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Estimates of eigenvalues of a boundary value problem with a parameter*

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Abstract. We study an eigenvalue problem for the Laplace operator with a boundary condition containing a parameter. We estimate the rate of convergence of the eigenvalues to the eigenvalues of the Dirichlet problem for large positive values of the parameter.
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1. Introduction

We consider the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega, \tag{1}$$

$$\frac{\partial u}{\partial \nu} + \alpha\sigma(x)u = 0 \quad \text{on} \quad \Gamma, \tag{2}$$

where $\Omega \subset R^n$, $n \geq 2$, is a bounded domain with boundary $\Gamma = \partial\Omega \in C^2$. By ν we denote the outward unit normal vector to Γ, α is a real parameter. The function $\sigma(x) \in C^1(\Gamma)$ is positive:

$$0 < \sigma_0 \leq \sigma(x) \leq \sigma_1,\ \sigma_0 = \inf_{x\in\Gamma} \sigma(x) \text{ and } \sigma_1 = \sup_{x\in\Gamma} \sigma(x).$$

Problem (1), (2) with $\sigma(x) = 1$ is known as the Robin (Fourier) problem for $\alpha > 0$ (see [6, Ch. 7, Par. 7.2]), and the generalized Robin problem for all α ([5]).

There is a sequence of eigenvalues $\lambda_1(\alpha) < \lambda_2(\alpha) \leq \ldots$ of problem (1) - (2) enumerated according to their multiplicities with

$$\lim_{k\to\infty} \lambda_k(\alpha) = +\infty.$$

We also consider the sequence of eigenvalues $0 < \lambda_1^D < \lambda_2^D \leq \ldots$ of the Dirichlet eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in} \quad \Omega, \tag{3}$$

$$u = 0 \quad \text{on} \quad \Gamma, \tag{4}$$

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with

$$\lim_{k\to\infty} \lambda_k^D = +\infty.$$

Note that the eigenvalues $\lambda_1(\alpha)$ and λ_1^D are simple and the corresponding eigenfunctions $u_{1,\alpha}(x)$ and $u_1^D(x)$ are positive.

In this paper, we estimate $\lambda_k(\alpha)$ for large values of α. We now give some known results.

It is easy to see that $\lambda_k(\alpha) \le \lambda_k^D$, $k = 1, 2, \dots$. These inequalities give the upper bound of $\lambda_k(\alpha)$ for all values of α. It was announced in ([2, Ch. 6, Par. 2, No. 1]) that for $n = 2$ and a smooth boundary $\lim\limits_{\alpha\to+\infty} \lambda_k(\alpha) = \lambda_k^D$.

Later the properties of the first eigenvalue $\lambda_1(\alpha)$ were studied more precisely. Consider the case $\sigma(x) = 1$. The following two-sided estimates:

$$\lambda_1^D \left(1 + \frac{\lambda_1^D}{\alpha q_1}\right)^{-1} \le \lambda_1(\alpha) \le \lambda_1^D \left(1 + \frac{4\pi}{\alpha|\Gamma|}\right)^{-1}, \qquad \alpha > 0,$$

were obtained in [12] for $n = 2$. Here $|\Gamma|$ is the length of Γ and q_1 is the first eigenvalue of the Steklov problem

$$\Delta^2 u = 0 \quad \text{in} \quad \Omega,$$
$$u = 0, \quad \Delta u - q\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma.$$

In [4], for any $n \ge 2$ we establish the following asymptotic expansion:

$$\lambda_1(\alpha) = \lambda_1^D - \frac{\int_\Gamma \left(\frac{\partial u_1^D}{\partial \nu}\right)^2 ds}{\int_\Omega \left(u_1^D\right)^2 dx}\, \alpha^{-1} + o\left(\alpha^{-1}\right), \qquad \alpha \to +\infty.$$

