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Golden maps between Golden Riemannian manifolds and constancy of certain maps

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Abstract. We first introduce Golden maps between Golden Riemannian manifolds, give an example and show that such map is harmonic. Then we investigate the constancy of certain maps from Golden Riemannian manifolds to various manifolds by imposing the holomorphic-like map condition. Then we consider the reverse case and show that all such maps are constant.

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1. Introduction

Manifolds equipped with certain differential-geometric structures possess rich geometric structures and such manifolds and maps between them have been studied widely in differential geometry. Indeed, almost complex manifolds, almost contact manifolds and almost product manifolds and maps between such manifolds have been studied extensively by many authors. Such manifolds are defined by a $(1,1)$-tensor field Φ such that the square of Φ satisfies certain conditions, like $\Phi^2 = -I$, $\Phi^2 = I$ or $\Phi^2 = -I + \eta \otimes \xi$, where η and ξ are 1-form and a vector field.

The number $\varphi = (1 + \sqrt{5})/2 = 1,618...$ which is a solution of the equation $x^2 - x - 1 = 0$, represents the golden ratio. Geometrically, Golden ratio implies that if a unit segment is divided into two subsegments, then both the ratio of the entire segment to the major subsegment and the ratio of the major subsegment to the minor subsegment must equal φ. The Golden ratio has been used in many different areas, particularly, in arts and architecture. Being inspired by the Golden ratio, the notion of Golden manifold M was defined in [4] by a tensor field Φ on M satisfying $\Phi^2 = \Phi + I$. The authors studied properties of Golden manifolds and they showed that Φ is an automorphism of the tangent bundle TM and its eigenvalues are φ and $1 - \varphi$. They also defined Golden Riemannian manifolds and investigated

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their submanifolds in [5]. Moreover, the integrability of Golden structures has been investigated in [8].

To compare two manifolds or to study the geometry of a manifold M by using another manifold N, it is useful to define a map F from M to N by imposing certain conditions on F. In this manner, holomorphic maps between two complex manifolds have nice geometric properties. For instance, any holomorphic map between Kaehler manifolds is harmonic [6]. By adapting the notion of holomophic maps, similar maps defined between almost contact manifolds [9] and other manifolds endowed with a geometric structure.

In this paper, we study a new map between Golden Riemannian manifolds by imposing a holomorphic-like condition for the first time as far as we know. We show that such map is a harmonic map, and then we obtain certain conditions for such maps to be totally geodesic. We also provide a simple elementary example. Moreover, we also check the existence of such maps between Golden Riemannian manifolds and another manifold equipped with a differentiable structure (almost complex, almost contact, almost product, almost para-contact) and surprisingly we find that there are no non-constant such maps.

2. Preliminaries

In this section, we give a brief information for almost complex manifolds, almost contact metric manifolds, almost product manifolds, almost para-contact metric manifolds, Golden Riemannian manifolds. We note that throughout this paper all manifolds and bundles, along with sections and connections, are assumed to be of class C^∞. A map is always a C^∞ map between manifolds.

2.1. Almost complex manifolds

Let M' be a $2n-$ dimensional real manifold. An almost complex structure J on M' is a tensor field $J : TM' \to TM'$ such that

$$J^2 = -I, \tag{1}$$

where I is the identity transformation. Then (M', J) is called an almost complex manifold [13].

A smooth map $\varphi : M_1' \to M_2'$ between almost complex manifolds (M_1', J_1) and (M_2', J_2) is called an almost complex (or holomorphic) map if $d\varphi\,(J_1 X) = J_2 d\varphi\,(X)$ for $X \in \Gamma\,(TM_1')\,,$ where J_1 and J_2 are complex structures of M_1' and M_2', respectively.

2.2. Almost contact metric manifolds

An $n-$dimensional differentiable manifold M is said to have an almost contact structure (ϕ, ξ, η) if it carries a tensor field ϕ of type $(1, 1)$, a vector field ξ and $1-$ form η on M, respectively, such that

$$\phi^2 = -I + \eta \otimes \xi, \phi\xi = 0, \eta \circ \phi = 0, \eta\,(\xi) = 1, \tag{2}$$

where I is the identity transformation. The almost contact structure is said to be normal if $N + 2d\eta \otimes \xi = 0$, where N is the Nijenhuis tensor of ϕ. Suppose that a Riemannian metric tensor g is given in M and satisfies the condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi). \tag{3}$$

Then (M, ϕ, ξ, η, g) is called an almost contact metric manifold [3].

