\alpha} h_{ij}^\beta \\ & - \sum_k K_{i\alpha kj} V_k - \sum_\beta K_{i\alpha j\beta} V_\beta \Big)\theta_j.\end{aligned} \tag{33}$$

Lemma 4.

$$\begin{aligned}\frac{\partial h_{ij}^\alpha}{\partial t} = V_{\alpha,ij} &+ \sum_k (L_{ik} h_{kj}^\alpha + L_{jk} h_{ki}^\alpha + h_{ijk}^\alpha V_k) \\ &+ \sum_\beta (N_{\alpha\beta} h_{ij}^\beta - K_{\alpha i\beta j} V_\beta) - \sum_{k,\beta} h_{ik}^\alpha h_{kj}^\beta V_\beta.\end{aligned} \tag{34}$$

Proof of Theorem 1. By reasoning as in [6], setting $i = j$ in (34) and summing over i by using $\sum_{i,k} L_{ik} h_{ki}^\alpha = 0$, we have

$$\begin{aligned}\frac{\partial H^\alpha}{\partial t} = \frac{1}{n}\Delta^\perp V_\alpha &+ \sum_k H_{,k}^\alpha V_k + \sum_\beta N_{\alpha\beta} H^\beta \\ &- \frac{1}{n}\sum_{i,k,\beta} h_{ik}^\alpha h_{ki}^\beta V_\beta - \frac{1}{n}\sum_{i,\beta} K_{\alpha i\beta i} V_\beta.\end{aligned} \tag{35}$$

Since $\sum_{i,j,\alpha,\beta} N_{\alpha\beta} h_{ij}^\alpha h_{ij}^\beta = 0$ and $\sum_{i,j,k,\alpha} L_{jk} h_{ki}^\alpha h_{ij}^\alpha = 0$, from (34) we have

$$\begin{aligned}\frac{1}{2}\frac{\partial S}{\partial t} = &\sum_{i,j,\alpha} h_{ij}^\alpha V_{\alpha,ij} + \frac{1}{2}\sum_k S_{,k} V_k \\ &- \sum_{i,j,\alpha,\beta} K_{\alpha i\beta j} h_{ij}^\alpha V_\beta - \sum_{i,j,k,\alpha,\beta} h_{ij}^\alpha h_{ik}^\alpha h_{kj}^\beta V_\beta.\end{aligned} \tag{36}$$

From (35) and $\sum_{\alpha,\beta} N_{\alpha\beta} H^\alpha H^\beta = 0$, we have

$$\begin{aligned}\frac{1}{2}\frac{\partial (nH^2)}{\partial t} = &\sum_\alpha H^\alpha \Delta^\perp V_\alpha + \frac{n}{2}\sum_k (H^2)_{,k} V_k \\ &- \sum_{i,j,\alpha,\beta} H^\alpha h_{ij}^\alpha h_{ij}^\beta V_\beta - \sum_{i,\alpha,\beta} H^\alpha K_{\alpha i\beta i} V_\beta.\end{aligned} \tag{37}$$

For $\varphi_t : M \to N_p^{n+p}(c)$, we consider the non-negative functional

$$W(\varphi_t) = \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} \theta_1 \wedge \cdots \wedge \theta_n. \tag{38}$$

From (31), we have

$$\begin{aligned}\tfrac{\partial}{\partial t}(\theta_1\wedge\cdots\wedge\theta_n)&=\textstyle\sum_i\theta_1\wedge\cdots\wedge\tfrac{\partial\theta_i}{\partial t}\wedge\cdots\wedge\theta_n\\&=\Big(\textstyle\sum_i V_{i,i}+n\sum_\alpha H^\alpha V_\alpha\Big)\theta_1\wedge\cdots\wedge\theta_n.\end{aligned}\tag{39}$$