The case $\alpha < 0$ has recently attracted attention (see, for instance, [9]). It was shown in [9] that for a piecewise-C^1 boundary

$$\liminf_{\alpha\to-\infty} \lambda_1(\alpha)/(-\alpha^2) \ge 1.$$

For C^1 boundaries it was proved ([10]) that

$$\lim_{\alpha\to-\infty} \lambda_1(\alpha)/(-\alpha^2) = 1.$$

The C^1-condition is optimal. In [9], the authors constructed plane triangle domains for which

$$\lim_{\alpha\to-\infty} \lambda_1(\alpha)/(-\alpha^2) > 1.$$

In [3], the authors proved that for C^1 boundaries

$$\lim_{\alpha\to-\infty} \frac{\lambda_k(\alpha)}{-\alpha^2} = 1 \tag{5}$$

for all $k = 1, 2, \dots$.

2. Main results

The main result of this paper reads as follows.

Theorem 1. *The eigenvalues $\lambda_k(\alpha)$, $k = 1, 2, \dots$, satisfy the estimates*

$$0 \leq \lambda_k^D - \lambda_k(\alpha) \leq C_1 \alpha^{-1/2} \left(\lambda_k^D\right)^2, \qquad \alpha > 0, \tag{6}$$

where the constant C_1 does not depend on k.

In the following theorem we gather the qualitative properties of eigenvalues of problem (1) - (2) (see also [2, Ch. 6] for i) and [9] for ii) and iii) for $\sigma(x) = 1$)

Theorem 2. *The eigenvalues have the following properties:*

i) $\lambda_k(\alpha)$, $k = 1, 2, \dots$, *are continuous functions of* α *and*

$$\lambda_k(\alpha_1) \leq \lambda_k(\alpha_2), \qquad \alpha_1 < \alpha_2; \tag{7}$$

ii) $\lambda_1(\alpha)$ *is a concave function of* α*:*

$$\lambda_1(\beta\alpha_1 + (1-\beta)\alpha_2) \geq \beta\lambda_1(\alpha_1) + (1-\beta)\lambda_1(\alpha_2), \qquad 0 < \beta < 1; \tag{8}$$

iii) $\lambda_1(\alpha)$ *is differentiable and*

$$\lambda_1'(\alpha) = \frac{\int_\Gamma \sigma u_{1,\alpha}^2 \, ds}{\int_\Omega u_{1,\alpha}^2 \, dx} > 0; \tag{9}$$

iv) the following estimate

$$\liminf_{\alpha \to -\infty} \frac{\lambda_1'(\alpha)}{-\alpha} \geq \sigma_1^2 \tag{10}$$

holds.

3. Operator treatment

In this section, we introduce two linear operators associated with problems (1) - (2) and (3) - (4) to derive the eigenvalue estimates (6).

Consider problem (1) - (2) in the space $H^1(\Omega)$ ([1, 11]). We define an eigenvalue of problem (1), (2) as a value λ for which there exists the non-zero function $u \in H^1(\Omega)$ satisfying the integral identity

$$\int_\Omega (\nabla u, \nabla v) \, dx + \alpha \int_\Gamma \sigma uv \, ds = \lambda \int_\Omega uv \, dx \tag{11}$$

for any $v \in H^1(\Omega)$. Relation (11) can be rewritten as

$$\int_\Omega ((\nabla u, \nabla v) + Muv) \, dx + \alpha \int_\Gamma \sigma uv \, ds = (\lambda + M) \int_\Omega uv \, dx, \quad M > 0. \tag{12}$$

Let us define an equivalent scalar product in the space $H^1(\Omega)$ by the formula

$$[u,v]_M = \int_\Omega ((\nabla u, \nabla v) + Muv)\,dx, \quad \|u\|_M^2 = [u,u]_M. \tag{13}$$