2.3. Almost product manifolds

Let N be an $n-$dimensional manifold with a tensor of type $(1,1)$ such that

$$F^2 = I, \tag{4}$$

where I is the identity transformation. Then we say that N is an almost product manifold with almost product structure F. We put

$$Q = \frac{1}{2}(I + F), \quad Q^{'} = \frac{1}{2}(I - F). \tag{5}$$

Then we have

$$Q + Q^{'} = I, \quad Q^2 = Q, \quad Q^{'2} = Q^{'}, \quad QQ^{'} = Q'Q = 0 \tag{6}$$

and

$$F = Q - Q'. \tag{7}$$

If an almost product manifold N admits a Riemannian metric g such that

$$g(FX, FY) = g(X, Y)$$

for any vector fields X and Y on N, then N is called an almost product Riemannian manifold [13].

2.4. Almost para-contact metric manifolds

An $n-$dimensional differentiable manifold N' is said to have an almost para-contact structure (ϕ', ξ', η') if it carries a tensor field ϕ' of type $(1,1)$, a vector field ξ' and a $1-$form η' on N', respectively, such that

$$\phi'^2 = I - \eta' \otimes \xi', \ \eta'(\xi') = 1, \ \phi'\xi' = 0, \quad \eta' \circ \phi' = 0 \tag{8}$$

where I is the identity tranformation. Suppose that a Riemannian metric tensor g' is given in N' and satisfies the condition

$$g'(\phi' X, \phi' Y) = g'(X, Y) - \eta'(X)\eta'(Y) \tag{9}$$

$$g'(X, \xi) = \eta'(X), \quad g'(X, \phi' Y) = -g'(\phi' X, Y) \tag{10}$$

for any vector fields X and Y on N', then $(N', \phi', \xi', \eta', g')$ is called an almost para-contact metric manifold [11, 14].

2.5. Golden Riemannian manifolds

Let $(\bar{M}, g)$ be a Riemannian manifold. A golden structure on $(\bar{M}, g)$ is an $(1,1)$ tensor field P which satisfies the equation

$$P^2 = P + I, \tag{11}$$

where I is the identity transformation. We say that the metric g is P compatible if

$$g(PX, Y) = g(X, PY) \tag{12}$$

for all $X, Y \in \Gamma(T\bar{M})$. If we substitute PX into X in (12), equation (12) may also written as

$$g(PX, PY) = g(P^2X, Y) = g((P+I)X, Y) = g(PX, Y) + g(X, Y).$$

The Riemannian metric (12) is called $P-$compatible and $(\bar{M}, P, g)$ is named a Golden Riemannian manifold [4]. It is known ([4]) that a Golden structure P is integrable if the Nijenhuis tensor N_P vanishes. In [8], the authors show that a Golden structure is integrable if and only if $\nabla P = 0$, where ∇ is the Levi-Civita connection of g.

2.6. Harmonic maps

Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $P : M \to N$ is a smooth mapping between them. Then the differential dP of P can be viewed as a section of the bundle $Hom(TM, P^{-1}TN) \to M$, where $P^{-1}TN$ is the pullback bundle which has fibres $(P^{-1}TN)_p = T_{P(p)}N$, $p \in M$. $Hom(TM, P^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of P is given by

$$\nabla dP(X, Y) = \nabla^P_X dP(Y) - dP(\nabla^M_X Y) \tag{13}$$

for $X, Y \in \Gamma(TM)$. It is known that the second fundamental form is symmetric. A smooth map $P : (M, g_M) \to (N, g_N)$ is said to be harmonic if trace $\nabla dP = 0$. The tension field of P is the section $\tau(P)$ of $\Gamma(P^{-1}TN)$ defined by

$$\tau(P) = \mathrm{div} dP = \sum_{i=1}^{m} \nabla dP(e_i, e_i), \tag{14}$$

where $\{e_1, ..., e_m\}$ is a local orthonormal frame on M. Then it follows that P is harmonic if and only if $\tau(P) = 0$. For more information, see [1].

3. Golden maps between Golden manifolds

In this section, we give a new notion, namely a Golden map, and show that such map is harmonic. We also investigate conditions for a Golden map to be totally geodesic.

Definition 1. *Let φ be a smooth map from a Golden Riemannian manifold (M, P, g) to a Golden Riemannian manifold (N, P', g'). Then φ is called a Golden map if the following condition is satisfied.*

$$d\varphi\, P = P' d\varphi\,. \tag{15}$$

We provide the following elementary example.

