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k–generalized Fibonacci numbers of the form $1 + 2^{n_1} + 4^{n_2} + \cdots + (2^k)^{n_k}$

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Abstract. A generalization of the well–known Fibonacci sequence is the k–generalized Fibonacci sequence $(F_n^{(k)})_{n\geq 2-k}$ whose first k terms are $0, \ldots, 0, 1$ and each term afterwards is the sum of the preceding k terms. In this paper, we investigate k–generalized Fibonacci numbers written in the form $1 + 2^{n_1} + 4^{n_2} + \cdots + (2^k)^{n_k}$, for non–negative integers n_i, with $n_k \geq \max_{1\leq i\leq k-1}\{n_i\}$.

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1. Introduction

The Fibonacci sequence denoted by $(F_n)_{n\geq 0}$ is the sequence of integers given by $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, for all $n \geq 0$.

The study of Fibonacci numbers having special representations has been of interest to many researchers and has generated an extensive literature. We only name a few of such studies. In 1963, Moser and Carlitz [16], and Rollet [22], proposed the problem of finding all square Fibonacci numbers. This problem was solved one year later by Cohn [4] and Wyler [25], independently. In 1965, Cohn [5] found all positive integer solutions (n, x) of the Diophantine equation $F_n = 2x^2$. Later, Robbins (see [18]) solved the equation $F_n = px^2$ for all primes $p \in [2, 10000]$ as well as for all primes p such that $p \equiv 3 \pmod 4$. In the subsequent work [19], he found all positive integer solutions (n, x) of the Diophantine equation $F_n = cx^2$ for all composite values of $c \in [2, 10000]$. Other studies concerning representations of Fibonacci numbers by quadratic and cubic polynomials are dealt with the Diophantine equations

- $F_n = k^2 + k + 2$, [10];
- $F_n = x^2 - 1$ and $F_n = x^3 \pm 1$, [20];

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- $F_n = px^2 + 1$ and $F_n = px^3 + 1$, [21].

Recently, Bugeaud, Mignotte and Siksek [1] confirmed that the only perfect powers (of exponent greater than 1) in the Fibonacci sequence are 0, 1, 8, and 144, which was a famous problem. Shortly after that, together with Luca [2], the same authors showed that the only Fibonacci numbers that are at distance 1 from a perfect power are $1, 2, 3, 5$, and 8.

Next we mention some results related to the problem discussed in this paper. Pethő and Tichy [17] showed that if p is a fixed prime, then there are only finitely many Fibonacci numbers of the form $p^a + p^b + p^c$ with integers $a > b > c \geq 0$. The proof of their result is ineffective in that it uses the finiteness of a number of non-degenerate solutions of $\mathcal{S}-$units equations. However, all solutions of such an equation can be found using the theory of lower bounds for linear forms in logarithms as in [6, 11, 23]. Regarding this type of representation, Luca and Szalay [13] and Luca and Stănică [12] showed that each of the Diophantine equations $F_n = p^a \pm p^b + 1$ and $F_n = p^a \pm p^b$ has only finitely many positive integer solutions (n, p, a, b), with p prime being also a variable. Marques and Togbé [14] found all Fibonacci numbers of the form $2^a + 3^b + 5^c$, with $c \geq \max\{a, b\} \geq 0$. Note that $2,\ 3,\ 5$ are $F_3,\ F_4,\ F_5$, respectively. In the same paper, the authors claim to have found all Fibonacci numbers of the form $y^a + y^b + y^c$, with positive integers $a,\ b,\ c$ and integer $y \in [2, 9]$.

Let $k \geq 2$ be an integer. One of many generalizations of the Fibonacci sequence, which is sometimes called the *k-generalized Fibonacci sequence* $(F_n^{(k)})_{n \geq -(k-2)}$, is given by the recurrence

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)}, \quad \text{for all} \quad n \geq 2,$$

with the initial conditions $F_{2-k}^{(k)} = F_{3-k}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. We refer to $F_n^{(k)}$ as the n^{th} *k-generalized Fibonacci number* or *k-Fibonacci number*. Note that for $k = 2$, we have $F_n^{(2)} = F_n$, the well-known n^{th} Fibonacci number. For $k = 3$, such numbers are called *Tribonacci* numbers. They are followed by the *Tetranacci* numbers for $k = 4$, and so on.

