MATHEMATICAL COMMUNICATIONS
Math. Commun. **19**(2014), 363–373

A characterization of linear operators that preserve isolation numbers

LeRoy B. Beasley[1] and Seok Zun Song[2,*]

[1] *Department of Mathematics and Statistics, Utah State University, Logan, Utah 84 322-3 900, USA*
[2] *Department of Mathematics, Jeju National University, Jeju 690-756, Korea*

Received May 2, 2013; accepted September 13, 2014

Abstract. We obtain characterizations of Boolean linear operators that preserve some of the isolation numbers of Boolean matrices. In particular, we show that the following are equivalent: (1) T preserves the isolation number of all matrices; (2) T preserves the set of matrices with isolation number one and the set of those with isolation number k for some $2 \leq k \leq \min\{m, n\}$; (3) for $1 \leq k \leq \min\{m, n\} - 1$, T preserves matrices with isolation number k, and those with isolation number $k+1$, (4) T maps J to J and preserves the set of matrices of isolation number 2; (5) T is a (P, Q)-operator, that is, for fixed permutation matrices P and Q, $m \times n$ matrix X, $T(X) = PXQ$ or, $m = n$ and $T(X) = PX^tQ$ where X^t is the transpose of X.

AMS subject classifications: 15A86, 15A04

Key words: Boolean matrix, Boolean rank, isolation number, Boolean linear operator

1. Introduction

The characterization of linear operators on vector spaces of matrices which leave functions, sets or relations invariant began over a century ago when Fröbenius [10], in 1897, characterized the linear operators that leave the determinant function invariant. Since then, several researchers have investigated the preservers of nearly every function, set and relation on matrices over fields. See [15, 17] for an excellent survey of preserver problems through 2001. For applications of linear preservers, see [14].

In the 1980's, research began on linear preserver problems over semirings, in particular linear operators on spaces of $(0, 1)$-matrices. (See for example [3].) Many functions, sets and relations concerning matrices do not depend upon the magnitude or nature of the individual entries of a matrix, but rather only on whether the entry is zero or nonzero. These combinatorially significant matrices have become increasingly important in recent years. The Boolean rank is of primary interest. Finding the Boolean rank of a $(0, 1)$-matrix is an NP-Complete problem, (See [16] noting that the Boolean rank is also called the Schein rank), and consequently finding bounds on the Boolean rank of a matrix is of interest to those researchers that

*Corresponding author. *Email addresses:* `leroy.b.beasley@usu.edu` (L. B. Beasley), `szsong@jejunu.ac.kr` (S. Z. Song)

 ©2014 Department of Mathematics, University of Osijek

would use the Boolean rank in their work. The isolation number of a $(0,1)$-matrix is the largest number of entries equal to 1, no two of which are in the same row, no two of which are in the same column, and no two of which are in a submatrix of all ones. If the $(0,1)$-matrix is the reduced adjacency matrix of a bipartite graph, the isolation number of the matrix is the maximum size of a non-competitive matching in the bipartite graph. This is related to the study of such combinatorial problems as the patient hospital problem, the stable marriage problem, etc. See `http://en.wikipedia.org/wiki/Stable_marriage_problem`. An additional reason for studying the isolation number is that it is a lower bound on the Boolean rank of a $(0,1)$-matrix, (see [12]). While finding the isolation number as well as finding the Boolean rank of a $(0,1)$-matrix is an NP-Complete problem ([1]), for some matrices finding the isolation number can be easier than finding the Boolean rank especially if the matrix is sparse:

Example 1. *Let*

$$\mathcal{X} = \begin{bmatrix} 1&1&1&\mathbf{1}&0&1&1&1&1 \\ 1&1&1&1&\mathbf{1}&1&1&1&1 \\ 1&1&1&0&0&\mathbf{1}&1&1&1 \\ 1&1&1&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0 \\ 1&1&1&0&0&0&0&0&0 \\ \mathbf{1}&1&0&0&0&0&0&0&0 \\ 0&\mathbf{1}&1&0&0&0&0&0&0 \\ 1&0&\mathbf{1}&0&0&0&0&0&0 \end{bmatrix}.$$

The Boolean rank of $\mathcal{X}$ is easily seen to be at most 6; however, to find that it is not 5 requires much calculation if the isolation number is not considered. However, the isolation number is easily seen to be 6, both computationally and visually, the bold ones in the matrix represent a set of isolated ones. Thus the Boolean rank is 6.

Note that if any of the non-bold ones are replaced by zeros, the resulting matrix still has Boolean rank 6 as well as isolation number 6.

