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An efficient hybrid pseudo-spectral method for solving optimal control of Volterra integral systems

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Abstract. In this paper, a new pseudo-spectral **(PS)** method is developed for solving optimal control problems governed by the non-linear Volterra integral equation **(VIE)**. The novel method is based upon approximating the state and control variables by the hybrid of block pulse functions and Legendre polynomials. The properties of hybrid functions are presented. The numerical integration and collocation method is utilized to discretize the continuous optimal control problem and then the resulting large-scale finite-dimensional non-linear programming **(NLP)** is solved by the existing well-developed algorithm in Mathematica software. A set of sufficient conditions is presented under which optimal solutions of discrete optimal control problems converge to the optimal solution of the continuous problem. The error bound of approximation is also given. Numerical experiments confirm efficiency of the proposed method especially for problems with non-sufficiently smooth solutions belonging to class C^1 or C^2.
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1. Introduction

Consider the following formulation of the optimal control problem governed by non-linear VIE:

Problem B: Specify the real valued continuous optimal control $u^*(t)$ and the corresponding optimal state $x^*(t)$, $t \in [0,1]$, that maximize (or minimize) the functional

$$J(x,u) = \int_0^1 F(t, x(t), u(t))dt, \tag{1}$$

subject to state dynamics

$$x(t) = y(t) + \int_0^t k(s, t, x(s), u(s))ds. \tag{2}$$

It is assumed that F, k and y are real valued and continuously differentiable with respect to their arguments and both x and u belong to Sobolev space $W^{l,\infty}$ (with

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$l \geq 2$) [6]. The interval $[0,1]$ can be transformed to the interval $[x_0, x_f]$ via an affine transformation. It is also supposed that optimal control of this problem is unique. Analytical discussions about the existence and uniqueness for optimal control of systems governed by non-linear VIE (2) can be found in [1, 10] and references therein.

The controlled VIE (2) appears in modelling many classes of phenomena [18]. Several authors have considered solving problem **B** with different methods, e.g., Vinokurov [26], Medhin [16], Schmidt [23, 24], and Belbas [2, 3, 4]. Some of them have obtained optimal control by using the Pontryagin's maximum principle and dynamic programming [2, 3]. Due to the difficulties in getting an analytical solution of these problems, the numerical methods are usually of interest. In [4], Belbas introduced iterative methods with their convergence for obtaining optimal control of a non-linear VIE by considering some conditions on the kernel of integral equation. Schmidt [24] proposed some direct and indirect numerical methods for solving optimal control problems governed by ODEs as well as integral equations. In [17], the authors introduced some hybrid methods based on steepest descent and two-step Newton methods for obtaining optimal control and the corresponding optimal state. Recently, orthogonal functions have also been used for solving optimal control of Volterra integral equations, e.g., in [12], a numerical iterative method is presented for calculating optimal control via triangular functions. In [18], Tohidi and Samadi investigated the use of Lagrange polynomials in optimal control problems for systems governed by the VIE and also surveyed the convergence of their proposed method. Their method has high efficiency, especially for problems with smooth solutions. In [13], a collocation method based on rationalized Haar wavelets was utilized to approximate optimal control and state variables. The numerical results of [13] are also compared with those obtained in this paper.

In recent years, the hybrid functions have been shown to be a powerful tool for the discretization of selected problems [9, 15, 21]. Among these hybrid functions, the hybrid function of block-pulse and Legendre polynomials has been shown to be computationally more effective [9]. The main advantage of these hybrid functions is their efficiency and simple applicability. The other advantage of these hybrid functions is that the orders of block pulse functions and Legendre polynomials are adjustable to obtain highly accurate numerical solutions. The approximations by hybrid functions have especially good accuracy for non-sufficiently smooth solutions belonging to class C^1 and C^2. We could increase the order of block pulse functions or degree of Legendre polynomials for achieving high precision. As a result of this, the dimension of the discrete problem, computational cost and CPU time become greater.

In this paper, we have introduced a PS method for obtaining optimal control of systems governed by the VIE. PS techniques have been shown to provide effective and flexible methods for solving different problems [19, 20, 22]. Our method consists of reducing the optimal control problem to a NLP by first expanding the state and control functions in terms of hybrid functions with unknown coefficients. The hybrid of block-pulse functions and Legendre polynomials is utilized for discretization by using the numerical integration and collocation method. After obtaining the finite dimensional programming problem, many well-developed constrained optimization methods can be used to dissolve this problem. In this article, we also provide the conditions for the convergence of the pseudo-spectral method for problem B.

This paper is organized as follows: In Section 2, we state some properties of hybrid functions. Section 3 is devoted to the application of the PS method to the optimal control problem. In Section 4, we analyze the convergence of the numerical technique applied to problem **B**. Section 5 contains numerical examples that show the efficiency and accuracy of the method. Section 6 ends this paper with a brief conclusion.

2. Properties of hybrid functions

2.1. Hybrid of block-pulse and Legendre polynomials

The orthogonal hybrid functions $b_{nm}(t)$, $n = 1, 2, \dots, N$ and $m = 0, 1, \dots, M-1$ are described on the interval $[0, 1)$ as

$$b_{nm}(t) = \begin{cases} P_m\Big(2Nt - 2n + 1\Big), & t \in \Big[\frac{n-1}{N}, \frac{n}{N}\Big), \\ 0, & \text{otherwise}, \end{cases}$$

where n and m are the order of block-pulse functions and Legendre polynomials, respectively [15]; here, $P_m(t)$, $m = 0, 1, \dots, M-1$, are the Legendre polynomials which satisfy the following recursive formula [15]:

$$P_0(t) = 1, \quad P_1(t) = t,$$

$$P_{m+1}(t) = \Big(\frac{2m+1}{m+1}\Big) t P_m(t) - \Big(\frac{m}{m+1}\Big) P_{m-1}(t), \qquad m = 1, 2, 3, \dots. \tag{3}$$

2.2. Function approximation

A function $f(t)$, $t \in [0, 1]$, may be expanded in terms of hybrid functions as follows [15]:

$$f(t) \simeq f_0(t) = \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} b_{nm}(t) = C^T B(t),$$

where $B(t)$ and C are column vectors given by:

$$\begin{aligned} B(t) &= [b_{10}(t), \dots, b_{1M-1}(t), b_{20}(t), \dots, b_{2M-1}(t), \dots, b_{N0}(t), \dots, b_{NM-1}(t)]^T, \quad (4) \\ C &= [c_{10}, \dots, c_{1M-1}, c_{20}, \dots, c_{2M-1}, \dots, c_{N0}, \dots, c_{NM-1}]^T. \end{aligned}$$