From (36) and (37), we see that

$$\begin{aligned}\frac{\partial\rho^n}{\partial t}=n\rho^{n-2}\Big\{&\sum_{i,j,\alpha}h^\alpha_{ij}V_{\alpha,ij}+\frac{1}{2}\sum_k(\rho^2)_{,k}V_k-\sum_{i,j,\alpha,\beta}K_{\alpha i\beta j}h^\alpha_{ij}V_\beta\\&-\sum_\alpha H^\alpha\Delta^\perp V_\alpha-\sum_{i,j,k,\alpha,\beta}h^\alpha_{ij}h^\alpha_{ik}h^\beta_{kj}V_\beta\\&+\sum_{i,j,\alpha,\beta}H^\alpha h^\alpha_{ij}h^\beta_{ij}V_\beta+\sum_{i,\alpha,\beta}H^\alpha K_{\alpha i\beta i}V_\beta\Big\}.\end{aligned}\tag{40}$$

From (38) - (40), we have

$$\begin{aligned}\frac{\partial w(\varphi_t)}{\partial t}=\int_M\rho^{n-2}\Big\{&\big[n\sum_{i,j,\alpha}h^\alpha_{ij}V_{\alpha,ij}-n\sum_\alpha H^\alpha\Delta^\perp V_\alpha+\frac{n}{2}\sum_k(\rho^2)_{,k}V_k\\&+\rho^2\sum_k V_{k,k}\big]+n\sum_\alpha\big[-\sum_{i,j,\beta}K_{\beta i\alpha j}h^\beta_{ij}-\sum_{i,j,k,\beta}h^\beta_{ij}h^\beta_{ik}h^\alpha_{kj}\\&+\sum_{i,j,\beta}H^\beta h^\beta_{ij}h^\alpha_{ij}+\sum_{i,\beta}H^\beta K_{\beta i\alpha i}+\rho^2H^\alpha\big]V_\alpha\Big\}dv.\end{aligned}\tag{41}$$

By the same reason as in [6], we see that

$$\begin{aligned}\frac{\partial w(\varphi_t)}{\partial t}=n\int_M\sum_\alpha\Big\{&\rho^{n-2}\big[-\sum_{i,j,k,\beta}h^\beta_{ij}h^\beta_{ik}h^\alpha_{kj}-\sum_{i,j,\beta}K_{\beta i\alpha j}h^\beta_{ij}\\&+\sum_{i,j,\beta}H^\beta h^\beta_{ij}h^\alpha_{ij}+\sum_{i,\beta}H^\beta K_{\beta i\alpha i}+\rho^2H^\alpha\big]\\&+\sum_{i,j}(\rho^{n-2}h^\alpha_{ij})_{ij}-\Delta^\perp\rho^{n-2}H^\alpha\Big\}V_\alpha dv.\end{aligned}\tag{42}$$

From (9), we see that

$$-\sum_{i,j,\beta}K_{\beta i\alpha j}h^\beta_{ij}+\sum_{i,\beta}H^\beta K_{\beta i\alpha i}=0.$$

Thus, by (30) and (42) with restriction to $t=0$, we obtain the Euler-Lagrange equation (2). This completes the proof of Theorem 1. □

4. Integral equalities of Willmore spacelike submanifolds

Define tensors

$$\tilde{h}_{ij}^{\alpha} = h_{ij}^{\alpha} - H^{\alpha}\delta_{ij}, \tag{43}$$

$$\tilde{\sigma}_{\alpha\beta} = \sum_{i,j} \tilde{h}_{ij}^{\alpha}\tilde{h}_{ij}^{\beta}, \quad \sigma_{\alpha\beta} = \sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta}. \tag{44}$$

Then the $(p \times p)$-matrix $(\tilde{\sigma}_{\alpha\beta})$ is symmetric and can be assumed to be diagonalized for a suitable choice of $e_{n+1}, \cdots, e_{n+p}$. We set

$$\tilde{\sigma}_{\alpha\beta} = \tilde{\sigma}_{\alpha}\delta_{\alpha\beta}. \tag{45}$$

By a direct calculation, we have

$$\sum_k \tilde{h}_{kk}^{\alpha} = 0, \quad \tilde{\sigma}_{\alpha\beta} = \sigma_{\alpha\beta} - nH^{\alpha}H^{\beta}, \quad \rho^2 = \sum_{\alpha} \tilde{\sigma}_{\alpha} = S - nH^2, \tag{46}$$

$$\sum_{i,j,k,\alpha} h_{kj}^{\beta}h_{ij}^{\alpha}h_{ik}^{\alpha} = \sum_{i,j,k,\alpha} \tilde{h}_{kj}^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ik}^{\alpha} + 2\sum_{i,j,\alpha} H^{\alpha}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ij}^{\beta} + H^{\beta}\rho^2 + nH^2H^{\beta}. \tag{47}$$