Example 1. *Let $\varphi : \mathbb{R}^4 \to \mathbb{R}^2$ be a map defined by*

$$\varphi\,(x_1, x_2, x_3, x_4) = \left(\frac{x_1 + x_2}{4}, \frac{x_3 + x_4}{4}\right).$$

Then, by direct calculations

$$\ker d\varphi\, = span\left\{X_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2},\; X_2 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\right\}$$

and

$$(\ker d\varphi\,)^{\perp} = span\left\{Z_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2},\; Z_2 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right\}.$$

Then considering Golden structures on $\mathbb{R}^4$ and $\mathbb{R}^2$ defined by

$$\Phi\,(a_1, a_2, a_3, a_4) = (\phi a_1, \phi a_2, (1-\phi)\, a_3, (1-\phi)\, a_4)$$

and

$$\Phi'\,(a_1, a_2) = (\phi a_1, (1-\phi)\, a_2)\,,$$

where ϕ and $1-\phi$ are eigenvalues of Golden structures [4]. It is easy to see that $d\varphi\,(PZ_1) = P'd\varphi\,(Z_1)$ and $d\varphi\,(PZ_2) = P'd\varphi\,(Z_2)$. Thus φ is a Golden map.

From now on, when we mention a Golden Riemannian manifold, we will assume that its almost Golden structure is integrable.

Lemma 1. *Let φ be a Golden map from a Golden Riemannian manifold (M, P, g) to a Golden Riemannian manifold (N, P', g') such that $d\varphi\, P = P'd\varphi$ is satisfied. Then we have*

$$(\nabla d\varphi\,)\,(X, PY) = (\nabla d\varphi\,)\,(PX, Y) \tag{16}$$

for $X, Y \in \Gamma\,(TM)$.

Proof. For $X, Y \in \Gamma\,(TM)$, from (13) and (11) we have

$$(\nabla d\varphi\,)\,(X, PY) = \nabla^{\varphi}_{X} d\varphi\, \left(P^2 Y - Y\right) - d\varphi\, \left(\nabla^{M}_{X} PY\right).$$

Then using (15) we get

$$(\nabla d\varphi\,)\,(X, PY) = \nabla^{\varphi}_{X} P'^2 d\varphi\,(Y) - \nabla^{\varphi}_{X} d\varphi\,(Y) - d\varphi\, \left(\nabla^{M}_{X} PY\right).$$

Since P' is integrable, from (11) we obtain

$$(\nabla d\varphi\,)\,(X, PY) = P'\nabla^{\varphi}_{X} d\varphi\,(Y) - d\varphi\, \left(\nabla^{M}_{X} PY\right).$$

Integrable P and (15) also imply

$$(\nabla d\varphi)(X, PY) = P'\left(\nabla^{\varphi}_X d\varphi(Y) - d\varphi\left(\nabla^M_X Y\right)\right).$$

Thus using (13) we arrive at

$$(\nabla d\varphi)(X, PY) = P'(\nabla d\varphi)(X, Y). \tag{17}$$

Since $\nabla d\varphi$ is symmetric, we obtain (16). □

The following theorem shows that a Golden map is a harmonic map.

Theorem 1. *Every Golden map from a Golden Riemannian manifold (M, P, g) to a Golden Riemannian manifold (N, P', g') is harmonic.*

Proof. First, from (16) and (17), we find

$$(\nabla d\varphi)(PX, PY) = P'^2(\nabla d\varphi)(X, Y). \tag{18}$$

Let $\{e_1, e_2, ..., e_n\}$ be a basis of T_pM, $p \in M$. Since P is an isomorphism, from [12] we know that $\{Pe_1, Pe_2, ..., Pe_n\}$ is also a basis of T_pM, $p \in M$. Then from (11) and (18), we have

$$\begin{aligned}\sum_{i=1}^{n}(\nabla d\varphi)(Pe_i, Pe_i) &= \sum_{i=1}^{n} P'(\nabla d\varphi)(e_i, e_i) + \sum_{i=1}^{n}(\nabla d\varphi)(e_i, e_i)\\ \tau(\varphi) &= P'\tau(\varphi) + \tau(\varphi),\end{aligned}$$

which gives

$$P'\tau(\varphi) = 0.$$

Since P' is an isomorphism on N, we get

$$\tau(\varphi) = 0,$$

which completes the proof. □

Remark 1. *We note that for any C^2 real valued function f defined on an open subset of a Riemannian manifold M, the equation $\triangle f = 0$ is called Laplace's equation and solutions are called harmonic functions on U. Let $F : M \longrightarrow N$ be a smooth map between Riemannian manifolds. Then F is called a harmonic morphism if, for every harmonic function $f : V \longrightarrow \mathbf{R}$ defined an open subset V of N with $F^{-1}(V)$ non-empty, the composition $f \circ F$ is harmonic on $F^{-1}(V)$. A smooth map $F : M \longrightarrow N$ between Riemannian manifolds is a harmonic morphism if and only if F is both harmonic and horizontally weakly conformal [7] and [10]. In this respect, a Golden map is a good candidate for a harmonic morphism.*

We now give a necessary and sufficient condition for a map φ to be totally geodesic. We recall that a map φ is totally geodesic if $\nabla d\varphi = 0$. A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total manifold into a geodesic in the base manifold in proportion to arc lengths.