A curious fact about the k–generalized Fibonacci sequence is that the k values after to the k initial values are powers of two. Indeed,

$$F_2^{(k)} = 1, \quad F_3^{(k)} = 2, \quad F_4^{(k)} = 4, \ldots, F_{k+1}^{(k)} = 2^{k-1}. \tag{1}$$

That is, $F_n^{(k)} = 2^{n-2}$, for all $2 \leq n \leq k+1$. Solutions of Diophantine equations on $k-$generalized Fibonacci numbers involving its first $k+1$ values will be called *trivial solutions.* The first $k-$generalized Fibonacci number that is not a power of two is $F_{k+2}^{(k)} = 2^k - 1$. Bravo and Luca showed in [3] that $F_n^{(k)} < 2^{n-2}$ for all $n \geq k+2$ and that except for trivial cases, there are no powers of two in any k–generalized Fibonacci sequence for $k \geq 3$, and that the only nontrivial power of two in the Fibonacci sequence is $F_6 = 8$.

In this paper, we find all k–generalized Fibonacci numbers of the form $1 + 2^{n_1} + 4^{n_2} + \cdots + (2^k)^{n_k}$, in non-negative integers n_i, with $n_k \geq \max\limits_{1 \leq i \leq k-1}\{n_i\}$. In other words, we look at the Diophantine equation

$$F_m^{(k)} = 1 + 2^{n_1} + 4^{n_2} + \cdots + (2^k)^{n_k}. \tag{2}$$

This equation is inspired by the work of Marques and Togbé [14] on the equation $F_n = F_3^a + F_4^b + F_5^c$ with $c \geq \max\{a, b\}$. Also, for every fixed k, equation (2) has only at most finitely many solutions $(m, n_1, \ldots, n_k)$ even without the restriction that $n_k \geq \max_{1\leq i\leq k-1}\{n_i\}$. These solutions can be computed using the theory of linear forms in logarithms of algebraic numbers because $\{F_n^{(k)}\}_{n\geq 1}$ is a non-degenerate linearly recurrent sequence whose dominant root has the property that is multiplicatively independent over the number 2 (see [23] and [11]). We do not know, however, how to prove a finiteness result when k is also a variable without the above size restriction on n_k. We prove the following theorem.

Main Theorem. *The only nontrivial solution of the Diophantine equation* (2) *in non–negative integers m, k, $n_1, \ldots, n_k$ with $k \geq 2$, $m \geq 2k + 3$, $n_k \geq 2$ and $n_k \geq \max_{1\leq i\leq k-1}\{n_i\}$, is $(m, k, n_1, n_2) = (8, 2, 2, 2)$. That is,*

$$F_8 = 1 + 2^2 + 4^2.$$

Before getting to the details, we give a brief description of our method. We first use lower bounds for linear forms in logarithms of algebraic numbers to bound m polynomially in terms of k. When k is small, we use the theory of continued fractions by means of a result of Dujella and Pethő to lower such bounds to cases that allow us to treat our problem computationally. When k is large, we use the fact that the dominant root of the k–generalized Fibonacci sequence is exponentially close to 2, to substitute this root by 2 in our calculations with linear form in logarithms obtaining in this way a simpler linear form in logarithms, which allows us to bound k and then complete the remaining calculations.