Terms not specifically defined here can be found in Brualdi and Ryser [8] for matrix terms, or Bondy and Murty [6] for graph theoretic terms.

2. Definitions and preliminary results

A *semiring* $\mathbb{S}$ is a set with two binary operations, addition $(+)$ and multiplication $(\cdot)$. There is a zero element and an identity element (for multiplication) in $\mathbb{S}$. That is, $(\mathbb{S},+)$ is closed, commutative and associative, but may not have additive inverses, except for the zero. $(\mathbb{S},\cdot)$ is closed, associative and commutative, but may not have multiplicative inverses, except for the identity. Further, the distributive laws hold, and we will only consider semirings which have no zero divisors, that is, nonzero elements, s, for which there is some nonzero element t in $\mathbb{S}$ such that $st = 0$. Of particular interest in this article is the *binary Boolean algebra*, $\mathbb{B} = \{0,1\}$ with the usual addition and multiplication, except that $1+1=1$. A semiring is *antinegative* if the only element with an additive inverse is the zero element.

In this article, we only consider matrices over the antinegative semiring $\mathbb{B}$.

Let $\mathcal{M}_{m,n}(\mathbb{B})$ be the set of all $m \times n$ matrices with entries in the binary Boolean algebra $\mathbb{B}$, and let $\mathcal{M}_m(\mathbb{B}) = \mathcal{M}_{m,m}(\mathbb{B})$. The usual definitions for adding and multiplying matrices apply to Boolean matrices as well.

The matrix $A^{(m,n)}$ denotes a matrix in $\mathcal{M}_{m,n}(\mathbb{B})$, I_n is an $n \times n$ identity matrix, $O_{m,n}$ is an $m \times n$ zero matrix, and $J_{m,n}$ is an $m \times n$ matrix all of whose entries are 1. Let $E_{i,j}^{(m,n)}$ be an $m \times n$ matrix whose (i,j)th entry is 1 and whose other entries are all 0, and we call $E_{i,j}^{(m,n)}$ a *cell*. We will suppress the superscripts and/or subscripts on these matrices when the orders are evident from the context and we write A, I, O, J, and E_{ij}, respectively. Further, we let the set of all cells be denoted $\mathcal{E}$. That is,

$$\mathcal{E} = \{E_{i,j} \in \mathcal{M}_{m,n}(\mathbb{B}) \mid i = 1, \ldots, m \text{ and } j = 1, \ldots, n\}.$$

The *Boolean rank*, $\beta(A)$, of a nonzero Boolean matrix A in $\mathcal{M}_{m,n}(\mathbb{B})$ is the minimal number k such that there exist $m \times k$ and $k \times n$ Boolean matrices $B \in \mathcal{M}_{m,k}(\mathbb{B})$ and $C \in \mathcal{M}_{k,n}(\mathbb{B})$ such that $A = BC$. The Boolean rank of the zero matrix is 0. It is well known that $\beta(A)$ is the least k such that A is the sum of k matrices of Boolean rank 1 ([3]). By observing that any matrix is a sum of its rows (or columns), which are rank one matrices, it follows that $1 \leq \beta(A) \leq m$ for all nonzero $A \in \mathcal{M}_{m,n}(\mathbb{B})$.

From now on we will assume that $2 \leq m \leq n$.

The Boolean rank of a zero-one matrix is equal to the biclique covering number of a bipartite graph, and hence has applications in graph theory. The Boolean rank has also been used in other applications as well, for example in investigating the exponent of primitive matrices, see [13].

By considering a minimal sum of rank one matrices for A and B such as $A = A_1 + \cdots + A_k$, and $B = B_1 + \cdots + B_l$, we have that $A + B = A_1 + \cdots + A_k + B_1 + \cdots + B_l$, so that $A + B$ has a rank at most $k + l$. This establishes the following lemma.

Lemma 1. *For matrices A and B in $\mathcal{M}_{m,n}(\mathbb{B})$, we have $\beta(A + B) \leq \beta(A) + \beta(B)$.*

If A and B are matrices in $\mathcal{M}_{m,n}(\mathbb{B})$, we say that B *dominates* A (written $A \leq B$ or $B \geq A$) if $b_{i,j} = 0$ implies $a_{i,j} = 0$ for all i and j. Equivalently, $A \leq B$ if and only if $A + B = B$. This provides a reflexive and transitive relation on $\mathcal{M}_{m,n}(\mathbb{B})$. Let A be a matrix. We let $|A|$ denote the number of nonzero entries in that matrix. So, $|I_n| = n$. We call $|A|$ the *weight* of A.