3. The proposed method

In this section, we consider the discretization process of problem **B**. The approximation process of the considered problem includes the discretization of both the cost function and the controlled integral equation constraint. For the approximation of problem **B**, the basic idea is to expand $x(t)$ and $u(t)$ in terms of hybrid functions

$$x(t) \simeq \overline{x}(t) = X^T B(t), \qquad u(t) \simeq \overline{u}(t) = U^T B(t), \tag{5}$$

where X, U and $B(t)$ are defined similarly to (4). By substituting (5) in (2), we obtain

$$\overline{x}(t) = y(t) + \int_0^t k(s, t, \overline{x}(s), \overline{u}(s))dt. \tag{6}$$

We discretize the VIE that exists in (6) by using the set of collocation nodes as follows:

$$\overline{x}(t_p) = y(t_p) + \int_0^{t_p} k(s, t_p, \overline{x}(s), \overline{u}(s))dt, \tag{7}$$

where t_p, $p = 1, 2, \ldots, MN$, can be Gauss-Chebyshev **(GC)** [14], Gauss-Legendre **(GL)**, zeros of Legendre polynomials, or equidistant nodes. If we use GL or GC nodes as collocation nodes, firstly these points should be transformed into the interval $[0, 1]$. The GL quadrature formula is utilized to approximate the integral term in (7). For this purpose, linear transformation must be made with the following form

$$c_p(\tau) = \frac{t_p}{2}(\tau + 1). \tag{8}$$

Then, (7) is converted to

$$\overline{x}(t_p) = y(t_p) + \frac{t_p}{2}\int_{-1}^{1} k(c_p(\tau), t_p, \overline{x}(c_p(\tau)), \overline{u}(c_p(\tau)))d\tau, \qquad p = 1, 2, \ldots, MN. \tag{9}$$

By applying the GL quadrature for approximating the integral involved in (9), we obtain

$$\overline{x}(t_p) = y(t_p) + \frac{t_p}{2}\sum_{j=1}^{MN} w_j k(c_p^j, t_p, \overline{x}(c_p^j), \overline{u}(c_p^j)), \qquad p = 1, 2, \ldots, MN, \tag{10}$$

where $c_p^j = c_p(\tau_j)$, and τ_js are the GL nodes, zeros of Legendre polynomial $P_{MN}(t)$ in $[-1, 1]$, and w_js are the corresponding weights. The quadrature weights, w_j, can be obtained by the following relation

$$w_j = \frac{2}{(1 - \tau_j^2)[P'_{MN}(\tau_j)]^2}, \qquad j = 1, \ldots, M. \tag{11}$$

Finally, the controlled Volterra integral (2) is reduced to MN non-linear algebraic equations given in (10).

For approximating the cost function stated in (1), we utilize the GL quadrature after the proper interval transformation

$$\begin{aligned} \int_0^1 F(t, x(t), u(t)) &= \frac{1}{2}\int_{-1}^{1} F(\frac{\tau + 1}{2}, \overline{x}(\frac{\tau + 1}{2}), \overline{u}(\frac{\tau + 1}{2}))d\tau \\ &\approx \sum_{j=1}^{MN} w'_j F(\tau'_j, \overline{x}(\tau'_j), \overline{u}(\tau'_j)), \end{aligned} \tag{12}$$

where $w'_j = \frac{1}{2}w_j$ and $\tau'_j = \frac{\tau_j+1}{2}$, and τ_j and w_j are GL nodes and weights stated in (11).

Finally, problem **B** is approximated by the following NLP

$$\min \overline{J}(X,U) \tag{13}$$

subject to

$$k_p(X,U) = -y(t_p), q \quad p = 1,2,\dots,MN, \tag{14}$$

where $X = (x_{10}, x_{11}, \dots, x_{NM-1})$ and $U = (u_{10}, u_{11}, \dots, u_{NM-1})$ are the unknown parameters of our discrete problem, and

$$\overline{J}(X,U) = \frac{1}{2}\sum_{j=1}^{MN} w_j F(\frac{\tau_j+1}{2}, \overline{x}(\frac{\tau_j+1}{2}), \overline{u}(\frac{\tau_j+1}{2})) = \sum_{j=1}^{MN} w'_j F(\tau'_j, \overline{x}(\tau'_j), \overline{u}(\tau'_j)),$$

and

$$k_p(X,U) = \frac{t_p}{2}\sum_{j=1}^{MN} w_j k(c_p^j, t_p, \overline{x}(c_p^j), \overline{u}(c_p^j)) - \overline{x}(t_p), \qquad p = 1,2,\dots,MN.$$

Following [8], we propose the following relaxation to guarantee the feasibility of discretization

$$|k_p(X,U) + y(t_p)| \le (M-2)^{\frac{3}{2}-l}, \qquad p = 1,2,\dots,MN. \tag{15}$$

In (15), note that with a given N, the order of the block pulse function, when M tends to infinity, the difference between (15) and (14) vanishes, because l is greater than or equal to 2. Thus, the optimal control problem **B** is approximated by a NLP with (13) as the objective function and (15) as constraints. This is summarized as:
Problem $\overline{\mathbf{B}}$: Find (X,U) that minimize

$$\overline{J}(X,U) = \sum_{j=1}^{MN} F(\tau'_j, \overline{x}(\tau'_j), \overline{u}(\tau'_j))w'_j, \tag{16}$$

subject to

$$|k_p(X,U) + y(t_p)| \le (M-2)^{\frac{3}{2}-l}, \qquad p = 1,2,\dots,MN. \tag{17}$$

When the continuous problem **B** is discretized, the infinite dimensional problem **B** is reduced to the finite dimensional non-linear optimization problem $\overline{\mathbf{B}}$. Many well-developed NLP techniques can be used to solve this problem [11]. The method used to solve the non-linear constrained optimization problem is based on the sequential quadratic programming **(SQP)** algorithm. It is an iterative method for non-linear optimization [11].