2. Some results on $k-$Fibonacci numbers

The characteristic polynomial of the k–generalized Fibonacci sequence is

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

The above polynomial has just one root $\alpha(k)$ outside the unit circle. It is real and positive so it satisfies $\alpha(k) > 1$. The other roots are strictly inside the unit circle. In particular, $\Psi_k(x)$ is irreducible in $\mathbb{Q}[x]$. Lemma 2.3 in [9] shows that

$$2(1 - 2^{-k}) < \alpha(k) < 2, \quad \text{for all} \quad k \geq 2. \tag{3}$$

This inequality was rediscovered by Wolfram [24]. We put $\alpha := \alpha(k)$. This is called the *dominant root* of $\Psi_k(x)$ for reasons that we present below. Dresden [7] gave the following Binet-like formula for $F_n^{(k)}$:

$$F_n^{(k)} = \sum_{i=1}^{k} \frac{\alpha^{(i)} - 1}{2 + (k+1)(\alpha^{(i)} - 2)} {\alpha^{(i)}}^{n-1}, \tag{4}$$

where $\alpha = \alpha^{(1)}, \ldots, \alpha^{(k)}$ are the roots of $\Psi_k(x)$. Dresden also showed that the contribution of the roots which are inside the unit circle to the right–hand side of (4) is very small. More precisely, he proved that

$$\left| F_n^{(k)} - \frac{\alpha - 1}{2 + (k+1)(\alpha - 2)} \alpha^{n-1} \right| < \frac{1}{2}, \quad \text{for all} \quad n \geq 1. \tag{5}$$

Other properties relevant to our work are the following. The inequality

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \tag{6}$$

holds for all $n \geq 1$ and $k \geq 2$ (see [3]). Further, the sequences

$$(F_n^{(k)})_{n \geq 1}, \quad (F_n^{(k)})_{k \geq 2} \quad \text{and} \quad (\alpha(k))_{k \geq 2} \tag{7}$$

are non decreasing. Particularly, $\alpha \geq \phi := (1 + \sqrt{5})/2$ for all $k \geq 2$.

We consider the function

$$f_k(z) := \frac{z - 1}{2 + (k+1)(z - 2)}, \quad \text{for} \quad k \geq 2.$$

If $z \in (2(1 - 2^{-k}), 2)$, a straightforward verification shows that $\partial_z f_k(z) < 0$. Indeed,

$$\partial_z f_k(z) = \frac{-k + 1}{(2 + (k+2)(z - 2))^2} < 0, \quad \text{for all} \quad k \geq 2.$$

Thus, from inequality (3), we conclude that

$$1/2 = f_k(2) \leq f_k(\alpha) \leq f_k\left(2(1 - 2^{-k})\right) = \frac{2^{k-1} - 1}{2^k - k - 1} \leq 1,$$

for all $k \geq 2$. Furthermore, one can check that the upper bound 1 on the right–hand side above can be replaced by 3/4 for all $k \geq 3$. Since we also have that $f_2((1 + \sqrt{5})/2) = 0.72360\ldots < 3/4$, we deduce that $f_k(\alpha) \leq 3/4$ holds for all $k \geq 2$. On the other hand, if $z = \alpha^{(i)}$ with $i = 2, \ldots, k$, then $|f_k(\alpha^{(i)})| < 1$ for all $k \geq 2$. Indeed, as $|\alpha^{(i)}| < 1$, then $|\alpha^{(i)} - 1| < 2$ and $|2 + (k+1)(\alpha^{(i)} - 2)| > k - 1$. Further, $f_2((1 - \sqrt{5})/2) = 0.2763\ldots$

Finally, in order to replace α by 2 in the final stage of our argument, we use an argument that is due to Bravo and Luca [3]. Namely, if $1 \leq r < 2^{k/2}$, then

$$\alpha^r = 2^r + \delta \qquad \text{and} \qquad f_k(\alpha) = f_k(2) + \eta,$$

with $|\delta| < 2^{r+1}/2^{k/2}$ and $|\eta| < 2k/2^k$. Thus,

$$\left| f_k(\alpha)\alpha^r - 2^{r-1} \right| < \frac{2^r}{2^{k/2}} + \frac{2^{r+1}k}{2^k} + \frac{2^{r+2}k}{2^{3k/2}}.$$

Furthermore, if $k > 10$, then $4k/2^k < 1/2^{k/2}$ and $8k/2^{3k/2} < 1/2^{k/2}$. Hence,

$$\left| f_k(\alpha)\alpha^r - 2^{r-1} \right| < \frac{2^{r+1}}{2^{k/2}}. \tag{8}$$

3. Preliminary considerations

Let us suppose that $(m, k, n_1, \ldots, n_k)$ is a solution of (2). Since $n_i \geq 0$ for all $i = 1, 2, \ldots, k$, we conclude that $F_m^{(k)} \geq k+1 \geq 3$ and so $m \geq 4$.