From (43), (46) and (47), the new Euler-Lagrange equation (2) can be rewritten as

Proposition 3. *Let M be an n-dimensional spacelike submanifold in $N_p^{n+p}(c)$. Then M is a Willmore spacelike submanifold if and only if for $n+1 \le \alpha \le n+p$*

$$\begin{aligned}\Box^{\alpha}(\rho^{n-2}) = {} & (n-1)\rho^{n-2}\Delta^{\perp}H^{\alpha} + 2(n-1)\sum_i (\rho^{n-2})_i H_{,i}^{\alpha} \\ & + (n-1)H^{\alpha}\Delta(\rho^{n-2}) - \rho^{n-2}\Big(\sum_{\beta} H^{\beta}\tilde{\sigma}_{\alpha\beta} + \sum_{i,j,k,\beta} \tilde{h}_{ij}^{\alpha}\tilde{h}_{ik}^{\beta}\tilde{h}_{kj}^{\beta}\Big).\end{aligned} \tag{48}$$

Setting $f = nH^{\alpha}$ in (24), we have

$$\begin{aligned}\Box^{\alpha}(nH^{\alpha}) &= \sum_{i,j} (nH^{\alpha}\delta_{ij} - h_{ij}^{\alpha})(nH^{\alpha})_{,ij} \\ &= \sum_i (nH^{\alpha})(nH^{\alpha})_{,ii} - \sum_{i,j} h_{ij}^{\alpha}(nH^{\alpha})_{,ij}.\end{aligned} \tag{49}$$

We also have

$$\begin{aligned}\frac{1}{2}\Delta(nH)^2 &= \frac{1}{2}\Delta\sum_{\alpha}(nH^{\alpha})^2 = \frac{1}{2}\sum_{\alpha}\Delta(nH^{\alpha})^2 \\ &= \frac{1}{2}\sum_{\alpha,i}[(nH^{\alpha})^2]_{,ii} = \sum_{\alpha,i}[(nH^{\alpha})_{,i}]^2 + \sum_{\alpha,i}(nH^{\alpha})(nH^{\alpha})_{,ii} \\ &= n^2|\nabla^{\perp}\vec{H}|^2 + \sum_{\alpha,i}(nH^{\alpha})(nH^{\alpha})_{,ii}.\end{aligned} \tag{50}$$

Therefore, from (49) and (50), we get

$$\begin{aligned}\sum_\alpha \square^\alpha(nH^\alpha) &= \frac{1}{2}\Delta(nH)^2 - n^2|\nabla^\perp \vec{H}|^2 - \sum_{i,j,\alpha} h^\alpha_{ij}(nH^\alpha)_{,ij} \\ &= \frac{1}{2}\Delta[n(n-1)H^2 - \rho^2 + S] - n^2|\nabla^\perp \vec{H}|^2 - \sum_{i,j,\alpha} h^\alpha_{ij}(nH^\alpha)_{,ij} \qquad (51) \\ &= \frac{1}{2}\Delta S + \frac{1}{2}n(n-1)\Delta H^2 - \frac{1}{2}\Delta\rho^2 - n^2|\nabla^\perp \vec{H}|^2 - \sum_{i,j,\alpha} h^\alpha_{ij}(nH^\alpha)_{,ij}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{2}\Delta S &= \sum_{i,j,k,\alpha}(h^\alpha_{ijk})^2 + \sum_{i,j,\alpha} h^\alpha_{ij}\Delta h^\alpha_{ij} \\ &= |\nabla h|^2 + \sum_{i,j,\alpha} h^\alpha_{ij}(nH^\alpha)_{,ij} + \sum_\alpha \sum_{i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk}) \qquad (52) \\ &\quad + \sum_{\alpha,\beta}\sum_{i,j,k} h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk}.\end{aligned}$$