Theorem 2. *Let φ be Golden map from a Golden Riemannian manifold (M, P, g) to a Golden Riemannian manifold (N, P', g'). Then φ is totally geodesic if and only if*

$$(\nabla d\varphi)(PX, PY) = P'(\nabla d\varphi)(X, Y), \tag{19}$$

for $X, Y \in \Gamma(TM)$.

Proof. It is obvious from (18). □

4. Constancy of some maps from Golden Riemannian manifolds

In this section we investigate constancy of certain maps from Golden Riemannian manifolds or to Golden manifolds by imposing holomorphic-like conditions. We first check the situation for a map between Golden Riemannian manifolds and almost complex manifolds.

Theorem 3. *Let φ be a smooth map from a Golden Riemannian manifold $\left(\bar{M}, P, g\right)$ to an almost complex manifold (M', J) such that the condition $d\varphi P = Jd\varphi$ is satisfied. Then φ is a constant map.*

Proof. Let $\left(\bar{M}, P, g\right)$ be a Golden Riemannian manifold and (M', J) an almost complex manifold. Suppose that $\varphi : \bar{M} \to M'$ satisfies

$$d\varphi(PX) = Jd\varphi(X), \quad X \in \Gamma\left(T\bar{M}\right). \tag{20}$$

Then apply J to the above equation and using (1) and (11), we get

$$d\varphi(PX) + d\varphi(X) = -d\varphi(X), \quad X \in \Gamma\left(T\bar{M}\right). \tag{21}$$

Applying J to (21) again and using (20), we have

$$d\varphi(PX) + d\varphi(X) + Jd\varphi(X) = -Jd\varphi(X), \quad X \in \Gamma\left(T\bar{M}\right). \tag{22}$$

Then (20) implies that

$$-3d\varphi(PX) = d\varphi(X), \quad X \in \Gamma\left(T\bar{M}\right). \tag{23}$$

From (21) and (23) we obtain

$$d\varphi(X) = 0,$$

which shows that φ is constant. □

In a similar way, we have the following result.

Theorem 4. *Let φ be a smooth map from an almost complex manifold (M', J) to a Golden Riemannian manifold $\left(\bar{M}, P, g\right)$ such that the condition $d\varphi J = Pd\varphi$ is satisfied. Then φ is a constant map.*

Remark 2. *Besides holomorphic maps, anti-holomorphic maps have also been studied by many authors under the condition $J'dF = -dFJ$, where F is a map between almost complex manifolds and J, J' are almost complex structures, respectively. By following similar computations above, one can find that the notion of anti-holomorphic maps does not work for Golden manifolds.*

The following result shows that a smooth map satisfying a compatible condition between Golden Riemannian manifolds and almost contact metric manifolds is also constant.

Theorem 5. *Let φ be a smooth map from a Golden Riemannian manifold $\left(\bar{M}, P, g\right)$ to an almost contact metric manifold (M, ϕ, ξ, η, g) such that the condition $d\varphi P = \phi d\varphi$ is satisfied. Then φ is a constant map.*

Proof. Let $\left(\bar{M}, P, g\right)$ be a Golden Riemannian manifold and (M, ϕ, ξ, η, g) an almost contact metric manifold. Suppose that $\varphi : \bar{M} \to M$ satisfies

$$d\varphi\left(PX\right) = \phi d\varphi\left(X\right), \quad X \in \Gamma\left(T\bar{M}\right). \tag{24}$$

Then apply ϕ to the above equation and using (2) and (11), we get

$$d\varphi\left(PX\right) = -2d\varphi\left(X\right) + \eta\left(d\varphi\left(X\right)\right)\xi, \quad X \in \Gamma\left(T\bar{M}\right). \tag{25}$$

Then applying ϕ to (25) again and using (24) and (2), we have

$$-3d\varphi\left(PX\right) = d\varphi\left(X\right). \tag{26}$$

From (25) and (26), we obtain

$$-5d\varphi\left(PX\right) = \eta\left(d\varphi\left(X\right)\right)\xi, \quad X \in \Gamma\left(T\bar{M}\right). \tag{27}$$