Now (12) transforms to

$$[u,v]_M + \alpha[Tu,v]_M = (\lambda + M)[Bu,v]_M,$$

where the linear self-adjoint non-negative operators $T : H^1(\Omega) \to H^1(\Omega)$ and $B : H^1(\Omega) \to H^1(\Omega)$ were defined by the bilinear forms

$$[Tu,v]_M = \int_\Gamma \sigma uv\,ds, \quad [Bu,v]_M = \int_\Omega uv\,dx, \quad u,v \in H^1(\Omega). \tag{14}$$

Hence we have an equation in the space $H^1(\Omega)$ with the norm $\|\cdot\|_M$:

$$(I + \alpha T)u = (\lambda + M)Bu. \tag{15}$$

Now we use the inequality ([11, Ch. 3, Par. 5, Formula 19])

$$\|v\|^2_{L_2(\Gamma)} \le \varepsilon \|\nabla v\|^2_{L_2(\Omega)} + C_\varepsilon \|v\|^2_{L_2(\Omega)}, \tag{16}$$

which is valid for $v \in H^1(\Omega)$ with an arbitrary $\varepsilon > 0$. Using (14), (16), we obtain

$$\begin{aligned}\|Tu\|_M^2 = [Tu,Tu]_M &= \int_\Gamma \sigma u Tu\,ds \le \sigma_1 \|u\|_{L_2(\Gamma)} \|Tu\|_{L_2(\Gamma)} \\ &\le \sigma_1 \varepsilon \left(\int_\Omega \left(|\nabla Tu|^2 + \frac{C_\varepsilon}{\varepsilon}(Tu)^2 \right) dx \right)^{1/2} \\ &\times \left(\int_\Omega \left(|\nabla u|^2 + \frac{C_\varepsilon}{\varepsilon} u^2 \right) dx \right)^{1/2} \le C_2 \varepsilon \|Tu\|_M \|u\|_M, \end{aligned} \tag{17}$$

where $\varepsilon > 0$, $M = M_\varepsilon$. It follows from (17) that

$$\|Tu\|_{M_\varepsilon} \le C_2 \varepsilon \|u\|_{M_\varepsilon},$$

and for any arbitrary small ε we have $\|\alpha T\|_{H^1(\Omega)\to H^1(\Omega)} < 1$ for $|\alpha| < 1/C_2\varepsilon$. Therefore, the inverse operator $(I + \alpha T)^{-1}$ is bounded and

$$\|(I + \alpha T)^{-1}\| \le (1 - |\alpha|\|T\|)^{-1}.$$

Hence, equation (15) is equivalent to

$$\left(I - (\lambda + M)(I + \alpha T)^{-1} B\right) u = 0.$$

The operator B is compact ([11, Ch. 3, Par. 5, Th. 3]) and the operator $(I + \alpha T)^{-1} B : H^1(\Omega) \to H^1(\Omega)$ is also compact. Hence the spectrum of problem (15) consists of real eigenvalues $\lambda_j(\alpha)$, $j = 1, 2, \ldots$, of finite multiplicity with the only limit point at the infinity. From (14), (15) we obtain the inequality

$$\lambda_j(\alpha) \ge -M_\varepsilon + (1 - |\alpha|\|T\|) \frac{\|u_{j,\alpha}\|^2_{M_\varepsilon}}{\|u_{j,\alpha}\|^2_{L_2(\Omega)}} \ge -M_\varepsilon$$

with the corresponding eigenfunction $u_{j,\alpha}$. Thus, $\lambda_j(\alpha) \to +\infty$, $j \to \infty$.

By the variational principle ([11, Ch. 4, Par. 1, No. 4]) we have

$$\lambda_k(\alpha) = \sup_{v_1,\dots,v_{k-1}\in L_2(\Omega)} \inf_{\substack{v \in H^1(\Omega)\\ (v,v_j)_{L_2(\Omega)}=0\\ j=1,\dots,k-1}} \frac{\int_\Omega |\nabla v|^2 dx + \alpha \int_\Gamma \sigma v^2 ds}{\int_\Omega v^2 dx}, \tag{18}$$

$$\lambda_k^D = \sup_{v_1,\dots,v_{k-1}\in L_2(\Omega)} \inf_{\substack{v \in \overset{\circ}{H}{}^1(\Omega)\\ (v,v_j)_{L_2(\Omega)}=0\\ j=1,\dots,k-1}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega v^2 dx}, \quad k = 1, 2, \dots. \tag{19}$$

To prove inequalities (6) we apply the following statement (see [6, Ch. 2, Th. 2.3.1]).