Putting (52) into (51), we have

$$\begin{aligned}\sum_\alpha \square^\alpha(nH^\alpha) &= |\nabla h|^2 - n^2|\nabla^\perp \vec{H}|^2 + \frac{1}{2}n(n-1)\Delta H^2 - \frac{1}{2}\Delta\rho^2 \\ &\quad + \sum_\alpha \sum_{i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk}) + \sum_{\alpha,\beta}\sum_{i,j,k} h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk}. \qquad (53)\end{aligned}$$

Multiplying (53) by ρ^{n-2} and taking the integral, using (25), we have

$$\begin{aligned}\sum_\alpha \int_M (nH^\alpha)\square^\alpha(\rho^{n-2})dv &= \int_M \rho^{n-2}(|\nabla h|^2 - n^2|\nabla^\perp \vec{H}|^2)dv \\ &\quad + \frac{1}{2}n(n-1)\int_M \rho^{n-2}\Delta H^2 dv - \frac{1}{2}\int_M \rho^{n-2}\Delta\rho^2 dv \\ &\quad + \int_M \rho^{n-2}\sum_\alpha \sum_{i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk})dv \qquad (54) \\ &\quad + \int_M \rho^{n-2}\sum_{\alpha,\beta}\sum_{i,j,k} h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk}dv.\end{aligned}$$

Taking the Willmore equation (48) into (54) and making use of the following:

$$\begin{aligned}
\int_M \rho^{n-2}\sum_\alpha H^\alpha \triangle^\perp H^\alpha dv &= \frac{1}{2}\int_M \rho^{n-2}\sum_\alpha \Delta^\perp (H^\alpha)^2 dv - \int_M \rho^{n-2}\sum_{i,\alpha}(H^\alpha_{,i})^2 dv\\
&= \frac{1}{2}\int_M \rho^{n-2}\Delta H^2 dv - \int_M \rho^{n-2}|\nabla\vec{H}|^2 dv,\\
\int_M H^2\Delta(\rho^{n-2})dv &= \int_M \sum_\alpha (H^\alpha)^2\sum_i(\rho^{n-2})_{,ii}dv\\
&= \sum_{\alpha,i}\int_M (H^\alpha)^2(\rho^{n-2})_{,ii}dv = -\sum_{\alpha,i}\int_M (\rho^{n-2})_i((H^\alpha)^2)_{,i}dv\\
&= -2\int_M \sum_\alpha H^\alpha\sum_i(\rho^{n-2})_i H^\alpha_{,i}dv,\\
-\frac{1}{2}\int_M \rho^{n-2}\Delta\rho^2 dv &= -\frac{1}{2}\sum_i\int_M \rho^{n-2}(\rho^2)_{,ii}dv\\
&= \frac{1}{2}\sum_i\int_M (\rho^2)_i(\rho^{n-2})_i dv = (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv,
\end{aligned}$$

by a direct calculation, we have the following:

Proposition 4. *Let M be an n-dimensional compact Willmore spacelike submanifold in $N_p^{n+p}(c)$. Then*

$$\begin{aligned}
&\int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp\vec{H}|^2)dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv\\
&\quad+\int_M \rho^{n-2}\sum_{\alpha,\beta} nH^\alpha(H^\beta\tilde{\sigma}_{\alpha\beta} + \sum_{i,j,k}\tilde{h}^\alpha_{ij}\tilde{h}^\beta_{ik}\tilde{h}^\beta_{kj})dv\\
&\quad+\int_M \rho^{n-2}\sum_\alpha\sum_{i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl}R_{lijk} + h^\alpha_{li}R_{lkjk})dv\\
&\quad+\int_M \rho^{n-2}\sum_{\alpha,\beta}\sum_{i,j,k} h^\alpha_{ij}h^\beta_{ki}R_{\alpha\beta jk}dv = 0.
\end{aligned}\tag{55}$$