4. Convergence analysis

Theorems 1 and 2 indicate the uniform convergence and accuracy estimation of the hybrid expansion. The proofs are given in [25].

Theorem 1 (See [25]). *If a continuous function $f(t)$ defined on $[0,1]$ has a bounded second derivative, then the hybrid expansion of the function converges uniformly to the function f.*

Theorem 2 (See [25]). *Let $f(t)$ be a continuous function defined on $[0,1]$, with the second derivative $|f^{''}(t)|$ bounded by M_1. Then we have the following accuracy estimation:*

$$||f(t) - C^T B(t)||^2 \leq \frac{3}{8} {M_1}^2 \Big(\sum_{i=n+1}^{\infty} \sum_{j=m}^{\infty} \frac{1}{i^5 (2j-3)^4} \Big).$$

The convergence analysis of the proposed PS method lies at the intersection of approximation theory, control theory and optimization. To examine the convergence analysis, we need to answer a few questions. For example, does the discretized problem $\overline{\mathbf{B}}$ have a feasible solution if a solution to the continuous-time problem $\mathbf{B}$ exists? Does a sequence of optimal solutions of discrete problems $\overline{\mathbf{B}}$ converge to continuous-time optimal solution of problem $\mathbf{B}$? In Theorem 3, the feasibility of discretized problem $\overline{\mathbf{B}}$ will be proved. First, we need the following definition and lemmas.

Definition 1 (See [6]). *A function $f : [0,1] \to \Re$ belongs to Sobolev space $W^{l,s}$, if its jth distributional derivative, $f^{(j)}$, lies in $L^s[0,1]$ for all $0 \leq j \leq l$ with the norm*

$$||f||_{W^{l,s}} = \sum_{j=0}^{l} ||f^{(j)}||_{L^s}, \tag{18}$$

where $||f||_{L^s}$ denotes the usual Lebesgue norm defined for $1 \leq s < \infty$ as follows:

$$\|f\|_{L^s} = \Big(\int_0^1 |f(t)|^s dt \Big)^{\frac{1}{s}},$$

and for $s = \infty$

$$\|f\|_{L^\infty} = \inf\{C \geq 0 : |f(t)| \leq C \textit{ for almost every } t \in [0,1]\}.$$

Lemma 1 (See [6]). *Given any function $f \in W^{l,\infty}$, there is a polynomial Q_M of degree M or less, such that*

$$|f(t) - Q_M(t)| \leq C C_0 M^{-l}, \qquad \forall t \in [0,1],$$

where C is a constant independent of M, l is the order of smoothness of f and $C_0 = ||f||_{W^{l,\infty}}$. ($Q_M(t)$ with the smallest norm $||f - Q_M||_{L^\infty}$ is called the Mth order best polynomial approximation of $f(t)$ in the norm of L^∞).

Proof. The proof is given in [6]. □

Remark 1. *The computational interval can be transformed from* $[0,1]$ *to* $[x_0, x_f]$ *via an affine transformation.*

Lemma 2 (See [6])**.** *Let* t_p*,* $1 \leq p \leq MN$*, be LG nodes and* w_p *the LG weights. Assume* $f(t)$ *is Rimann integrable. Then*

$$\int_{-1}^{1} f(t)dt = \lim_{MN \to \infty} \sum_{p=1}^{MN} f(t_p)w_p.$$

We have developed the basic idea of [18] for proving the next two theorems.

Theorem 3. *Given any feasible solution* $(x(t), u(t))$ *for problem* $\boldsymbol{B}$*. Suppose* $x(t)$ *and* $u(t)$ *belong to* $W^{l,\infty}$ *with* $l \geq 2$*. Then, there exists a positive integer* M' *such that for any* $M \geq M'$ *problem* $\overline{\overline{\mathbf{B}}}$ *has a feasible solution* $(\overline{x}_p, \overline{u}_p) = (\overline{x}(t_p), \overline{u}(t_p))$*, such that for* $t_p \in I_n = [\frac{n-1}{N}, \frac{n}{N})$*,* $n = 1, 2, \ldots, N$*, the feasible solutions satisfy*

$$|x(t_p) - \overline{x}_p| \leq \frac{1}{N} C_1^n (M-2)^{1-l}, \qquad p = 1, \ldots, MN,$$

and

$$|u(t_p) - \overline{u}_p| \leq \frac{1}{N} C_2^n (M-2)^{1-l}, \qquad p = 1, \ldots, MN,$$

where t_p *are GC or GL nodes and* C_1^n *and* C_2^n *are positive constants independent of* M*.*

Proof. Assume that $p_{M-2}^n(t)$ and $q_{M-2}^n(t)$ are the $(M-2)$th order best polynomial approximations of $x'(t)$ and $u'(t)$ in the interval $\overline{I}_n = [\frac{n-1}{N}, \frac{n}{N}]$ and norm of L^∞. By utilizing Lemma 1, there exist positive constants C_1^n and C_2^n independent of M such that for $n = 1, \ldots, N$ we possess

$$|x'(t) - p_{M-2}^n(t)| \leq C_1^n (M-2)^{1-l}, \qquad \forall t \in \overline{I}_n,$$

and

$$|u'(t) - q_{M-2}^n(t)| \leq C_2^n (M-2)^{1-l}, \qquad \forall t \in \overline{I}_n.$$

Define

$$\overline{x}(t) = \int_{\frac{n-1}{N}}^{t} p_{M-2}^n(\tau)d\tau + x(\frac{n-1}{N}), \quad t \in I_n, n = 1, \ldots, N, \tag{19}$$

$$\overline{u}(t) = \int_{\frac{n-1}{N}}^{t} q_{M-2}^n(\tau)d\tau + u(\frac{n-1}{N}), \qquad t \in I_n, n = 1, \ldots, N. \tag{20}$$

For $n = 1, \ldots, N$, we have

$$|x(t) - \overline{x}(t)| = \Big| \int_{\frac{n-1}{N}}^{t} \Big(x'(\tau) - p_{M-2}^{n}(\tau)\Big) d\tau \Big| \leq \frac{1}{N} C_1^n (M-2)^{1-l}, \quad t \in I_n, \tag{21}$$

$$|u(t) - \overline{u}(t)| = \Big| \int_{\frac{n-1}{N}}^{t} \Big(u'(\tau) - q_{M-2}^{n}(\tau)\Big) d\tau \Big| \leq \frac{1}{N} C_2^n (M-2)^{1-l}, \quad t \in I_n. \tag{22}$$