Theorem 3. *Let T_1 and T_2 be two linear self-adjoint, compact and positive operators on a separable Hilbert space H. Assume also that $\mu_k(T_1)$ and $\mu_k(T_2)$ are their k-th respective eigenvalues. Then*

$$|\mu_k(T_1) - \mu_k(T_2)| \le \|T_1 - T_2\|. \tag{20}$$

Now we give the proof of Theorem 1.

Proof. Consider the boundary value problem

$$-\Delta u + u = h \quad \text{in} \quad \Omega, \tag{21}$$

$$\frac{\partial u}{\partial \nu} + \alpha\sigma(x)u = 0 \quad \text{on} \quad \Gamma, \quad \alpha > 0, \tag{22}$$

with $h \in L_2(\Omega)$. A weak solution $u \in H^1(\Omega)$ of problem (21), (22) satisfy the integral identity

$$\int_\Omega ((\nabla u, \nabla v) + uv)dx + \alpha \int_\Gamma \sigma uv\, ds = \int_\Omega hv\, dx \tag{23}$$

for all $v \in H^1(\Omega)$. Let us define the scalar product in the space $H^1(\Omega)$ as

$$(u,v)_{H^1(\Omega),\alpha} = \int_\Omega ((\nabla u, \nabla v) + uv)dx + \alpha \int_\Gamma \sigma uv\, ds \tag{24}$$

and the corresponding norm by

$$\|u\|^2_{H^1(\Omega),\alpha} = (u,u)_{H^1(\Omega),\alpha}.$$

Due to (16), scalar product (24) is equivalent to the standard one

$$(u,v)_{H^1(\Omega)} = \int_\Omega ((\nabla u, \nabla v) + uv)dx. \tag{25}$$

Using (23), (24), we obtain the relation

$$(u,v)_{H^1(\Omega),\alpha} = (h,v)_{L_2(\Omega)}. \tag{26}$$

Hence, consider the linear functional

$$l_h(v) = (h, v)_{L_2(\Omega)}.$$

This functional is bounded on the space $H^1(\Omega)$:

$$|l_h(v)| \leq \|h\|_{L_2(\Omega)} \|v\|_{L_2(\Omega)}. \tag{27}$$

Now, by the Riesz lemma there exists the unique function $u \in H^1(\Omega)$ satisfying integral identity (23). Applying (26) with $v = u$, we obtain

$$\|u\|^2_{H^1(\Omega),\alpha} \leq \|h\|_{L_2(\Omega)} \|u\|_{H^1(\Omega),\alpha}.$$

Therefore,

$$\|u\|_{L_2(\Omega)} \leq \|u\|_{H^1(\Omega),\alpha} \leq \|h\|_{L_2(\Omega)}, \tag{28}$$

and we can define the bounded linear operator $A_\alpha : L_2(\Omega) \to L_2(\Omega)$ such that $u = A_\alpha h$ and $\|A_\alpha\| \leq 1$. Moreover, if the domain Ω with C^2 boundary is bounded, then the space $H^1(\Omega)$ embeds compactly into the space $L_2(\Omega)$ ([6, Ch. 1, Th. 1.1.1]). It means that the operator A_α is compact. Note that

$$\begin{aligned}
(h, A_\alpha g)_{L_2(\Omega)} &= \int_\Omega h A_\alpha g \, dx = \int_\Omega hv \, dx \\
&= \int_\Omega ((\nabla u, \nabla v) + uv) dx + \alpha \int_\Gamma \sigma uv \, ds \\
&= \int_\Omega ug \, dx = (A_\alpha h, g)_{L_2(\Omega)}, \qquad f, g \in L_2(\Omega),
\end{aligned} \tag{29}$$