In general, for a matrix $A=(a_{ij})$ we denote by $N(A)$ the square of the norm of A, that is,

$$N(A) = \mathrm{trace}(A\cdot A^t) = \sum_{i,j}(a_{ij})^2.$$

Clearly, $N(A) = N(T^tAT)$ for any orthogonal matrix T. From (18), we have

$$\begin{aligned}\sum_{\alpha,\beta}\sum_{i,j,k} h_{ij}^{\alpha}h_{ki}^{\beta}R_{\alpha\beta jk} &= \sum_{\alpha,\beta}\sum_{i,j,k,l} h_{ij}^{\alpha}h_{ki}^{\beta}(h_{kl}^{\beta}h_{lj}^{\alpha} - h_{jl}^{\beta}h_{lk}^{\alpha})\\ &= \frac{1}{2}\sum_{\alpha,\beta,j,k}\Big(\sum_l h_{jl}^{\alpha}h_{lk}^{\beta} - \sum_l h_{jl}^{\beta}h_{lk}^{\alpha}\Big)^2\\ &= \frac{1}{2}\sum_{\alpha,\beta,j,k}\Big(\sum_l \tilde{h}_{jl}^{\alpha}\tilde{h}_{lk}^{\beta} - \sum_l \tilde{h}_{jl}^{\beta}\tilde{h}_{lk}^{\alpha}\Big)^2\\ &= \frac{1}{2}\sum_{\alpha,\beta} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha),\end{aligned}\tag{56}$$

where $\tilde{A}_\alpha := (\tilde{h}_{ij}^{\alpha}) = (h_{ij}^{\alpha} - H^\alpha\delta_{ij})$.

By using (12), (18), (44), (46), (47) and (56), we conclude that

$$\begin{aligned}&\sum_{\alpha}\sum_{i,j,k,l} h_{ij}^{\alpha}(h_{kl}^{\alpha}R_{lijk} + h_{li}^{\alpha}R_{lkjk})\\ &= nc\rho^2 + \sum_{\alpha,\beta}\sum_{i,j,k,l} h_{ij}^{\alpha}h_{ij}^{\beta}h_{lk}^{\alpha}h_{lk}^{\beta} - n\sum_{\alpha,\beta}\sum_{i,j,k} H^\beta h_{kj}^{\beta}h_{ij}^{\alpha}h_{ik}^{\alpha} - \sum_{\alpha,\beta,i,j,k} h_{ji}^{\alpha}h_{ik}^{\beta}R_{\beta\alpha jk}\\ &= nc\rho^2 + \sum_{\alpha,\beta}\sigma_{\alpha\beta}^2 - n\sum_{\alpha,\beta}\sum_{i,j,k} H^\beta\tilde{h}_{kj}^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ik}^{\alpha} - 2n\sum_{\alpha,\beta}\sum_{i,j} H^\alpha H^\beta\tilde{h}_{ij}^{\alpha}\tilde{h}_{ij}^{\beta}\\ &\quad -n\sum_{\beta}(H^\beta)^2\rho^2 - n^2H^2\sum_{\beta}(H^\beta)^2 + \frac{1}{2}\sum_{\alpha,\beta} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha)\\ &= nc\rho^2 + \sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}^2 - nH^2\rho^2 - n\sum_{\alpha,\beta}\sum_{i,j,k} H^\beta\tilde{h}_{kj}^{\beta}\tilde{h}_{ij}^{\alpha}\tilde{h}_{ik}^{\alpha}\\ &\quad + \frac{1}{2}\sum_{\alpha,\beta} N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha).\end{aligned}\tag{57}$$

Putting (56) and (57) into (55), we have the following:

Proposition 5. *Let M be an n-dimensional compact Willmore spacelike submanifold in $N_p^{n+p}(c)$. Then*

$$\begin{aligned}&\int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp\vec{H}|^2)dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2dv\\ &\quad + n\int_M \rho^{n-2}\Big(\sum_{\alpha,\beta} H^\alpha H^\beta\tilde{\sigma}_{\alpha\beta} - H^2\rho^2\Big)dv + nc\int_M \rho^n dv\\ &\quad + \int_M \rho^{n-2}\sum_{\alpha,\beta}(N(\tilde{A}_\alpha\tilde{A}_\beta - \tilde{A}_\beta\tilde{A}_\alpha) + \tilde{\sigma}_{\alpha\beta}^2)dv = 0.\end{aligned}\tag{58}$$

Corollary 2. *Let M be an n-dimensional compact Willmore spacelike hypersurface in $N_p^{n+p}(c)$. Then*

$$\int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla H|^2)dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2dv + \int_M \rho^n(nc+\rho^2)dv = 0. \tag{59}$$

5. Proofs of Theorems

From Remark 1, in the proofs of Theorem 2 - Theorem 4, we should assume that $n \neq 3, 5$.