The VID system dynamics (2) of problem **B** can be written in the form

$$x'(t) = k_1(t, x(t), u(t)) + \int_0^t k_2(t, s, x(s), u(s)) ds, \tag{23}$$

in which

$$k_1(t, x(t), u(t)) = y'(t) + k(t, t, x(t), u(t))$$

and

$$k_2(t, s, x(s), u(s)) = \frac{\partial k}{\partial t}(t, s, x(s), u(s)).$$

From (21) and (22), it follows that both $x(t_p)$ and $\overline{x}_p$ (also $u(t_p)$ and $\overline{u}_p$) are contained in some compact set. On this compact set, because partial derivatives of k are continuous, they are then Lipschitz continuous (i.e., k_1 and k_2 are jointly Lipschitz with respect to their variables $x(.)$ and $u(.)$ in the interval I_n, $1 \leq n \leq N$). In the approximation of system dynamics with the proposed method, we overlook the approximation of the integral involved in the constraint, and just consider the collocation process and approximation of state and control variables in terms of hybrid functions, so we obtain

$$\overline{x}'(t_p) = k_1(t_p, \overline{x}(t_p), \overline{u}(t_p)) + \int_0^{t_p} k_2(t_p, s, \overline{x}(s), \overline{u}(s)) ds. \tag{24}$$

According to (15), we relax (24) to the form

$$|\overline{x}'(t_p) - k_1(t_p, \overline{x}(t_p), \overline{u}(t_p)) - \int_0^{t_p} k_2(t_p, s, \overline{x}(s), \overline{u}(s)) ds| \leq (M-2)^{\frac{3}{2}-l}. \tag{25}$$

Now, we demonstrate that $\overline{x}(t_p)$ and $\overline{u}(t_p)$ are feasible solutions of (25). For $t_p \in I_n$, we have

$$
\begin{aligned}
|\overline{x}'(t_p) - k_1 & (t_p, \overline{x}(t_p), \overline{u}(t_p)) - \int_0^{t_p} k_2(t_p, s, \overline{x}(s), \overline{u}(s))ds| \\
& \leq |\overline{x}'(t_p) - x'(t_p)| + |k_1(t_p, x(t_p), u(t_p)) - k_1(t_p, \overline{x}(t_p), \overline{u}(t_p))| \\
& \quad + |\int_0^{t_p} k_2(t_p, s, x(s), u(s))ds - \int_0^{t_p} k_2(t_p, s, \overline{x}(s), \overline{u}(s))ds| \\
& \quad + \overbrace{|x'(t_p) - k_1(t_p, x(t_p), u(t_p)) - \int_0^{t_p} k_2(t_p, s, x(s), u(s))|}^{\textit{Since } x(t) \textit{ and } u(t) \textit{ are feasible solutions of (23) equal to zero}} \\
& \leq |p^n_{M-2}(t_p) - x'(t_p)| + L^n_{k_1}\{|x(t_p) - \overline{x}(t_p)| + |u(t_p) - \overline{u}(t_p)|\} \\
& \quad + L^n_{k_2} \int_0^{t_p} (|x(s) - \overline{x}(s)| + |u(s) - \overline{u}(s)|)ds \\
& \leq \{C^n_1 + L^n_{k_1}\frac{1}{N}(C^n_1 + C^n_2) + L^n_{k_2}\frac{n}{N^2}(\max_{1\leq i\leq n} C^i_1 + \max_{1\leq i\leq n} C^i_2)\}(M-2)^{1-l},
\end{aligned}
$$

where $L^n_{k_1}$ and $L^n_{k_2}$ are Lipschitz constants of k_1 and k_2 in the interval I_n, respectively. There exists a positive integer M_n such that for all $M > M_n$ we have

$$
\{C^n_1 + L^n_{k_1}\frac{1}{N}(C^n_1 + C^n_2) + L^n_{k_2}\frac{n}{N^2}\max_{1\leq i\leq n}(C^i_1 + C^i_2)\}(M-2)^{1-l} \leq (M-2)^{\frac{3}{2}-l}.
$$

By assuming

$$
M' = \max_{1\leq n\leq N} M_n,
$$

it is sufficient to take

$$
M' = \max_{1\leq n\leq N} M_n.
$$

Thus, it follows that $\overline{x}(t_p)$ and $\overline{u}(t_p)$ are feasible solutions of (25). □

Let $X^* = [x^*_{10}, x^*_{11}, \ldots, x^*_{NM-1}]^T$ and $U^* = [u^*_{10}, u^*_{11}, \ldots, u^*_{NM-1}]^T$ be the optimal solutions of problem $\overline{\mathbf{B}}$. The approximate optimal control and state are

$$
\overline{x}^*(t) = \sum_{n=1}^{N}\sum_{m=0}^{M-1} x^*_{nm} b_{nm}(t), \quad \overline{u}^*(t) = \sum_{n=1}^{N}\sum_{m=0}^{M-1} u^*_{nm} b_{nm}(t).
$$

In the next theorem, we will prove the convergence of the sequence

$$
\{(\overline{x}^*(t_p), \overline{u}^*(t_p)) : (1 \leq p \leq MN) \text{ and } (N \in \mathbb{N})\}^\infty_{M=M'}.
$$

Assume that the following conditions are satisfied:

H_1: The function sequence $\{(\overline{x}'^*(t), \overline{u}'^*(t))\}^\infty_{M=M'}$ has a subsequence that uniformly converges to the continuous functions $\{(p(t), q(t))\}$ on each interval I_n for $1 \leq n \leq N$.