with $u = A_\alpha h$, $v = A_\alpha g$, $u, v \in H^1(\Omega)$. Relation (29) means that A_α is a self-adjoint operator. Now, by relation (29) we have

$$\begin{aligned}
(h, A_\alpha h)_{L_2(\Omega)} &= \int_\Omega uh \, dx \\
&= \int_\Omega (|\nabla u|^2 + u^2) dx + \alpha \int_\Gamma \sigma u^2 \, ds = \|u\|^2_{H^1(\Omega),\alpha} > 0, \qquad h \neq 0.
\end{aligned}$$

Hence, the operator A_α is positive. Finally, A_α is a self-adjoint positive compact operator in the Hilbert space $H = L_2(\Omega)$. By the well-known theorem ([6, Ch. 1, Th. 1.2.1]), A_α has a sequence of eigenvalues $\{\mu_k(\alpha)\}$, $k = 1, 2, \ldots$ with finite multiplicities such that $\mu_k(\alpha) > 0$, $\mu_k(\alpha) \searrow 0$, $k \to \infty$. Let us denote by $u_{k,\alpha}(x) \in L_2(\Omega)$ the eigenfunction satisfying $A_\alpha u_{k,\alpha} = \mu_k(\alpha) u_{k,\alpha}$. Thus,

$$\mu_k(\alpha) \left(u_{k,\alpha}, v\right)_{H^1(\Omega),\alpha} = \left(u_{k,\alpha}, v\right)_{L_2(\Omega)}$$

and

$$\mu_k(\alpha) \left(\int_\Omega ((\nabla u_{k,\alpha}, \nabla v) + u_{k,\alpha} v) dx + \alpha \int_\Gamma \sigma u_{k,\alpha} v \, ds \right) = \int_\Omega u_{k,\alpha} v \, dx.$$

It can be seen that

$$\mu_k(\alpha) = \frac{1}{\lambda_k(\alpha) + 1}.$$

Let us note that for $\alpha > 0$ we have

$$\mu_k(\alpha) \le \frac{1}{\lambda_1(\alpha) + 1} < 1,$$

so $\|A_\alpha\| < 1$.

Furthermore, consider a Dirichlet problem

$$-\Delta y + y = h \quad \text{in} \quad \Omega, \tag{30}$$

$$y = 0 \quad \text{on} \quad \Gamma. \tag{31}$$

For $h \in L_2(\Omega)$ a weak solution $y \in \overset{o}{H}{}^1(\Omega)$ of problem (30), (31) satisfies the integral identity

$$\int_\Omega ((\nabla y, \nabla v) + yv)dx = \int_\Omega hv\, dx \tag{32}$$

for all $v \in \overset{o}{H}{}^1(\Omega)$. Define the scalar product in the space $\overset{o}{H}{}^1(\Omega)$ by (25). Using (25), (32), we obtain the relation

$$(y, v)_{\overset{o}{H}{}^1(\Omega)} = l_h(v). \tag{33}$$

Now, by (27) and the Riesz lemma there exists the unique function $y \in \overset{o}{H}{}^1(\Omega)$ satisfying integral identity (32). Using (32) with $v = y$, we obtain

$$\|y\|^2_{\overset{o}{H}{}^1(\Omega)} \le \|h\|_{L_2(\Omega)} \|y\|_{\overset{o}{H}{}^1(\Omega)}. \tag{34}$$