Proof of Theorem 2. (1) For $p = 1$, from Lemma 2 and (59), we have

$$\begin{aligned} 0 = &\int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2)dv + \int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^\perp \vec{H}|^2 dv \\ &+(n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv + \int_M \rho^n(nc+\rho^2)dv \geq \int_M \rho^n(nc+\rho^2)dv. \end{aligned} \tag{60}$$

(*i*) If $c = 1$, since $nc + \rho^2 > 0$, from (60), it follows that $\rho^2 = 0$ and M is totally umbilical. If $c = 0$, since $nc + \rho^2 = \rho^2$, from (60), we easily see that $\rho^2 = 0$, thus M is totally umbilical.

(*ii*) If $c = -1$ and $\rho^2 \geq n$, since $nc + \rho^2 = -n + \rho^2 \geq 0$, from (60), we have $\rho^2 = 0$ and M is totally umbilical or $\rho^2 = n$. In the latter case, since $\rho^2 = n > 0$, from (60) we have that

$$\int_M \rho^{n-2}(\frac{3n^2}{n+2} - n)|\nabla^\perp \vec{H}|^2 dv = 0 \quad \text{and} \quad \int_M \rho^{n-2}(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla^\perp \vec{H}|^2)dv = 0.$$

Thus $\nabla^\perp \vec{H} = 0$ and $\nabla h = 0$, that is, $H = constant$ and the second fundamental form of M is parallel. It easily follows that M is an isoparametric spacelike hypersurface with two distinct constant principal curvatures. By the congruence Theorem of Abe, Koike and Yamaguchi (see Theorem 5.1 of [1]), we know that M is isometric to Example 1. This is impossible since M is compact.

(2) For $p \geq 2$, from (45), we get

$$\sum_{\alpha,\beta} \tilde{\sigma}_{\alpha\beta}^2 = \sum_\alpha \tilde{\sigma}_\alpha^2 \geq \frac{1}{p}\Big(\sum_\alpha \tilde{\sigma}_\alpha\Big)^2 = \frac{1}{p}\rho^4. \tag{61}$$

From (58) and (61) and

$$\sum_{\alpha,\beta} N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) \geq 0, \tag{62}$$

$$\sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} = \sum_\alpha (H^\alpha)^2 \tilde{\sigma}_\alpha \geq 0, \tag{63}$$

we have

$$\begin{aligned} 0 = &\int_M \rho^{n-2}(|\nabla h|^2 - n|\nabla^\perp \vec{H}|^2)dv + (n-2)\int_M \rho^{n-2}|\nabla\rho|^2 dv \\ &+ n\int_M \rho^{n-2}\Big(\sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} - H^2\rho^2\Big)dv + nc\int_M \rho^n dv \\ &+ \int_M \rho^{n-2}\sum_{\alpha,\beta}(N(\tilde{A}_\alpha \tilde{A}_\beta - \tilde{A}_\beta \tilde{A}_\alpha) + \tilde{\sigma}_{\alpha\beta}^2)dv \\ \geq &\int_M \rho^n \left\{\frac{1}{p}\rho^2 + nc - nH^2\right\} dv. \end{aligned} \tag{64}$$

In particular, if

$$\rho^2 \geq np(H^2 - c),$$

from (64), we see that $\rho^2 = 0$ and M is totally umbilical or $\rho^2 = np(H^2 - c)$. In the latter case, from (64) we have that

$$\int_M \rho^{n-2} \sum_{\alpha,\beta} H^\alpha H^\beta \tilde{\sigma}_{\alpha\beta} dv = 0,$$

that is

$$\int_M \rho^{n-2} \sum_\alpha (H^\alpha)^2 \tilde{\sigma}_\alpha dv = 0. \tag{65}$$