Again applying ϕ to (27), we get

$$\phi d\varphi\left(X\right) = 0, \quad X \in \Gamma\left(T\bar{M}\right). \tag{28}$$

Using (28) in (27) we conclude that

$$\eta\left(d\varphi\left(X\right)\right) = 0, \quad X \in \Gamma\left(T\bar{M}\right). \tag{29}$$

Then applying φ to (29) we get

$$-d\varphi\left(X\right) + \eta\left(d\varphi\left(X\right)\right)\xi = 0$$

which gives

$$d\varphi\left(X\right) = 0.$$

This completes the proof. □

In a similar way, we have the following result.

Theorem 6. *Let φ be a smooth map from an almost contact metric manifold (M, ϕ, ξ, η, g) to a Golden Riemannian manifold $\left(\bar{M}, P, g\right)$ such that the condition $d\varphi\phi = Pd\varphi$ is satisfied. Then φ is a constant map.*

We now check a similar situation for a map between Golden Riemannian manifolds and almost product manifolds.

Theorem 7. *Let φ be a smooth map from a Golden Riemannian manifold $\left(\bar{M}, P, g\right)$ to an almost product manifold (N, F) such that the condition $d\varphi P = F d\varphi$ is satisfied. Then φ is a constant map.*

Proof. Let $\left(\bar{M}, P, g\right)$ be a Golden Riemannian manifold and (N, F) an almost product manifold. Suppose that $\varphi : \bar{M} \to N$ satisfies

$$d\varphi\left(PX\right) = F d\varphi\left(X\right), \quad X \in \Gamma\left(T\bar{M}\right). \tag{30}$$

Then apply F to the above equation and using (4) and (11), we get

$$d\varphi\left(PX\right) = 0, \quad X \in \Gamma\left(T\bar{M}\right). \tag{31}$$

Applying F to (31) again and using (30), we have

$$d\varphi\left(PX\right) = -d\varphi(X), \quad X \in \Gamma\left(T\bar{M}\right). \tag{32}$$

From (32) and (31) we obtain

$$d\varphi\left(X\right) = 0,$$

which shows that φ is constant. □

In a similar way, we have the following result.

Theorem 8. *Let φ be a smooth map from an almost product manifold (N, F) to a Golden Riemannian manifold $\left(\bar{M}, P, g\right)$ such that the condition $d\varphi F = P d\varphi$ is satisfied. Then φ is a constant map.*

Finally, we check the same problem for almost para-contact metric manifolds.

Theorem 9. *Let φ be a smooth map from a Golden Riemannian manifold $\left(\bar{M}, P, g\right)$ to an almost para-contact metric manifold $(N', \phi', \xi', \eta', g')$ such that the condition $d\varphi P = \phi' d\varphi$ is satisfied. Then φ is a constant map.*

Proof. Let $\left(\bar{M}, P, g\right)$ be a Golden Riemannian manifold and $(N', \phi', \xi', \eta', g')$ an almost para-contact metric manifold. Suppose that $\varphi : \bar{M} \to N'$ satisfies

$$d\varphi\left(PX\right) = \phi' d\varphi\left(X\right), \quad X \in \Gamma\left(T\bar{M}\right). \tag{33}$$

Then apply ϕ' to the above equation and using (8) and (11), we get

$$d\varphi\left(PX\right) + d\varphi\left(X\right) = d\varphi\left(X\right) - \eta'\left(d\varphi\left(X\right)\right)\xi', \quad X \in \Gamma\left(T\bar{M}\right). \tag{34}$$

Then applying ϕ' to (34) again and using (33) and (8), we have

$$d\varphi\left(PX\right) = -d\varphi\left(X\right). \tag{35}$$

From (34) and (35), we obtain

$$d\varphi\left(X\right) = \eta'\left(d\varphi\left(X\right)\right)\xi', \quad X \in \Gamma\left(T\bar{M}\right). \tag{36}$$

Again applying ϕ' to (36), we get

$$\phi' d\varphi\left(X\right) = 0, \quad X \in \Gamma\left(T\bar{M}\right). \tag{37}$$

From (37) we have $d\varphi\left(X\right) = 0$, which shows that φ is constant. □

In a similar way, we have the following result.

Theorem 10. *Let φ be a smooth map from an almost para-contact metric manifold $(N', \phi', \xi', \eta', g')$ to a Golden Riemannian manifold $\left(\bar{M}, P, g\right)$ such that the condition $d\varphi\phi' = Pd\varphi$ is satisfied. Then φ is a constant map.*

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