H_2: Define

$$\widehat{x}(t) = \int_{\frac{n-1}{N}}^{t} p(\tau)d\tau + \widehat{x}(\frac{n-1}{N}), \qquad t \in I_n, 1 \le n \le N, \tag{26}$$

$$\widehat{u}(t) = \int_{\frac{n-1}{N}}^{t} q(\tau)d\tau + \widehat{u}(\frac{n-1}{N}), \qquad t \in I_n, 1 \le n \le N. \tag{27}$$

For $1 \le n \le N$ it is presumed

$$\lim_{M\to\infty} \overline{x}^*(\frac{n-1}{N}) = \widehat{x}(\frac{n-1}{N}), \quad \lim_{M\to\infty} \overline{u}^*(\frac{n-1}{N}) = \widehat{u}(\frac{n-1}{N}). \tag{28}$$

Theorem 4. *Let*

$$\{(\overline{x}^*(t_p), \overline{u}^*(t_p)) : (1 \le p \le MN) \text{ and } (N \in \mathbb{N})\}_{M=M'}^{\infty}$$

be a sequence of optimal solutions to problem $\overline{B}$. Assume that the conditions H_1 and H_2 are satisfied. Then the pair of $(\widehat{x}(t), \widehat{u}(t))$ is the optimal solution of problem B.

Proof. For a simpler notation, we presume that the sequence $\{(\overline{x}'^*(t), \overline{u}'^*(t))\}_{M=M'}^{\infty}$ uniformly converges to the continuous functions $\{(p(t), q(t))\}_{M=M_1}^{\infty}$ on the interval I_n for $1 \le n \le N$. By using

$$|\overline{x}^*(t) - \widehat{x}(t)| \le \int_{\frac{n-1}{N}}^{t} |\overline{x}'^*(\tau) - p(\tau)|d\tau + |\overline{x}^*(\frac{n-1}{N}) - \widehat{x}(\frac{n-1}{N})|, \qquad t \in I_n. \tag{29}$$

It is clear that $\lim_{M\to\infty} \overline{x}^*(t) = \widehat{x}(t)$ uniformly on I_n. By a similar argument, one can get $\lim_{M\to\infty} \overline{u}^*(t) = \widehat{u}(t)$ for $t \in I_n$. The proof of this theorem is divided into three parts. First, we represent that $(\widehat{x}(t), \widehat{u}(t))$ is a feasible solution of problem B. Then, we prove the convergence of the cost function $\overline{J}(X^*, U^*)$ to the cost function $J(\widehat{x}, \widehat{u})$, and finally show that $(\widehat{x}(t), \widehat{u}(t))$ is an optimal solution of problem B.

Step 1: To prove that $(\widehat{x}(t), \widehat{u}(t))$ is a feasible solution of problem B, we must indicate that $(\widehat{x}(t), \widehat{u}(t))$ for $t \in I_n$, $1 \le n \le N$, satisfies the state constraint (2). By the contradiction argument, we presume that $(\widehat{x}(t), \widehat{u}(t))$ is not a solution of integral (2). Then, there is a time t' so that

$$\widehat{x}(t') - y(t') - \int_0^{t'} k(s, t', \widehat{x}(s), \widehat{u}(s))ds \ne 0. \tag{30}$$

Let $t' \in I_n$. Since the zeros of orthogonal polynomials are dense [7] with increasing M, there exists a sequence $\{t_i\}$ of LG or CG that satisfies

$$\lim_{i\to\infty} t_i = t'. \tag{31}$$

Because $\overline{x}^*(t)$, $\overline{u}^*(t)$, $y(t)$ and $k(s, t, \overline{x}^*(s), \overline{u}^*(s))$ are continuous on I_n with respect to their arguments, with increasing i which implies $M \to \infty$, we have $\overline{x}^*(t) \to \widehat{x}(t)$,

$\overline{u}^*(t) \to \widehat{u}(t)$ and $k(s,t,\overline{x}^*(s),\overline{u}^*(s)) \to k(s,t,\widehat{x}(s),\widehat{u}(s))$. So we obtain

$$0 = \lim_{i\to\infty} \overbrace{\left(\overline{x}^*(t_i) - y(t_i) - \int_0^{t_i} k(s,t_i,\overline{x}^*(s),\overline{u}^*(s))\right)}^{=0} ds \tag{32}$$

$$= \widehat{x}(t') - y(t') - \int_0^{t'} k(s,t',\widehat{x}(s),\widehat{u}(s))ds.$$

This contradicts (30). Therefore, $(\widehat{x}(t),\widehat{u}(t))$ must be a feasible solution of problem B.

Step 2: In this step, we will show that

$$\lim_{MN\to\infty} \overline{J}(X^*,U^*) = J(\widehat{x}(t),\widehat{u}(t)), \tag{33}$$

where

$$\overline{J}(X^*,U^*) = \sum_{j=1}^{MN} w'_j F(\tau'_j,\overline{x}^*(\tau'_j),\overline{u}^*(\tau'_j)), \tag{34}$$

$$J(\widehat{x}(t),\widehat{u}(t)) = \int_0^1 F(t,\widehat{x}(t),\widehat{u}(t))dt, \tag{35}$$

and τ'_j and w'_j are introduced in Section (3). Since $\{(\overline{x}^*(t),\overline{u}^*(t))\}_{M=M'}^{\infty}$ converges uniformly to $\{(\widehat{x}(t),\widehat{u}(t))\}$ on the interval I_n for $1 \le n \le N$, we obtain

$$\lim_{M\to\infty} |\overline{x}^*(\tau'_j) - \widehat{x}(\tau'_j)| = 0, \quad \lim_{M\to\infty} |\overline{u}^*(\tau'_j) - \widehat{u}(\tau'_j)| = 0. \tag{36}$$

Because F is a continuously differentiable function, it is jointly Lipschitz with respect to its variables x and u in the interval I_n, $1 \le n \le N$. By assuming that $\tau'_j \in I_n$, we have

$$\begin{aligned} &|F(\tau'_j,\widehat{x}(\tau'_j),\widehat{u}(\tau'_j)) - F(\tau'_j,\overline{x}^*(\tau'_j),\overline{u}^*(\tau'_j))| \\ &\qquad \le L_F^n\Big(|\widehat{x}(\tau'_j) - \overline{x}^*(\tau'_j)| + |\widehat{u}(\tau'_j) - \overline{u}^*(\tau'_j)|\Big), \end{aligned} \tag{37}$$

where L_F^n is a Lipschitz constant of F on the interval I_n for $1 \le n \le N$. Since $F(t,\widehat{x}(t),\widehat{u}(t))$ is the continuous function on I_n, we can conclude from Lemma 2 that