We make some considerations on n_k. If $n_k = 0$, then $F_m^{(k)} = k+1$. Thus, either $k = 2$ and $m = 4$, or $k \geq 3$ and $m \leq k+1$, obtaining that in this case the solutions are given by

$$(m, k, n_1, \ldots, n_k) = (4, 2, 0, \ldots, 0) \quad \text{or} \quad (t+2, M_t, 0, \ldots, 0),$$

where M_t is the t^{th} Mersenne number and $t \geq 2$. If $n_k = 1$, then $m \leq k+3$, which follows from the fact that $F_m^{(k)} \leq 2^{k+1} - 1 < F_{k+4}^{(k)} = 2^{k+2} - 8$ for all $k \geq 2$ together with (7). But, for $m \leq k+2$ this leads to a contradiction:

$$2^k + k \leq 1 + 2^{n_1} + 4^{n_2} + \cdots + 2^k = F_m^{(k)} \leq 2^k - 1.$$

If $m = k+3$, then by (2)

$$2^{k+1} - 3 = 1 + 2^{n_1} + 4^{n_2} + \cdots + (2^{k-1})^{n_{k-1}} + 2^k \leq 2^{k+1} - 1.$$

A simple deduction involving binary expansions shows that our equation is not possible for $n_i = 0$ or 1 and $k \geq 2$. Thus, when $n_k = 1$, equation (2) has no solutions.

The above argument also shows that for $n_k \geq 2$, equation (2) has no trivial solutions. In fact,

$$4^k < F_m^{(k)} < 2^{m-2},$$

so $m > 2k+2$. In this way, our problem is reduced to studying Diophantine equation (2) in integers $k \geq 2,\ m \geq 2k+3$ and

$$n_k \geq \max\{2, n_i : 1 \leq i \leq k-1\}.$$

To conclude this section, we present an inequality relating to m, n_k and k. By equation (2), we obtain

$$2^{kn_k} < 1 + 2^{n_1} + 4^{n_2} + \cdots + (2^k)^{n_k} = F_m^{(k)} < 2^{m-2}.$$

Moreover, by inequality (6),

$$\begin{aligned}\alpha^{m-2} \leq F_m^{(k)} = 1 + 2^{n_1} + 4^{n_2} + \cdots + (2^k)^{n_k} &\leq \frac{2^{(k+1)n_k} - 1}{2^{n_k} - 1} \\ &< \frac{2^{(k+1)n_k}}{2^{n_k - 1}} = 2^{kn_k + 1}\end{aligned}$$

Thus,

$$kn_k + 2 < m < 1.5kn_k + 3.5. \tag{9}$$

Here, we used the fact that $\log 2/\log\alpha \leq \log 2/\log\phi < 1.5$. Estimate (9) is essential for our purpose.

4. An inequality for m in terms of k

¿From now on, $k \geq 2$, $m \geq 2k+3$ and $n_k \geq 2$ are integers satisfying (2). We see easily that $m \geq 7$. In order to find an upper bound for m, we use a result of E. M. Matveev on the lower bound for nonzero linear forms in logarithms algebraic numbers.

Let γ be an algebraic number of degree d over $\mathbb{Q}$ with the minimal primitive polynomial over the integers

$$f(X) := a_0 \prod_{i=1}^{d} (X - \gamma^{(i)}) \in \mathbb{Z}[X],$$

where the leading coefficient a_0 is positive. The logarithmic height of γ is given by

$$h(\gamma) := \frac{1}{d} \left(\log a_0 + \sum_{i=1}^{d} \log \max\{|\gamma^{(i)}|, 1\} \right).$$

We use the following theorem of Matveev [15].