Therefore,

$$\|y\|_{L_2(\Omega)} \le \|y\|_{\overset{o}{H}{}^1(\Omega)} \le \|h\|_{L_2(\Omega)}, \tag{35}$$

and we can define the bounded linear operator $A^D : L_2(\Omega) \to L_2(\Omega)$ such that $u = A^D h$ and $\|A^D\| \le 1$. If the domain Ω is bounded, then the space $\overset{o}{H}{}^1(\Omega)$ embeds compactly into the space $L_2(\Omega)$ ([6, Ch. 1, Th. 1.1.1]). Hence, the operator A^D is compact. Note that

$$\begin{aligned}(h, A^D g)_{L_2(\Omega)} &= \int_\Omega h A^D g\, dx = \int_\Omega hv\, dx = \int_\Omega ((\nabla y, \nabla v) + yv)dx \\ &= \int_\Omega yg\, dx = (A^D h, g)_{L_2(\Omega)}, \quad f, g \in L_2(\Omega),\end{aligned} \tag{36}$$

with $y = A^D h$, $v = A^D g$, $y, v \in \overset{o}{H}{}^1(\Omega)$. Relation (36) means that A^D is a self-adjoint operator. Now, by (36) we have

$$(h, A^D h)_{L_2(\Omega)} = \int_\Omega yh\, dx = \int_\Omega (|\nabla y|^2 + y^2)dx = \|y\|^2_{\overset{o}{H}{}^1(\Omega)} > 0, \quad h \ne 0.$$

Hence, the operator A^D is positive. Finally, A^D is a self-adjoint positive compact operator in the Hilbert space $H = L_2(\Omega)$. By ([6, Ch. 1, Th. 1.2.1]), there exists a sequence of eigenvalues $\{\mu_k^D\}$, $k = 1, 2, \ldots$, of the operator A^D with finite multiplicities such that $\mu_k^D > 0$, $\mu_k^D \searrow 0$, $k \to \infty$. Denote by $y_k(x) \in L_2(\Omega)$ the respective eigenfunction satisfying $A^D y_k = \mu_k^D y_k$. Thus, $\mu_k^D (y_k, v)_{\overset{\circ}{H}{}^1(\Omega)} = (y_k, v)_{L_2(\Omega)}$ and

$$\mu_k^D \int_\Omega ((\nabla y_k, \nabla v) + y_k v) dx = \int_\Omega y_k v dx.$$

Then,

$$\mu_k^D = \frac{1}{\lambda_k^D + 1}.$$

Note that

$$\mu_k^D \le \frac{1}{\lambda_1^D + 1} < 1,$$

so $\|A^D\| < 1$.

Now we estimate the norm $\|A_\alpha - A^D\|_{L_2(\Omega)\to L_2(\Omega)}$ for large positive values of α.

Let us remind that in domains with C^2-class boundaries and positive $\sigma(x) \in C^1(\Gamma)$ the functions $u = A_\alpha h$ and $y = A^D h$ are strong solutions and belong to $H^2(\Omega)$ ([11, Ch. 4, Par. 2, Th. 4]). Moreover, the following estimate

$$\|y\|_{H^2(\Omega)} \le C_2 \|h\|_{L_2(\Omega)} \tag{37}$$

holds. Now we use estimate (16) with $\varepsilon = 1$:

$$\|y\|_{L_2(\Gamma)} \le C_3 \|y\|_{H^1(\Omega)}. \tag{38}$$

Combining (37) and (38) we derive the inequality

$$\|\nabla y\|_{L_2(\Gamma)} \le C_4 \|y\|_{H^2(\Omega)}. \tag{39}$$

Since $\left|\frac{\partial y}{\partial \nu}\right| \le |\nabla y|$ on Γ, from (37), (39) we obtain the estimate

$$\left\|\frac{\partial y}{\partial \nu}\right\|_{L_2(\Gamma)} \le C_5 \|h\|_{L_2(\Omega)}. \tag{40}$$

Suppose that $w = \left(A^D - A_\alpha\right) h$. By (21), (22), (30), (31) the function w is a solution of the boundary value problem

$$-\Delta w + w = 0 \quad \text{in} \quad \Omega, \tag{41}$$

$$\frac{\partial w}{\partial \nu} + \alpha \sigma w = \frac{\partial y}{\partial \nu} \quad \text{on} \quad \Gamma. \tag{42}$$