A set of indices, $\mathcal{I}$, of A is called a set of *isolated ones* of A if (1) $(i,j) \in \mathcal{I}$ implies $a_{i,j} = 1$; (2) whenever $(i,j), (k,l) \in \mathcal{I}$ we have that $i \neq k$, $j \neq l$, and the submatrix on rows i and k and on columns j and l is not $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. The isolation number was first defined and used by Gregory and Pullman in 1983 ([11]). They were studying Boolean and nonnegative factorizations of matrices, especially prime matrices.

The *isolation number* of A, $\iota(A)$, is the cardinality of a largest set of isolated ones of A. In [2], it was shown that $\iota(A) = 1$ if and only if $\beta(A) = 1$, and $\iota(A) = 2$ if and only if $\beta(A) = 2$.

Since any two entries of A corresponding to distinct isolated ones can lie in no single Boolean rank one submatrix, we have:

Lemma 2. *Let $A \in \mathcal{M}_{m,n}(\mathbb{B})$. Then $\iota(A) \leq \beta(A)$.*

However, as the following example illustrates, unless $i(A) = 1, 2,$ or m, the isolation number may be significantly less than the Boolean rank of a matrix.

Example 2. *Let $n \geq 3$ and let $D_n \in \mathcal{M}_n(\mathbb{B})$ be the matrix $J \setminus I$, that is, D_n is the matrix all of whose entries are ones except that all diagonal entries are zero. Then, $\iota(D_n) = 3$, for any $n \geq 3$. However, $\beta(A) = k$ where $k = \min \left\{ k \mid n \leq \binom{k}{\lceil \frac{k}{2} \rceil} \right\}$* (see [9]). *So $\iota(D_{20}) = 3$ while $\beta(D_{20}) = 6$.*

Let $A, B \in \mathcal{M}_{m,n}(\mathbb{B})$. Since those indices in any set of isolated ones for $A + B$ whose corresponding entries in A (or B) are ones is a set of isolated ones for A (or B, resp.), it follows that:

Lemma 3. *For matrices A and B in $\mathcal{M}_{m,n}(\mathbb{B})$, we have $\iota(A + B) \leq \iota(A) + \iota(B)$.*

In [2] the structure of a matrix $A \in \mathcal{M}_{m,n}(\mathbb{B})$ whose isolation number is m was provided. Thus, the set of matrices with isolation number m can be systematically described, while the authors are not aware of any systematic description of the set of matrices of Boolean rank m.

A mapping $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ is called a *Boolean linear operator* if for any $X, Y \in \mathcal{M}_{m,n}(\mathbb{B})$, $T(X + Y) = T(X) + T(Y)$, and $T(O) = O$.

Let $f : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathbb{S}$ be a mapping where $\mathbb{S}$ is any set. Let S be a subset of $\mathcal{M}_{m,n}(\mathbb{B})$.

For a Boolean linear operator $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ we say that T

(1) *preserves* f if for any $k \in \mathbb{S}$, $f(T(X)) = k$ whenever $f(X) = k$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B})$;

(2) *strongly preserves* f if for any $k \in \mathbb{S}$, $f(T(X)) = k$ if and only if $f(X) = k$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B})$;

and for $\mathbb{S} = \mathcal{M}_{m,n}(\mathbb{B})$,

(3) *preserves* S if $T(X) \in S$ whenever $X \in S$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B})$;

(4) *strongly preserves* S if $T(X) \in S$ if and only if $X \in S$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B})$.

By an abuse of language we use the expression "T preserves Boolean rank k (isolation number k)" to mean "T preserves the set of matrices of Boolean rank k (isolation number k)".

A Boolean linear operator $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ is called a (P, Q)-*operator* if there are permutation matrices $P \in \mathcal{M}_m(\mathbb{B})$ and $Q \in \mathcal{M}_n(\mathbb{B})$ such that $T(X) = PXQ$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B})$, or when $m = n$, $T(X) = PX^tQ$ for all $X \in \mathcal{M}_m(\mathbb{B})$, where X^t is the transpose of X.

In [3, Theorem 5.3], it was shown that if $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ preserves Boolean ranks one and two, then T is a (P, Q)-operator. Since $\iota(A) = 1$ if and only if $\beta(A) = 1$, and $\iota(A) = 2$ if and only if $\beta(A) = 2$, we have:

Theorem 1. *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator. Then the following are equivalent:*

1. T preserves the isolation number of matrices,

2. T preserves isolation numbers one and two,

3. T is a (P,Q)-operator.

3. Preservers of isolation number two with $T(J)=J$

Lemma 4. *Let $T:\mathcal{M}_{m,n}(\mathbb{B})\to\mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator with $m,n\geq 3$. If T preserves isolation number 2 and $T(J)=J$, then T is bijective on the set of cells (hence, bijective on $\mathcal{M}_{m,n}(\mathbb{B})$).*

Proof. Suppose that E is a cell and $T(E)=O$. Then $\iota(J-E)=2$. But, $T(J-E)=T(J)=J$, contradicting that T preserves isolation number 2.