Theorem 1. *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\gamma_1, \ldots, \gamma_t$ positive real numbers of $\mathbb{K}$, and $b_1, \ldots, b_t$ rational integers. Put*

$$\Lambda := \gamma_1^{b_1} \cdots \gamma_t^{b_t} - 1 \qquad \text{and} \qquad B \geq \max\{|b_1|, \ldots, |b_t|\}.$$

Let $A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$ be real numbers, for $i = 1, \ldots, t$. Then, assuming that $\Lambda \neq 0$, we have

$$|\Lambda| > \exp(-1.4 \times 30^{t+3} \times t^{4.5} \times D^2 (1 + \log D)(1 + \log B) A_1 \cdots A_t).$$

By using formula (4) and estimate (5), we can write

$$F_m^{(k)} = f_k(\alpha)\alpha^{m-1} + e_k(m), \qquad \text{where} \qquad |e_k(m)| < 1/2. \tag{10}$$

Hence, equation (2) can be rewritten as

$$f_k(\alpha)\alpha^{m-1} - 2^{kn_k} = 1 + 2^{n_1} + 4^{n_2} + \cdots + (2^{k-1})^{n_{k-1}} - e_k(m). \tag{11}$$

Dividing both sides of equation (11) by 2^{kn_k} and taking absolute values, we get

$$\left| f_k(\alpha)\alpha^{m-1} 2^{-kn_k} - 1 \right| < \frac{2^{kn_k} - 1}{2^{kn_k}(2^{n_k} - 1)} + \frac{1}{2^{kn_k+1}} < \frac{3}{2^{n_k}}. \tag{12}$$

We apply Theorem 1 with the parameters $t := 3$, $\gamma_1 := f_k(\alpha)$, $\gamma_2 := \alpha$, $\gamma_3 := 2$, $b_1 := 1$, $b_2 := m-1$, $b_3 := kn_k$. Hence, $\Lambda := f_k(\alpha)\alpha^{m-1}2^{-kn_k} - 1$ and from (12), we have that

$$|\Lambda| < \frac{3}{2^{n_k}}. \tag{13}$$

The algebraic number field $\mathbb{K} := \mathbb{Q}(\alpha)$ contains γ_1, γ_2, and γ_3 and has degree k over $\mathbb{Q}$; i.e., $D = k$. We show that $\Lambda \neq 0$. Otherwise, we get the relation $f_k(\alpha)\alpha^{m-1} =$

2^{kn_k}. Conjugating this relation by an automorphism σ of the Galois group of $\Psi_k(x)$ over $\mathbb{Q}$ with $\sigma(\alpha) = \alpha^{(i)}$ for some $i > 1$, we get that $2^{kn_k} = f_k(\alpha^{(i)})(\alpha^{(i)})^{m-1}$. Then $|f_k(\alpha^{(i)})| > 16$, which is impossible. Hence, $\Lambda \neq 0$.

Knowing that $\mathbb{Q}(\alpha) = \mathbb{Q}(f_k(\alpha))$ and $|f_k(\alpha^{(i)})| \leq 1$ for $i = 1, \ldots, k$ and $k \geq 2$, we obtain that $h(\gamma_1) = (\log a_0)/k$, where a_0 is the leading coefficient of the minimal primitive polynomial over the integers of γ_1. Put

$$g_k(x) = \prod_{i=1}^{k} \left(x - f_k(\alpha^{(i)})\right) \in \mathbb{Q}[x]$$

and $\mathcal{N} = \mathrm{N}_{\mathbb{K}/\mathbb{Q}}(2 + (k+1)(\alpha - 2)) \in \mathbb{Z}$. We conclude that $\mathcal{N} g_k(x) \in \mathbb{Z}[x]$ vanishes at $f_k(\alpha)$. Thus, a_0 divides $|\mathcal{N}|$. But

$$\begin{aligned}
|\mathcal{N}| = \left| \prod_{i=1}^{k} \left(2 + (k+1)(\alpha^{(i)} - 2)\right) \right| &= (k+1)^k \left| \prod_{i=1}^{k} \left(2 - \frac{2}{k+1} - \alpha^{(i)}\right) \right| \\
&= (k+1)^k \left| \Psi_k \left(2 - \frac{2}{k+1}\right) \right| \\
&= \frac{2^{k+1}k^k - (k+1)^{k+1}}{k-1} < 2^k k^k.
\end{aligned}$$