Multiplying equation (41) by w and integrating it over Ω with respect to boundary condition (42), we get the relation

$$\int_\Omega \left(|\nabla w|^2 + w^2\right) dx + \frac{1}{\alpha} \int_\Gamma \left(\frac{\partial w}{\partial \nu}\right)^2 \frac{ds}{\sigma} = \frac{1}{\alpha} \int_\Gamma \frac{\partial w}{\partial \nu} \frac{\partial y}{\partial \nu} \frac{ds}{\sigma}, \qquad \alpha > 0. \tag{43}$$

Then we obtain the inequality

$$\|w\|_{L_2(\Omega)}^2 + \frac{1}{\alpha}\left\|\frac{\partial w}{\partial \nu}\right\|_{L_2(\Gamma)}^2 \le \frac{C_6}{\alpha}\left\|\frac{\partial w}{\partial \nu}\right\|_{L_2(\Gamma)}\left\|\frac{\partial y}{\partial \nu}\right\|_{L_2(\Gamma)}$$

and, consequently,

$$\|w\|_{L_2(\Omega)}^2 + \frac{1}{\alpha}\left\|\frac{\partial w}{\partial \nu}\right\|_{L_2(\Gamma)}^2 \le \frac{1}{2\alpha}\left\|\frac{\partial w}{\partial \nu}\right\|_{L_2(\Gamma)}^2 + \frac{C_6^2}{2\alpha}\left\|\frac{\partial y}{\partial \nu}\right\|_{L_2(\Gamma)}^2.$$

Therefore, we have the estimate

$$\|w\|_{L_2(\Omega)} \le \frac{C_6}{\sqrt{2\alpha}}\left\|\frac{\partial y}{\partial \nu}\right\|_{L_2(\Gamma)}, \quad \alpha > 0. \tag{44}$$

Combining (44) with (40), we get

$$\|w\|_{L_2(\Omega)} \le C_7\alpha^{-1/2}\|h\|_{L_2(\Omega)}, \quad \alpha > 0,$$

with the constant C_6 independent of α. Thus, for all $h \in L_2(\Omega)$ we have the estimate

$$\left\|\left(A^D - A_\alpha\right)h\right\|_{L_2(\Omega)} \le C_7\alpha^{-1/2}\|h\|_{L_2(\Omega)}$$

and

$$\left\|A^D - A_\alpha\right\| \le C_7\alpha^{-1/2}, \quad \alpha > 0. \tag{45}$$

Now we apply (20) to the operators $T_1 = A_\alpha$, $T_2 = A^D$. Then, by the relations

$$\mu_k(\alpha) = \frac{1}{\lambda_k(\alpha)+1}, \quad \mu_k^D = \frac{1}{\lambda_k^D+1},$$

and inequalities (20), (45) we get the estimate

$$\left|\frac{1}{\lambda_k(\alpha)+1} - \frac{1}{\lambda_k^D+1}\right| \le C_7\alpha^{-1/2}. \tag{46}$$

Therefore,

$$\left|\lambda_k^D - \lambda_k(\alpha)\right| \le C_7\alpha^{-1/2}\left(\lambda_k^D+1\right)\left(\lambda_k(\alpha)+1\right). \tag{47}$$

and taking into account inequalities (49) (see Section 4), we obtain the estimate

$$0 \le \lambda_k^D - \lambda_k(\alpha) \le C_7\alpha^{-1/2}\left(\lambda_k^D+1\right)^2 \le C_1\alpha^{-1/2}\left(\lambda_k^D\right)^2. \tag{48}$$

The proof of Theorem 1 is completed. □

4. General properties of eigenvalues

In this Section, we give the proof of Theorem 2.