Suppose that for some cell E, $|T(E)|>1$. Then there is some cell F such that $T(J)=T(J-F)$; however, as above, $\iota(J-F)=2$. But, $T(J-F)=T(J)=J$, again a contradiction. Thus, the image of a cell is a cell.

Suppose that for some cells E and F, $T(E)=T(F)$. Let $N=J-(E+F)$. Then $J=T(J)=T(N+E+F)=T(N)+T(E)+T(F)=T(N)+T(E)=T(N+E)$ since $T(E)=T(F)$. But, since $N+E=J-F$, that says that $T(J-F)=T(J)=J$ and as above we get a contradiction since $\iota(J-F)=2$ and $\iota(J)=1$.

Thus, T is bijective on the set of cells. □

A matrix $L\in\mathcal{M}_{m,n}(\mathbb{B})$ is called a *line matrix* if $L=\sum\limits_{l=1}^{n}E_{i,l}$ or $L=\sum\limits_{s=1}^{m}E_{s,j}$ for some $i\in\{1,\ldots,n\}$ or for some $j\in\{1,\ldots,n\}$; $R_i=\sum\limits_{l=1}^{n}E_{i,l}$ is the ith *row matrix* and $C_j=\sum\limits_{s=1}^{m}E_{s,j}$ is the jth *column matrix.* A matrix in $\mathcal{M}_{m,n}(\mathbb{B})$ is a *double star* if it is a sum of a row matrix and a column matrix which share a diagonal entry. That is, $D_k=R_k+C_k$ is a double star for all $k=1,\cdots,n$. Two cells are called *collinear* if they are dominated by a line matrix.

Lemma 5. *Let $T:\mathcal{M}_{m,n}(\mathbb{B})\to\mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator with $m,n\geq 3$. If T preserves isolation number 2 and T is bijective on the set of cells, then T maps lines to lines.*

Proof. Suppose that the image of a line is not a line. Then, there are two collinear cells whose images are not collinear. Without loss of generality, we may assume that $T(E_{1,1}+E_{1,2})=E_{1,1}+E_{2,2}$.

Let $\mathcal{S}_2$ be the set of all matrices of weight 2 in $\mathcal{M}_{m,n}(\mathbb{B})$. That is, $A\in\mathcal{S}_2$ if and only if there are distinct cells E and F such that $E+F=A$. Thus, $T(\mathcal{S}_2)=\mathcal{S}_2$ since T is bijective. If $\mathcal{G}$ is the subset of $\mathcal{S}_2$ consisting of all those members with isolation number 2, then $T(\mathcal{G})=\mathcal{G}$. Since T is bijective, $T(\mathcal{S}_2\setminus\mathcal{G})=\mathcal{S}_2\setminus\mathcal{G}$. Since $T(E_{1,1}+E_{1,2})=E_{1,1}+E_{2,2}$, $E_{1,1}+E_{1,2}\notin\mathcal{G}$ and $E_{1,1}+E_{2,2}\in\mathcal{G}$, we have a contradiction. Thus, T maps lines to lines. □

Lemma 6. *Let $T:\mathcal{M}_{m,n}(\mathbb{B})\to\mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator. Then, T is bijective and maps lines to lines if and only if T is a (P,Q)-operator.*

Proof. The sufficiency is obvious. For the necessity, let $\mathcal{L} = \{R_i | 1 \leq i \leq m\} \cup \{C_j | 1 \leq j \leq n\}$. Then, since T is bijective, T is bijective on $\mathcal{L}$.

If $m \neq n$, since T is bijective, the image of each R_i must be some R_k, and the image of each C_j must be some C_l. This is easily seen by a counting argument.

If $m = n$ and the image of one row is a row and the image of another row is a column, say without loss of generality that $T(R_1) = R_1$ and $T(R_2) = C_1$. Then, $|R_1 + R_2| = 2n$ while $|R_1 + C_1| = 2n - 1$, an impossibility since T is bijective.

Thus for $m = n$, either the image of every row is a row and hence the image of every column is a column since T is bijective on $\mathcal{L}$. If the image of every row is a column and the image of every column is a row, composing T with the transpose operator gives an operator that maps rows to rows and columns to columns.