Hence, $h(\gamma_1) < \log(2k) \leq 2\log k$ for all $k \geq 2$. Further, $h(\gamma_2) = (\log \alpha)/k$ and $h(\gamma_3) = \log 2$. Thus, we can take $A_1 := 2k \log k$, $A_2 := 0.7$ and $A_3 := 0.7k$. Finally, from (9), we can take $B := m - 1$.

Theorem 1 gives the following lower bound for $|\Lambda|$:

$$\exp\left(-1.4 \times 30^6 \times 3^{4.5} k^2 (1 + \log k)(1 + \log(m-1))(2k \log k)(0.7)(0.7k)\right),$$

which is by inequality (13) smaller than $3/2^{n_k}$. Taking logarithms on both sides and performing respective calculations, we get that

$$\begin{aligned}
n_k &< \frac{\log 3}{\log 2} + \frac{1.4 \times 30^6 \times 3^{4.5} \times 0.7^2 \times 2 \times 2.5 \times 1.5}{\log 2} k^4 (\log k)^2 \log m \\
&< 7.6 \times 10^{11} k^4 (\log k)^2 \log m, \qquad (14)
\end{aligned}$$

where we used the fact that $1 + \log k < 2.5 \log k$ and $1 + \log(m-1) < 1.5 \log m$ for all $k \geq 2$ and $m \geq 7$.

By inequality (9), $m < 1.5kn_k + 3.5$, and inserting this bound into (14), we conclude that

$$\begin{aligned}
m &< 1.5k(7.6 \times 10^{11} k^4 (\log k)^2 \log m) + 3.5 \\
&< 1.2 \times 10^{12} k^5 (\log k)^2 \log m,
\end{aligned}$$

or, equivalently,

$$\frac{m}{\log m} < 1.2 \times 10^{12} k^5 (\log k)^2. \qquad (15)$$

Now, as the function $x \mapsto x/\log x$ is increasing for all $x > e$, we can easily show that the inequality $\dfrac{x}{\log x} < A$ yields $x < 2A \log A$, whenever $A > 3$. Applying this argument to inequality (15), with $A := 1.2 \times 10^{12} k^5 (\log k)^2$ and $x := m$, we obtain

$$\begin{aligned} m &< 2(1.2 \times 10^{12} k^5 (\log k)^2) \log(1.2 \times 10^{12} k^5 (\log k)^2) \\ &< 1.2 \times 10^{14}\, k^5 (\log k)^3, \end{aligned}$$

where we have used that $\log(1.2 \times 10^{12} k^5 (\log k)^2) < 48 \log k$ holds for all $k \geq 2$. We record what we have just proved.

Lemma 1. *If $(m, k, n_1, \ldots, n_k)$ is a solution of (2), with $k \geq 2$, $m \geq 2k + 3$ and $n_k \geq 2$, then the inequality*

$$kn_k + 3 \leq m < 1.2 \times 10^{14}\, k^5 (\log k)^3 \tag{16}$$

holds.

5. The case of small k

Here, we treat the case $k \in [2, 182]$ showing that in such range equation (2) has a solution only when $k = 2$ and the only solution then is

$$F_8 = 1 + 2^2 + 4^2.$$

We make use of the following result due to Dujella and Pethő which is a generalization of a result of Baker and Davenport (see [8]). Our aim here is to reduce the upper bound of m obtained for each $k \in [2, 182]$ by using inequality (16) and afterwards conclude by performing a computational search.