$$\begin{aligned} \int_0^1 F(t,\widehat{x}(t),\widehat{u}(t))dt &= \frac{1}{2}\int_{-1}^1 F(\frac{\tau+1}{2},\widehat{x}(\frac{\tau+1}{2}),\widehat{u}(\frac{\tau+1}{2}))d\tau \\ &= \frac{1}{2}\sum_{n=1}^N \int_{2(\frac{n-1}{N})-1}^{2(\frac{n}{N})-1} F(\frac{\tau+1}{2},\widehat{x}(\frac{\tau+1}{2}),\widehat{u}(\frac{\tau+1}{2}))d\tau \\ &= \frac{1}{2}\sum_{n=1}^N \lim_{k_n\to\infty}\sum_{j=1}^{k_n} F(\frac{\tau_j^n+1}{2},\widehat{x}(\frac{\tau_j^n+1}{2}),\widehat{u}(\frac{\tau_j^n+1}{2}))w_j^n \\ &= \lim_{MN\to\infty}\sum_{j=1}^{MN} F(\tau'_j,\widehat{x}(\tau'_j),\widehat{u}(\tau'_j))w'_j, \end{aligned} \tag{38}$$

where $\sum_{n=1}^{N} k_n = MN$, τ_j^n are the LG nodes, zeros of Legendre polynomial P_{NM}, that are in the interval $[2(\frac{n-1}{N})-1, 2(\frac{n}{N})-1)$, and w_j^n are the LG weights corresponding to τ_j^n. It is worth mentioning that k_n is the number of LG nodes located in the interval $[2(\frac{n-1}{N})-1, 2(\frac{n}{N})-1)$. According to [7], since the zeros of Legendre polynomials are dense, when NM tends to infinity, each $k_n, 1 \le n \le N$, goes to infinity and so the last equality of (38) is satisfied.

Therefore,

$$\int_0^1 F(t,\widehat{x}(t),\widehat{u}(t))dt = \lim_{M\to\infty}\Big(\sum_{j=1}^{MN} F(\tau_j',\overline{x}^*(\tau_j'),\overline{u}^*(\tau_j'))w_j' + \sum_{j=1}^{MN}\Big[F(\tau_j',\widehat{x}(\tau_j'),\widehat{u}(\tau_j')) - F(\tau_j',\overline{x}^*(\tau_j'),\overline{u}^*(\tau_j'))\Big]w_j'\Big). \tag{39}$$

From (36), (37) and $\sum_{j=1}^{MN} w_j = 2$, we have

$$\begin{aligned}\lim_{M\to\infty}\Big|\sum_{j=1}^{MN}\ &[F(\tau_j',\widehat{x}(\tau_j'),\widehat{u}(\tau_j')) - F(\tau_j',\overline{x}^*(\tau_j'),\overline{u}^*(\tau_j'))]w_j'\Big| \\ &\le \sum_{n=1}^{N}\lim_{k_n\to\infty}\sum_{j=1}^{k_n}|F(\tau_j'^{n},\widehat{x}(\tau_j'^{n}),\widehat{u}(\tau_j'^{n})) - F(\tau_j'^{n},\overline{x}^*(\tau_j'^{n}),\overline{u}^*(\tau_j'^{n}))|w_j'^{n} \\ &\le \sum_{n=1}^{N} L_F^n \lim_{k_n\to\infty}\sum_{j=1}^{k_n}(|\widehat{x}(\tau_j'^{n}) - \overline{x}^*(\tau_j'^{n})|) + |\widehat{u}(\tau_j'^{n}) - \overline{u}^*(\tau_j'^{n})|)w_j'^{n} = 0.\end{aligned} \tag{40}$$

By using (39) and (40), we get

$$\int_0^1 F(t,\widehat{x}(t),\widehat{u}(t))dt = \lim_{MN\to\infty}\sum_{j=1}^{MN} F(\tau_j',\overline{x}^*(\tau_j'),\overline{u}^*(\tau_j'))w_j', \tag{41}$$

which confirms (33).

Step 3: Now let $x^*(t)$ and $u^*(t)$ be the optimal solutions of problem B, where both of them belong to $W^{l,\infty}$, $l \ge 2$. According to Theorem 3, there exists a sequence of a feasible solution,

$$\{(\overline{x}(t_p),\overline{u}(t_p)), 1 \le p \le MN\}_{M=M'}^{\infty},$$

for problem $\overline{B}$ which converges to

$$\{(x^*(t_p),u^*(t_p)), 1 \le p \le MN\}.$$

Let

$$J(x^*(t),u^*(t)) = \int_0^1 F(t,x^*(t),u^*(t))dt$$

and

$$\overline{J}(X,U) = \sum_{j=1}^{MN} F(\tau_j',\overline{x}(\tau_j'),\overline{u}(\tau_j'))w_j'.$$

By

$$J(\widehat{x}(t), \widehat{u}(t)) = \lim_{MN\to\infty} \sum_{j=1}^{MN} F(\tau'_j, \overline{x}^*(\tau'_j), \overline{u}^*(\tau'_j))w'_j$$

and the property of an optimal solution, we have

$$\begin{aligned} J(x^*, u^*) &\leq \lim_{MN\to\infty} \sum_{j=1}^{MN} F(\tau'_j, \overline{x}^*(\tau'_j), \overline{u}^*(\tau'_j))w'_j \\ &\leq \lim_{MN\to\infty} \sum_{j=1}^{MN} F(\tau'_j, \overline{x}(\tau'_j), \overline{u}(\tau'_j))w'_j. \end{aligned} \tag{42}$$

By using the same logic as in Step 2, it is easy to demonstrate that

$$J(x^*(t), u^*(t)) = \lim_{MN\to\infty} \overline{J}(X, U). \tag{43}$$

Since $\{(\overline{x}(t_p), \overline{u}(t_p))\}_{M=M'}^{\infty}$ converges to $\{(x^*(t_p), u^*(t_p))\}$, (42) and (43) imply that

$$J(x^*, u^*) = J(\widehat{x}, \widehat{u}).$$

Thus, $(\widehat{x}, \widehat{u})$ is an optimal solution to problem B. □

5. Illustrative examples

We give three examples to demonstrate the applicability, efficiency and accuracy of our method. The numerical experiments are implemented in Mathematica 7. In order to analyze the error of the method, the following notations are introduced:

$$\| E^x \|_\infty = \max_{1\leq p\leq MN} |E^x(t_p)|, \quad |E^u\|_\infty = \max_{1\leq p\leq MN} |E^u(t_p)|,$$

where

$$\begin{aligned} E^x(t) &= \overline{x}^*(t) - x^*(t), \\ E^u(t) &= \overline{u}^*(t) - u^*(t), \\ E^J &= |J^* - \overline{J}^*| \end{aligned}$$

and t_p for $1 \leq p \leq MN$ are collocation nodes. We have utilized different collocation points. The choice of Legendre collocation points has given superior results to the ones obtained from equidistant or Chebyshev points. For ease of reference, we use the following notations in this section:

HCM: Hybrid collocation method (the proposed method in this paper),

LCM: Lagrange collocation method (the presented method in [18]).