If $\rho^2 = 0$, that is, M is totally umbilical; if $\rho^2 \neq 0$, from (65) it follows that

$$\sum_\alpha (H^\alpha)^2 \tilde{\sigma}_\alpha = 0.$$

Thus, we see that $H^\alpha = 0$ and $H = 0$. If $c = 1$, we have $\rho^2 = -np < 0$, a contradiction; if $c = 0$, we have $\rho^2 = 0$, also a contradiction since we assume that $\rho^2 \neq 0$; if $c = -1$, we have $\rho^2 = np$. Since $H = 0$ and M is maximal, it follows that $S = np$. From a result of T Ishihara [7] (see Theorem 1.3 of [7]), M is isometric to

$$H^{n_1}\left(\sqrt{\frac{n_1}{n}}\right) \times \cdots \times H^{n_{p+1}}\left(\sqrt{\frac{n_{p+1}}{n}}\right),$$

where $n_1 + \cdots + n_{p+1} = n$. This is impossible since M is compact. This completes the proof of Theorem 2. □

Proof of Theorem 3. For a fixed $\alpha, n+1 \leq \alpha \leq n+p$, we can take a local orthonormal frame field $\{e_1, \ldots, e_n\}$ such that $h^\alpha_{ij} = \lambda^\alpha_i \delta_{ij}$, then $\tilde{h}^\alpha_{ij} = \mu^\alpha_i \delta_{ij}$ with $\mu^\alpha_i = \lambda^\alpha_i - H^\alpha$, $\sum_i \mu^\alpha_i = 0$. Thus

$$\begin{aligned} \sum_{\alpha,i,j,k,l} h^\alpha_{ij}(h^\alpha_{kl} R_{lijk} + h^\alpha_{li} R_{lkjk}) &= \frac{1}{2} \sum_{\alpha,i,j} (\lambda^\alpha_i - \lambda^\alpha_j)^2 R_{ijij} \\ &= \frac{1}{2} \sum_{\alpha,i,j} (\mu^\alpha_i - \mu^\alpha_j)^2 R_{ijij} \geq nK\rho^2, \end{aligned} \tag{66}$$

and the equality in (66) holds if and only if $R_{ijij} = K$ for any $i \neq j$.

Let $\sum_i (\tilde{h}^\beta_{ii})^2 = \tau_\beta$. Then $\tau_\beta \leq \sum_{i,j} (\tilde{h}^\beta_{ij})^2 = \tilde{\sigma}_\beta$. Since $\sum_i \tilde{h}^\beta_{ii} = 0, \sum_i \mu^\alpha_i = 0$ and

$\sum_i(\mu_i^\alpha)^2=\tilde{\sigma}_\alpha$, from Lemma 1 we have that

$$\begin{aligned}\sum_{\alpha,\beta}\sum_{i,j,k}H^\alpha\tilde{h}^\alpha_{ij}\tilde{h}^\beta_{kj}\tilde{h}^\beta_{ik}&=\sum_{\beta,\alpha}\sum_{i,j,k}H^\beta\tilde{h}^\beta_{ij}\tilde{h}^\alpha_{kj}\tilde{h}^\alpha_{ik}=\sum_{\alpha,\beta}H^\beta\sum_i\tilde{h}^\beta_{ii}(\mu_i^\alpha)^2\\&\geq-\frac{n-2}{\sqrt{n(n-1)}}\sum_{\alpha,\beta}|H^\beta|\tilde{\sigma}_\alpha\sqrt{\tau_\beta}\\&\geq-\frac{n-2}{\sqrt{n(n-1)}}\sum_\alpha\tilde{\sigma}_\alpha\sum_\beta|H^\beta|\sqrt{\tilde{\sigma}_\beta}\\&\geq-\frac{n-2}{\sqrt{n(n-1)}}\rho^2\sqrt{\sum_\beta(H^\beta)^2\sum_\beta\tilde{\sigma}_\beta}=-\frac{n-2}{\sqrt{n(n-1)}}H\rho^3.\end{aligned}\tag{67}$$