For a real number x, we put $||x|| = \min\{|x - n| \, : \, n \in \mathbb{Z}\}$ for the distance from x to the nearest integer.

Lemma 2. *Let M be a positive integer and p/q a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := ||\mu q|| - M||\gamma q||$. If $\epsilon > 0$, then there is no solution to the inequality*

$$0 < m\gamma - n + \mu < AB^{-m},$$

in positive integers m and n with $\log(Aq/\epsilon)/\log B \leq m \leq M$.

In order to apply Lemma 2, we let

$$\Gamma := (m - 1) \log \alpha - kn_k \log 2 + \log f_k(\alpha).$$

Returning to Λ given by expression (12), we see that $e^{\Gamma} - 1 = \Lambda$. We note that Γ is positive since Λ is positive, which can be deduced by looking at the right-hand side of equation (11).

Thus,

$$0 < \Gamma < e^{\Gamma} - 1 < \frac{3}{2^{n_k}}.$$

Replacing Γ by its formula and dividing both sides by $\log 2$, we get

$$0 < (m-1)\left(\frac{\log\alpha}{\log 2}\right) - kn_k + \frac{\log f_k(\alpha)}{\log 2} < \frac{3}{2^{n_k}\log 2} < 5\times 2^{\frac{2.5}{1.5k}}\left(2^{\frac{1}{1.5k}}\right)^{-(m-1)}, \quad (17)$$

where we used that $n_k > (m-3.5)/(1.5k)$, which follows from inequality (9). We put

$$\gamma_k := \frac{\log\alpha}{\log 2}, \qquad \mu_k := \frac{\log f_k(\alpha)}{\log 2},$$

and

$$A_k := 5\times 3.18^{1/k}, \qquad B_k := 1.58^{1/k}.$$

The fact that α is a unit in $\mathcal{O}_{\mathbb{K}}$ ensures that γ_k is an irrational number. Even more, γ_k is transcendental by the Gelfond-Schneider theorem. Inequality (17) can be rewritten as

$$0 < (m-1)\gamma_k - kn_k + \mu_k < A_k B_k^{-(m-1)}. \quad (18)$$

Now, we take $M := \lfloor 1.2\times 10^{14}k^5(\log k)^3\rfloor$ which is an upper bound on m by inequality (16), and apply Lemma 2 to inequality (18) for each $k\in[2,182]$.

By means of computer search with Mathematica we found the values of

$$m_k := \lfloor \log(A_k q/\epsilon)/\log B_k\rfloor$$

(see Table 1) which corresponds to upper bounds on $m-1$, according to Lemma 2.

Thus, gathering all the information obtained and considering inequality (9), our problem is reduced to search solutions for (2) in the following range:

$$k\in[2,182], \quad m\in[2k+3, m_k+1], \quad n_k\in[2,(m_k-1)/k]. \quad (19)$$

k	m_k	k	m_k	k	m_k	k	m_k	k	m_k	k	m_k	k	m_k
2	17	28	3506	54	8365	80	15507	106	32374	132	51132	158	72554
3	285	29	3756	55	8505	81	15827	107	32932	133	51328	159	73451
4	392	30	3796	56	8844	82	19241	108	33587	134	51949	160	74389
5	542	31	3947	57	9143	83	19668	109	34297	135	53259	161	75502
6	632	32	4076	58	9847	84	20161	110	34838	136	53814	162	76422
7	753	33	4232	59	9979	85	20928	111	35627	137	54368	163	77243
8	923	34	4405	60	10475	86	21146	112	37020	138	55179	164	78246
9	1041	35	4522	61	10674	87	21633	113	36772	139	56083	165	79259
10	1108	36	4649	62	10857	88	22511	114	37453	140	56820	166	80211
11	1251	37	4861	63	11298	89	23176	115	38230	141	57614	167	81368
12	1393	38	4962	64	11659	90	23190	116	38909	142	59817	168	82173
13	1483	39	5164	65	12178	91	23840	117	39475	143	59259	169	83163
14	1617	40	5352	66	12347	92	24284	118	40263	144	60383	170	84215
15	1791	41	5420	67	12739	93	25063	119	40850	145	61012	171	85231
16	1866	42	5548	68	13113	94	25375	120	41805	146	61844	172	86208
17	2069	43	5592	69	13501	95	25884	121	42310	147	62723	173	87505
18	2358	44	5873	70	13895	96	26539	122	43101	148	63756	174	88219
19	2340	45	6033	71	14365	97	27020	123	43706	149	64660	175	89398
20	2448	46	6068	72	14745	98	27551	124	44413	150	65364	176	90235
21	2527	47	6190	73	15135	99	28121	125	45164	151	66187	177	91424
22	2760	48	6548	74	15593	100	28696	126	46300	152	67133	178	92302
23	2870	49	6937	75	15993	101	29292	127	46835	153	68067	179	93492
24	2973	50	7041	76	16490	102	29916	128	47351	154	69035	180	94399
25	3061	51	7335	77	19875	103	30482	129	48169	155	70523	181	95778
26	3295	52	7921	78	17471	104	31111	130	48930	156	70810	182	96654
27	3391	53	7917	79	17777	105	31683	131	49718	157	71599		

Table 1: *Reducing m on k*

Finally, we note that if $(m, k, n_1, \ldots, n_k)$ is a solution of the equation (2) and s is the number of n_i's which are zero, then $0 \leq s \leq k-1$ ($n_k \geq 2$), and the following hold:

i) $F_m^{(k)} - s$ is odd;

ii) k divides the greatest exponent of 2 in the binary representation of $F_m^{(k)} - s$;

iii) $F_m^{(k)} - s$ has $k+1-s$ digits of 1 in base 2 and the remaining digits equal to zero.

Hence, we search for all $k-$Fibonacci numbers $F_m^{(k)}$, with k and m in the range given by (19), which satisfy the above conditions. A new computational search with Mathematica revealed that $s = 0$ and

k	2	6	14	30	62	126
m	8	15	31	63	127	255

Comparing the representation in base 2 of each $F_m^{(k)}$ with the shape of the right–hand side of equation (2), we conclude that the only nontrivial solution of the equation (2) is that given by the Main Theorem. With this, we completed the analysis of the case when k is small.

6. The case of large k

We now assume that $k > 182$ and show that the equation (2) has no nontrivial solutions. From (16) we have that

$$m < 1.2 \times 10^{14} k^5 (\log k)^3 < 2^{k/2}.$$

Then, combining inequality (8) with $r = m-1$, equality (11) and the fact that $n_k \geq 2$, we conclude that

$$\begin{aligned} |2^{m-2} - 2^{kn_k}| &< |2^{m-2} - f_k(\alpha)\alpha^{m-1}| + |f_k(\alpha)\alpha^{m-1} - 2^{kn_k}| \\ &< \frac{2^m}{2^{k/2}} + \frac{2^{kn_k}}{3} + \frac{1}{2}. \end{aligned}$$

Now, dividing both sides by 2^{m-2}, we get

$$|1 - 2^{kn_k-(m-2)}| < \frac{4}{2^{k/2}} + \frac{1}{3 \times 2^{m-2-kn_k}} + \frac{1}{2^{m-1}}. \tag{20}$$

On the other hand, by (9), the left–hand side in (20) is greater than or equal to $1/2$. So, in summary, from (20) and the previous observation, we have that

$$\frac{4}{2^{k/2}} + \frac{1}{3 \times 2^{m-2-kn_k}} + \frac{1}{2^{m-1}} > \frac{1}{2}. \tag{21}$$

However, inequality (21) is impossible, given that $k > 182$, $m \geq 7$ and $m-2-kn_k \geq 1$.

Thus, we have shown that there are no solutions $(m, k, n_1, \ldots, n_k)$ to Diophantine equation (2) with $k > 182$, $m \geq 2k+3$ and $n_k \geq 2$, which completes the proof of our Main Theorem.

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