In both cases, letting σ be a permutation such that $T(R_i) = R_{\sigma(i)}$ and τ be a permutation such that $T(C_j) = C_{\tau(j)}$, we have that T is a (P,Q)-operator where P is the permutation matrix corresponding to σ and Q is the permutation matrix corresponding to τ. $\square$

Theorem 2. *Let $m, n \geq 3$ and $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator. Then, T preserves isolation number 2 and $T(J) = J$ if and only if T is a (P,Q)-operator.*

Proof. By Lemma 4 we have that T is bijective. By Lemma 5, T maps lines to lines. By Lemma 6 we have that T is a (P,Q)-operator. Conversely, clearly every (P,Q)-operator preserves isolation number 2 and $T(J) = J$. $\square$

4. Preservers of isolation numbers one and k

Throughout this section we will use without reference the fact that $\beta(A) = 1$ if and only if $\iota(A) = 1$. So if T preserves isolation number one, T preserves Boolean rank one.

In this section, we provide characterizations of Boolean linear operators $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ that preserve isolation numbers 1 and k, where $1 < k \leq m \leq n$.

Lemma 7. *Let E be a cell $E \in \mathcal{M}_{m,n}(\mathbb{B})$, and Z a matrix such that $E \leq Z$ and let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator. If $|T(Z)| \leq |Z|$ and $|T(E)| \geq 2$, then there exists a cell $F \in \mathcal{M}_{m,n}(\mathbb{B})$ such that $T(Z \setminus F) = T(Z)$.*

Proof. Suppose that $E = E_1$ and Z is a matrix such that $|T(Z)| \leq |Z|$. Further, suppose that $E_1 \leq Z$ and $|T(E_1)| > 1$. If $T(E_1) \neq T(Z)$, there is some cell $E_2 \leq Z$ such that $|T(E_1 + E_2)| > |T(E_1)|$. Continuing in this manner, if possible, we find cells $E_1, E_2, \cdots, E_i$ such that $E_1 + E_2 + \cdots + E_i \leq Z$ and $|T(E_1 + E_2 + \cdots + E_i)| > |T(E_1 + E_2 + \cdots + E_{i-1})|$. Since $|Z|$ and $|T(Z)|$ are finite, there exists some $j < |T(Z)|$ such that $T(E_1 + E_2 + \cdots + E_j) = T(Z)$. It now follows that there is some cell $F \leq Z$ such that $T(Z \setminus F) = T(Z)$. $\square$

Let $\mathcal{N}_k$ be the set of all Boolean rank one matrices in $\mathcal{M}_{m,n}(\mathbb{B})$ which are dominated by a matrix whose isolation number is k. Suppose that w is the largest weight of any matrix in $\mathcal{N}_k$. Let $\mathcal{N}_k^+$ be the set of all elements of $\mathcal{N}_k$ that are of weight

w. Since $X \in \mathcal{N}_k^+$ implies $PXQ \in \mathcal{N}_k^+$ for any permutation matrices, P and Q of appropriate orders, the following is easily seen.

Lemma 8. *Let E be a cell in $\mathcal{M}_{m,n}(\mathbb{B})$. Then there is an element of $\mathcal{N}_k^+$ dominating E.*

Lemma 9. *If T preserves isolation number one, then $\iota(T(A)) \le \beta(A)$ for all $A \in \mathcal{M}_{m,n}(\mathbb{B})$.*

Proof. Let $A \in \mathcal{M}_{m,n}(\mathbb{B})$ and suppose $\beta(A) = l$. Then $A = A_1 + A_2 + \cdots + A_l$, where $\beta(A_i) = \iota(A_i) = 1$. Then, using Lemma 2, $\iota(T(A)) \le \beta(T(A)) = \beta(T(\sum_{i=1}^{l} A_i)) \le \sum_{i=1}^{l} \beta(T(A_i)) = \sum_{i=1}^{l} 1 = l$. That is, $\iota(T(A)) \le \beta(A)$. □

An operator $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ is *singular* if $T(X) = O$ for some nonzero $X \in \mathcal{M}_{m,n}(\mathbb{B})$; otherwise T is *nonsingular.* Notice that if T is a (P,Q)-operator, then T is nonsingular. Further, due to the nature of Boolean operators, we note that a nonsingular operator need not be invertible, unlike the field case.

Lemma 10. *If $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ is a Boolean linear operator which preserves isolation numbers 1 and k for some $1 < k \le m$, then T maps cells to cells.*

Proof. If $k = m$, then T preserves Boolean ranks 1 and m and by [4, Theorem 3.5] T is a (P,Q)-operator and hence it maps cells to cells. Thus, assume that $k < m$.