5.1. Example 1

Find the optimal control function $u^*(t)$ and the corresponding optimal state $x^*(t)$ which minimizes

$$J = \int_0^1 \Big(x(t) - (t-\frac{1}{2})|t-\frac{1}{2}|\Big)^2 + \Big(u(t) - (t-\frac{1}{2})|t-\frac{1}{2}|\Big)^2 dt, \tag{44}$$

subject to

$$x(t) = y(t) + t\int_0^t (x(s) + s^2 u(s))ds, \tag{45}$$

where

$$y(t) = -\frac{t}{32} + (t-\frac{1}{2})|t-\frac{1}{2}|.$$

The exact optimal control and state functions are

$$u^*(t) = (t-\frac{1}{2})|t-\frac{1}{2}|$$

and

$$x^*(t) = (t-\frac{1}{2})|t-\frac{1}{2}|.$$

Trivially, the optimal value of the cost functional is $J^* = 0$.

M	$\|\|E^u\|\|_\infty$	$\|\|E^x\|\|_\infty$	E^J	E^J [18]
2	3.4983E-05	6.0019E-05	3.2876E-09	5.4794E-05
3	3.8740E-06	5.8651E-06	1.4796E-11	1.1056E-06
4	1.2695E-06	1.9143E-06	1.5638E-12	8.2445E-08
5	5.3148E-07	7.9997E-07	2.7207E-13	1.7443E-08
6	2.5490E-07	3.9125E-07	6.4955E-14	3.2877E-09

Table 1: *Numerical results of Example 1*

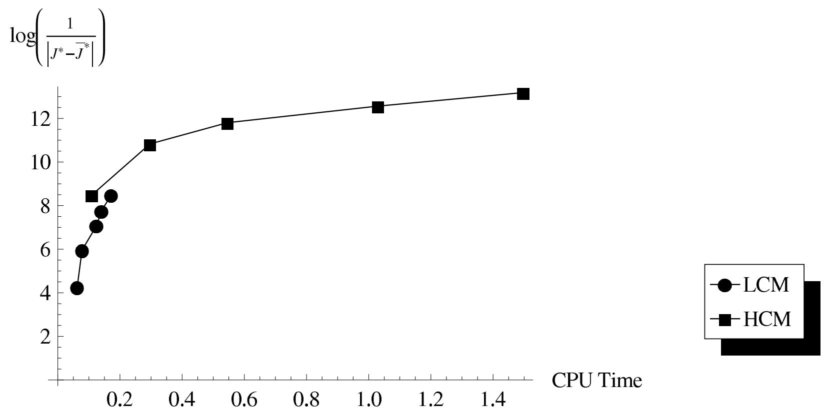

Figure 1: *Efficiency comparison of HCM and LCM for Example 1*

It should be mentioned that optimal control and state in this example belong to class C^1. Table 1 exhibits numerical results of this example by using HCM for different values of M and $N = 4$. In Table 1, one can compare the absolute errors in values of the objective function by utilizing HCM with those acquired from LCM. We used various values of M such as 2, 3, 4, 5 and 6 and obtained $\log_{10}\frac{1}{E^J}$ for each of the selective values of M in some CPU times and plotted Figure 1. The horizontal and vertical axes of this figure display CPU time and $\log_{10}\frac{1}{E^J}$, respectively. By adverting to Figure 1, we know that for the same CPU time in both graphs, HCM has better accuracy with respect to LCM. Although HCM requires more computational time, its absolute error is less.

5.2. Example 2

Consider the minimization of the functional

$$J = \int_0^1 (x(t) - \sin(t))^2 + (u(t) - t)^2 dt \tag{46}$$

subject to [5]

$$x(t) = y(t) + \int_0^t u(s)(x(s) + t)ds, \tag{47}$$

where

$$y(t) = t\cos(t) - \frac{1}{2}t^3.$$

The optimal value of the cost functional is $J^* = 0$. Optimal control $u^*(t)$ and the corresponding optimal state $x^*(t)$ are as follows:

$$\begin{cases} x^*(t) = \sin(t), \\ u^*(t) = t. \end{cases}$$

Solutions of this problem in this case are smooth.

M	$\|\|E^u\|\|_\infty$	$\|\|E^x\|\|_\infty$	E^J	E^J [18]
1	8.9932E-02	1.0318E-01	1.1054E-02	3.1560E-05
2	3.8865E-04	7.1106E-04	2.3296 E-07	5.9668E-06
3	7.9314E-05	1.0310E-04	5.6973E-09	4.9400E-08
4	2.2525E-06	3.0010E-06	3.3217E-12	4.2260E-11
5	1.1730E-07	8.5603E-08	6.8699E-15	1.9397E-13
6	1.1095E-09	4.0778E-10	1.2832E-17	8.2905E-17

Table 2: *Numerical results of Example 2*

Table 2 gives the results obtained by HCM and LCM. The approximate solutions (for $N = 2$ and $M = 5$) for both state and control functions together with the exact solutions are depicted in Figure 2. For comparing the efficiency of HCM and LCM proposed in [18], we plot Figure 3 in the same way as explained for Figure 1 in Example 1. In this figure, the horizontal axis is CPU time and the vertical axis is $Log_{10}\frac{1}{E^J}$. Although absolute errors for HCM listed in Table 2 have smaller values with respect to LCM, computing time for HCM is more. A sharp slope of LCM with respect to a low slope of HCM shows that LCM has better accuracy.