From (55), (56), (62), (63), (66) and (67), we have

$$\begin{aligned}0\geq&\int_M\rho^{n-2}\sum_\alpha n(H^\alpha)^2\tilde{\sigma}_\alpha-\int_M\rho^{n-2}\frac{n(n-2)}{\sqrt{n(n-1)}}H\rho^3dv\\&+\int_M\rho^{n-2}nK\rho^2dv\geq\int_M n\rho^n\left\{K-\frac{n-2}{\sqrt{n(n-1)}}H\rho\right\}dv.\end{aligned}\tag{68}$$

In particular, if

$$K\geq\frac{n-2}{\sqrt{n(n-1)}}H\rho,$$

from (68) we see that $\rho^2=0$ and M is totally umbilical or

$$K=\frac{n-2}{\sqrt{n(n-1)}}H\rho.$$

In the latter case, from (68), we know that (65) holds. If $\rho^2=0$, that is, M is totally umbilical; if $\rho^2\neq0$, it follows from (65) that $\sum_\alpha(H^\alpha)^2\tilde{\sigma}_\alpha=0$. Thus, we see that $H^\alpha=0$ and $H=0$. It also follows from (66) that $R_{ijij}=K$ for any $i\neq j$. Since

$$K=\frac{n-2}{\sqrt{n(n-1)}}H\rho=0,$$

we have $R_{ijij}=0$ for any $i\neq j$. From the Gauss equation (11), we have $n(n-1)c+S=0$. If $c=1$, we have $n(n-1)+S=0$, a contradiction; if $c=0$, we have $S=0$, thus $\rho^2=0$, also a contradiction since we assume that $\rho^2\neq0$; if $c=-1$, we have $S=n(n-1)$. Since M is maximal and $S=np$, where $p=n-1$, from a result of T Ishihara [7], M is isometric to

$$H^{n_1}(\sqrt{\frac{n_1}{n}})\times\cdots\times H^{n_n}(\sqrt{\frac{n_n}{n}}),$$

where $n_1+\cdots+n_n=n$. This is impossible since M is compact. This completes the proof of Theorem 3. □

Proof of Theorem 4. From (12) and (43), we have

$$Q \leq R_{ii} = (n-1)c - (n-2)\sum_{\alpha} H^{\alpha}\tilde{h}_{ii}^{\alpha} - (n-1)H^2 + \sum_{\alpha,j}(\tilde{h}_{ij}^{\alpha})^2.$$

Thus

$$\rho^2 = \sum_{\alpha,i,j}(\tilde{h}_{ij}^{\alpha})^2 \geq nQ - n(n-1)(c-H^2), \tag{69}$$

and

$$\sum_{\alpha,\beta}\tilde{\sigma}_{\alpha\beta}^2 \geq \frac{1}{p}\rho^4 \geq \frac{1}{p}\rho^2[nQ - n(n-1)(c-H^2)]. \tag{70}$$

From (58), (62), (63) and (70), we have

$$\begin{aligned} 0 \geq & \int_M \rho^{n-2}\sum_{\alpha} n(H^{\alpha})^2\tilde{\sigma}_{\alpha} - n\int_M \rho^{n-2}H^2\rho^2 dv \\ & + nc\int_M \rho^n dv + \int_M \rho^{n-2}\frac{1}{p}\rho^2[nQ - n(n-1)(c-H^2)] \\ \geq & \frac{n}{p}\int_M \rho^n\left\{Q - (n-p-1)(c-H^2)\right\} dv. \end{aligned} \tag{71}$$

In particular, if

$$Q \geq (n-p-1)(c-H^2),$$

from (71), we see that $\rho^2 = 0$ and M is totally umbilical or $Q = (n-p-1)(c-H^2)$. In the latter case, from (71), we know that (65) holds. If $\rho^2 = 0$, that is, M is totally umbilical; if $\rho^2 \neq 0$, it follows from (65) that

$$\sum_{\alpha}(H^{\alpha})^2\tilde{\sigma}_{\alpha} = 0.$$

Thus, we see that $H^{\alpha} = 0$ and $H = 0$. It also follows from (71) that the equality in (69) holds, that is, $\rho^2 = nQ - n(n-1)(c-H^2) = nQ - n(n-1)c$. Since we also know that $Q = (n-p-1)c$, we see that $\rho^2 = -npc$, by reasoning as in the proof of Theorem 2, we know that this is impossible. This completes the proof of Theorem 4. □

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