Since T preserves isolation number 1, T is nonsingular. Suppose that the image of some cell dominates two or more cells. Say, $E = E_1$ is such a cell and $|T(E_1)| > 1$. By Lemma 8, there is $Z \in \mathcal{N}_k^+$ that dominates E_1. That is, $E_1 \le Z$ and $|T(E_1)| > 1$. Since Z is of isolation number one and T preserves both isolation number one and isolation number k, $T(Z) \in \mathcal{N}_k$. Thus, $|T(Z)| \le |Z|$ since $Z \in \mathcal{N}_k^+$. By Lemma 7, there is some cell $F \le Z$ such that $T(Z \setminus F) = T(Z)$. Without loss of generality, we may assume that $F = E_{1,1}$ and that $Z = \begin{bmatrix} J_{p,q} & O \\ O & O \end{bmatrix}$.

If $q = n$, then we must have $p = m - k + 1$ (otherwise we could add a row of ones to Z and still be in $\mathcal{N}_k$). For $A = \begin{bmatrix} O & O \\ O & I_{k-1} \end{bmatrix}$, $A + Z$ is of isolation number k and dominates Z. Let $B = (A + Z) \setminus (E_{1,1} + E_{m,n})$. Then $\iota(B) = k$, while $\iota(B + E_{1,1}) = k - 1$. Also, since $\beta(B + E_{1,1}) = k - 1$ and , since T preserves Boolean rank one, $\beta(T(B+E_{1,1})) \le k-1$. Thus, by Lemma 9, $\iota(T(B+E_{1,1})) \le k-1$. Further, $\iota(T(B)) = k$, since T preserves isolation number k. But $T(B) = T(B + E_{1,1})$, a contradiction. Thus, the image of a cell is a cell.

If $p = m$, a similar argument shows that T maps cells to cells.

Now, assume that $p < m$ and $q < n$. Since $Z \in \mathcal{N}_k^+$, we must have that $(m - p) + (n - q) \ge k - 1$. Let $s = m - p$ and $t = n - q$. Let

$$B = \begin{bmatrix} O_{p-t+1,q+1-s} & O_{p-t+1,s} & O_{p-t+1,t-1} \\ O_{t-1,q+1-s} & O_{t-1,s} & I_{t-1} \\ O_{s,q+1-s} & I_s & O_{s,t-1} \end{bmatrix}.$$

Then $\iota(Z + B) = k - 1$ and $\iota((Z \setminus E_{1,1}) + B) = k$, a contradiction since $T(Z + B) = T((Z \setminus E_{1,1}) + B)$ and cannot have isolation number both k and something strictly less than k. □

Lemma 11. *If $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ is a Boolean linear operator which preserves isolation numbers 1 and k for some $1 < k \leq m$, then T is a bijection on $\mathcal{E}$ and hence invertible on $\mathcal{M}_{m,n}(\mathbb{B})$.*

Proof. We only need to show that T is injective on $\mathcal{E}$. By Lemma 10, the image of a cell is a cell. Suppose that T is not bijective on the set of cells. Then, without loss of generality, we may assume that $T(E_{1,1}) = T(E_{i,j})$ and $i \leq 2$. But then, for $Z = \begin{bmatrix} J_{m-k+2,n} \\ O_{k-2,n} \end{bmatrix}$ and $A = \begin{bmatrix} O & O \\ O & I_{k-2} \end{bmatrix}$, let $X = Z + A$, and $Y = (Z \setminus E_{1,1}) + A$. Then $\iota(X) = \beta(X) = k - 1$ while $\iota(Y) = k$ and $T(X) = T(Y)$. Since T preserves isolation number k, $i(T(Y)) = k$, and by Lemma 9, $\iota(T(X)) \leq \beta(X) = k - 1$. That is, $i(T(X)) \leq k - 1 < k = i(T(Y)) = i(T(X))$, a contradiction. Thus T is bijective on the set of cells. □

From Theorem 3.1 in [3] we have:

Theorem 3 (See [3]). *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator; then T preserves Boolean rank 1 and is invertible if and only if T is a (P, Q)-operator.*

Since a matrix has Boolean rank one if and only if it has isolation number one, we have:

Corollary 1. *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator; then T preserves isolation number 1 and is invertible if and only if T is a (P, Q)-operator.*

Theorem 4. *For a Boolean linear operator $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$, the following are equivalent:*

(1) T preserves the isolation number;

(2) T preserves isolation numbers 1 and k for some $1 < k \leq m$;

(3) T is a (P, Q)-operator.