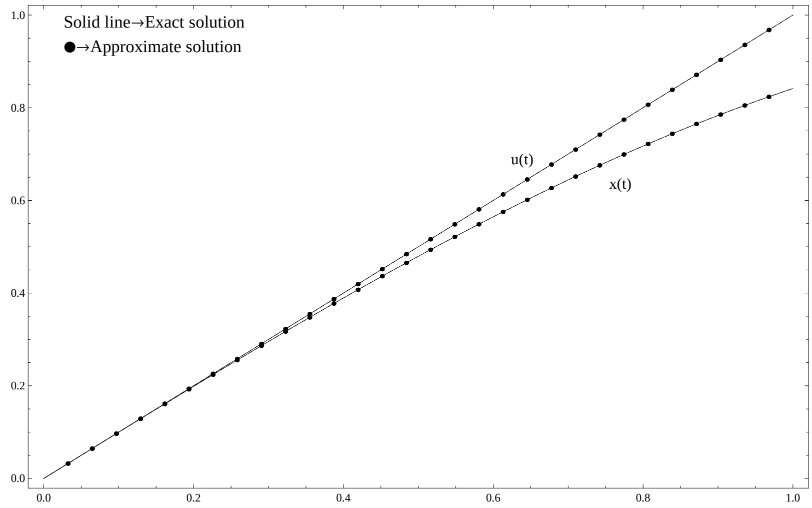

Figure 2: *The exact and approximate optimal control and state for Example 2*

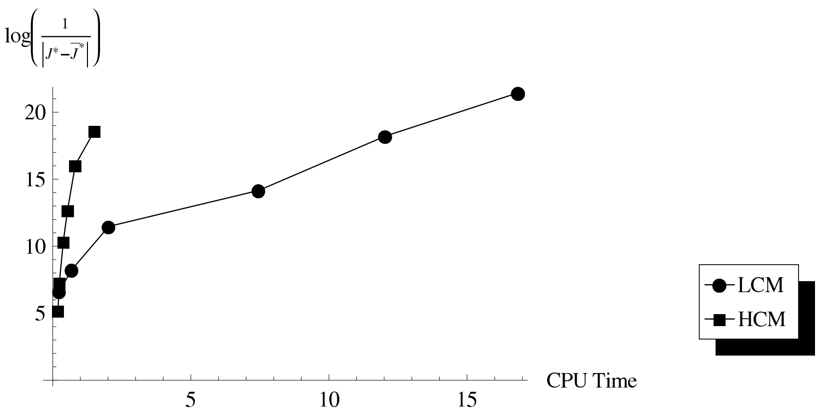

Figure 3: *Efficiency comparison of HCM and LCM for Example 2*

5.3. Example 3

Consider the minimization of the cost functional

$$J = \int_0^1 (x(t) - e^{-t^2})^2 + (u(t) - t)^2 dt$$

subject to the controlled Volterra integral

$$x(t) = y(t) - \int_0^t u(s)tx(s)ds,$$

where

$$y(t) = e^{-t^2} + \frac{t(1 - e^{-t^2})}{2}.$$

This problem has optimal solutions $u^*(t) = t$ and $x^*(t) = e^{-t^2}$. The results of solving this example by HCM and LCM are given in Table 3. Figure 4 shows the exact and approximate optimal control and state for $N = 2$ and $M = 5$. The absolute errors of examples 5.2 and 5.3 are compared with those obtained by the method [13] in Table 4, in which p is the number of rationalized Haar wavelets.

M	$\|\|E^u\|\|_\infty$	$\|\|E^x\|\|_\infty$	E^J	E^J [18]
1	2.2010E-03	8.6413E-03	4.9698E-05	1.3600E-03
2	3.2213E-04	1.6130E-03	9.6934E-07	5.8680E-06
3	1.2791E-04	4.6045E-04	6.4649E-08	5.2080E-07
4	6.1710E-06	1.9625E-05	2.0778E-10	1.1391E-09
5	3.3455E-06	4.5577E-06	8.5037E-12	1.1597E-10
6	4.0963E-06	3.0045E-07	3.4206E-14	2.0071E-13
7	1.8362E-08	3.4903E-08	2.0262E-16	1.5714E-14

Table 3: *Numerical results of Example 3*

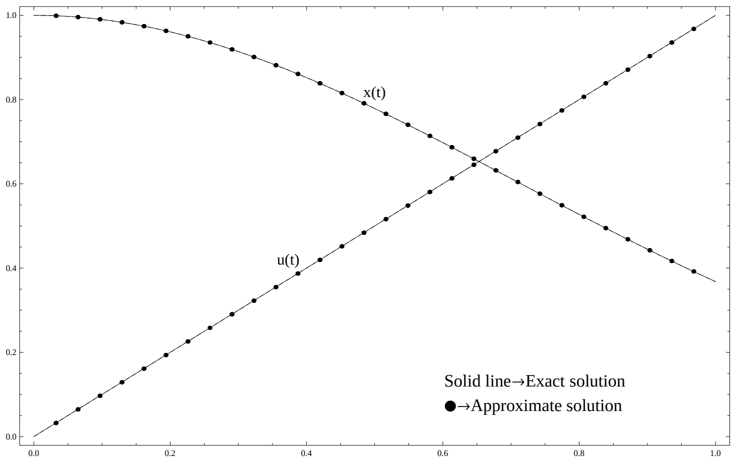

Figure 4: *The exact and approximate optimal control and state*

$Methods$	$E^J(Example2)$	$E^J(Example3)$
Method of [13]		
$p = 8$	$4.3214E-04$	$4.3033E-05$
$p = 16$	$1.1927E-04$	$1.2303E-05$
Present method		
$M = 4, N = 2$	$3.3217E-12$	$2.07781E-10$
$M = 8, N = 2$	$1.4554E-18$	$2.02621E-16$

Table 4: *Comparison of E^J for HCM and the method presented in [13]*

6. Conclusion

In this paper, we proposed an advanced numerical PS method for solving optimal control of Volterra integral equation by means of hybrid functions via the collocation method. The problem has been reduced to a finite dimensional parametric optimization and there exist many effective algorithms which can be applied to solve the NLP. Illustrative examples have shown the validity, applicability and efficiency of the proposed method especially for solutions belonging to class C^1 and C^2. The method is in the case of optimal control of systems governed by the VIE which is applicable in the field of practical science and engineering [13]. We believe that the proposed approach can be extended to solving optimal control of Volterra integro-differential systems by using an operational matrix of integration or derivative.

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