Proof. If $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ preserves isolation numbers 1 and k for some $1 < k \leq m$, then T is a bijection on $\mathcal{E}$ and hence invertible on $\mathcal{M}_{m,n}(\mathbb{B})$ by Lemma 11. Thus T is a (P, Q)-operator by Corollary 1.

The other implications are obvious. □

5. Preservers of adjacent isolation numbers

A *tournament matrix* is a square matrix $M \in \mathcal{M}_n(\mathbb{B})$ such that $M + M^t = J - I$, that is, a $(0, 1)$-matrix M, such that $m_{i,i} = 0$ and for $i \neq j$, $m_{i,j} = 1$ if and only if $m_{j,i} = 0$. It may be noted that if M is an $r \times r$ tournament matrix and $A = \begin{bmatrix} I_r + M & J_{r,n-r} \\ J_{m-r,r} & J_{m-r,n-r} \end{bmatrix}$, then $\iota(A) = r$ and, if $B > A$, then $\iota(B) < \iota(A)$.

Lemma 12. *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator. If $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ preserves isolation numbers k and $k + 1$ for some $1 \leq k \leq m - 1$, then T is a bijection on $\mathcal{E}$.*

Proof. By virtue of Theorem 1, we may assume that $k \geq 2$.

Suppose that $T(X) = O$ for some nonzero $X \in \mathcal{M}_{m,n}(\mathbb{B})$. Then, for some $E_{i,j}$, $T(E_{i,j}) = O$. Without loss of generality, we may assume that $i \neq j$ and $1 \leq i,j \leq k+1$. Let M be a $(k+1) \times (k+1)$ tournament matrix such that $M \not\geq E_{i,j}$. Let $\Xi = \begin{bmatrix} I_{k+1}+M & J_{k+1,n-k-1} \\ J_{m-k-1,k+1} & J_{m-k-1,n-k-1} \end{bmatrix}$. Then, $\iota(\Xi) = k+1$ while $\iota(\Xi + E_{i,j}) = k$, so that $\iota(T(\Xi)) = k+1$ and $\iota(T(\Xi + E_{i,j})) = k$. But since $T(E_{i,j}) = O$, $T(\Xi + E_{i,j}) = T(\Xi)$, a contradiction. Thus T is nonsingular.

Let $L = T^q$, where T^q is idempotent. Then L is idempotent and preserves isolation numbers k and $k+1$. Since T is nonsingular, so is L. Suppose for some (i,j), $|L(E_{i,j})| > 1$. Then, $L(E_{i,j}) = Z + F$, where F is a cell and, as in the previous paragraph, $\Xi \geq E_{i,j}$ and $\Xi \not\geq F$. Thus, $\iota(\Xi + E_{i,j}) = k+1$ while $\iota(\Xi + F) = k$ so that $\iota(L(\Xi + E_{i,j})) = k+1$ while $\iota(L(\Xi + F)) = k$.

Now, $L(\Xi) = L(\Xi + E_{i,j}) = L(\Xi) + L(E_{i,j}) = L(\Xi) + Z + F$. Since L is idempotent, $L(\Xi) = L^2(\Xi) = L(L(\Xi) + Z + F) = L(L(\Xi) + Z) + L(F) = L((L(\Xi) + Z + F) + Z) + L(F) = L(L(\Xi) + Z + F + Z) + L(F) = L(L(\Xi) + Z + F) + L(F) = L^2(\Xi) + L(F) = L(\Xi) + L(F) = L(\Xi + F)$. But, $\iota(L(\Xi)) = k+1$ while $\iota(L(\Xi + F)) = k$, a contradiction. Thus, L, and hence T maps cells to cells.

Now, suppose that E and F are cells and $T(E) = T(F)$. If $F \neq E$, we may assume that $\Xi \geq E$ and $\Xi \not\geq F$. But then, $\iota(\Xi + E) = k+1$ while $\iota(\Xi + F) = k$ so that $\iota(T(\Xi + E)) = k+1$ and $\iota(T(\Xi + F)) = k$. But, $T(\Xi + E) = T(\Xi + F)$, a contradiction. □

Lemma 13. *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator. If $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ preserves isolation numbers k and $k+1$ for some $1 \leq k \leq m-1$, then T maps lines to lines.*

Proof. By Lemma 12, T is bijective on the set of cells. Suppose T does not map lines to lines, then it must map some pair of noncollinear cells to a pair of collinear cells since T is bijective on the set of cells. Without loss of generality, we may assume that $T(E_{1,1} + E_{2,2}) = E_{1,1} + E_{1,2}$. Let $E_3, E_4, \cdots, E_{k+1}$ be cells such that $T(E_{j,j}) = E_j$ for $3 \leq j \leq k+1$. Then, $\iota(E_{1,1} + E_{1,2} + E_3 + E_4 + \cdots + E_{k+1}) \leq k$. Since $\iota(E_{1,1} + E_{2,2} + E_{3,3} + \cdots + E_{k+1,k+1}) = k+1$, $\iota(T(E_{1,1} + E_{2,2} + E_{3,3} + \cdots + E_{k+1,k+1})) = k+1$. But $T(E_{1,1} + E_{2,2} + E_{3,3} + \cdots + E_{k+1,k+1}) = E_{1,1} + E_{1,2} + E_3 + E_4 + \cdots + E_{k+1}$ which has the isolation number less than $k+1$, a contradiction. Thus, T maps lines to lines. □

Theorem 5. *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator. If $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ preserves isolation numbers k and $k+1$ for some $1 \leq k \leq m-1$, then T is a (P,Q)-operator.*

Proof. By Lemma 12, T is bijective. By Lemma 13, T maps lines to lines. The theorem now follows by applying Theorem 6. □

6. Summary

In [5], the authors prove the following.

Theorem 6 (See [5, Theorem 3.8]). *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a Boolean linear operator. Then T strongly preserves isolation number k for any $1 \le k \le \min\{m,n\}$ if and only if T is a (P,Q)-operator.*

A compilation of the above theorem and Theorems 2, 4, and 5 summarizes the results in this article.

Theorem 7. *For a Boolean linear operator $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$, the following are equivalent:*

1. *T preserves the isolation number;*
2. *T preserves isolation numbers 1 and k for some $1 < k \le m$;*
3. *$T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ preserves isolation numbers k and $k+1$ for some $1 \le k \le m-1$;*
4. *T preserves isolation number 2 and $T(J) = J$;*
5. *T strongly preserves isolation number k for any $1 \le k \le \min\{m,n\}$;*
6. *T is a (P,Q)-operator.*

Acknowledgement

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2012R1A1A2042193).

References

[1] K. Akiko, *Complexity of the sex-equal stable marriage problem*, Japan J. Indust. Appl. Math. **10**(1993), 1–19. (English summary)

[2] L. B. Beasley, *Isolation number versus Boolean rank*, Linear Algebra Appl. **436**(2012), 3469–3474.

[3] L. B. Beasley, N. J. Pullman, *Boolean rank preserving operators and Boolean rank-1 spaces*, Linear Algebra Appl. **59**(1984), 55–77.

[4] L. B. Beasley, S. Z. Song, *Linear operators that preserve Boolean rank*, in press.

[5] L. B. Beasley, S. Z. Song, Y. B. Jun, *Linear operators that preserve graphical properties of matrices: isolation numbers*, in press.

[6] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, New York, 2008.

[7] D. E. Brown, *Finding the isolation number of a graph is NP-complete*, personal communication (david.e.brown@usu.edu).

[8] R. Brualdi, H. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, New York, 1991.

[9] D. de Caen, D. Gregory, N. Pullman, *The Boolean rank of zero one matrices*, in: *Proceedings of the Third Caribbean Conference on Combinatorics and Computing*, (C. C. Cadogan, Ed.), Barbados, (1981), 169–173.

[10] G. Fröbenius, *Uber die Darstellung der entlichen Gruppen durch Linear Substitutionen*, S. B. Deutsch. Akad. Wiss. Berlin, (1897), 994–1015. ,v

[11] D. A. GREGORY, N. J. PULLMAN, *Semiring rank: Boolean and nonnegative rank factorizations*, J. Combin. Inform. System Sci. **8**(1983), 223–233.

[12] D. GREGORY, N. J. PULLMAN, K. F. JONES, J. R. LUNDGREN, *Biclique coverings of regular bigraphs and minimum semiring ranks of regular matrices*, J. Combin. Theory Ser. B **51**(1991), 73–89.

[13] D. A. GREGORY, S. J. KIRKLAND, N. J. PULLMAN, *A bound on the exponent of a primitive matrix using Boolean rank*, Linear Algebra Appl. **217**(1995), 101–116.

[14] A. E. GUTERMAN, A. V. MIKHALEV, *General algebra and linear transformations preserving matrix invariants*, J. Math. Sci. **128**(2005), 3384–3395.

[15] C.-K. LI, S. J. PIERCE, *Linear preserver problems*, Amer. Math. Monthly **108**(2001), 591–605.

[16] G. MARKOWSKY, *Ordering D-classes and computing the Schein rank is hard*, Semigroup Forum **44**(1992), 373–375.

[17] S. J. PIERCE ET AL., *A survey of linear preserver problems*, Linear and Multilinear Algebra **33**(1992), 1–129.