Proof. Due to (18), $\lambda_k(\cdot)$ is an increasing function. Using (19) and the inclusion $\overset{o}{H}{}^1(\Omega) \subset H^1(\Omega)$, we have

$$\begin{aligned}
\lambda_k(\alpha) &= \sup_{v_1,\ldots,v_{k-1}\in L_2(\Omega)} \inf_{\substack{v\in H^1(\Omega)\\ (v,v_j)_{L_2(\Omega)}=0\\ j=1,\ldots,k-1}} \frac{\int_\Omega |\nabla v|^2 dx + \alpha\int_\Gamma \sigma v^2 ds}{\int_\Omega v^2 dx}\\
&\le \sup_{v_1,\ldots,v_{k-1}\in L_2(\Omega)} \inf_{\substack{v\in \overset{o}{H}{}^1(\Omega)\\ (v,v_j)_{L_2(\Omega)}=0\\ j=1,\ldots,k-1}} \frac{\int_\Omega |\nabla v|^2 dx + \alpha\int_\Gamma \sigma v^2 ds}{\int_\Omega v^2 dx}\\
&= \sup_{v_1,\ldots,v_{k-1}\in L_2(\Omega)} \inf_{\substack{v\in \overset{o}{H}{}^1(\Omega)\\ (v,v_j)_{L_2(\Omega)}=0\\ j=1,\ldots,k-1}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega v^2 dx} = \lambda_k^D. \qquad (49)
\end{aligned}$$

The continuity of $\lambda_k(\alpha)$ was proved in ([2, Ch. 6, Par. 2, No. 6]).

Inequality (8) can be proved by the following:

$$\begin{aligned}
\lambda_1(\beta\alpha_1 + (1-\beta)\alpha_2) &= \inf_{v\in H^1(\Omega)} \frac{\int_\Omega |\nabla v|^2\,dx + (\beta\alpha_1 + (1-\beta)\alpha_2)\int_\Gamma \sigma v^2\,ds}{\int_\Omega v^2\,dx}\\
&\ge \beta \inf_{v\in H^1(\Omega)} \frac{\int_\Omega |\nabla v|^2\,dx + \alpha_1\int_\Gamma \sigma v^2\,ds}{\int_\Omega v^2\,dx}\\
&\quad + (1-\beta) \inf_{v\in H^1(\Omega)} \frac{\int_\Omega |\nabla v|^2\,dx + \alpha_2\int_\Gamma \sigma v^2\,ds}{\int_\Omega v^2\,dx}\\
&= \beta\lambda_1(\alpha_1) + (1-\beta)\lambda_1(\alpha_2), \qquad 0<\beta<1.
\end{aligned}$$

The eigenvalue $\lambda_1(\alpha)$ is simple for all $-\infty < \alpha < \infty$. The family of self-adjoint operators $(I+\alpha T)^{-1}B$ in the space $H^1(\Omega)$ with norm (13) satisfies the conditions of the asymptotic perturbation theorem ([7, Ch. 8, Par. 4, Th. 2.9]). It means that the eigenvalue $\lambda_1(\alpha)$ is a differentiable function of α. So

$$\lim_{j\to\infty} \frac{\lambda_1(\alpha_j) - \lambda_1(\alpha)}{\alpha_j - \alpha} = \lambda_1'(\alpha) \qquad (50)$$

for an arbitrary sequence $\alpha_j \to \alpha$, $j\to\infty$, $\alpha_j \ne \alpha$. Let $\alpha_j\to\alpha$, $j\to\infty$, and $\|u_{1,\alpha_j}\|_{L_2(\Omega)} = 1$, $u_{1,\alpha_j} \ge 0$. Therefore, $\|u_{1,\alpha_j}\|_{H^1(\Omega)} \le C_8$. By (11), the functions u_{1,α_j} satisfy

$$\int_\Omega (\nabla u_{1,\alpha_j}, \nabla v)\,dx + \alpha_j \int_\Gamma \sigma u_{1,\alpha_j} v\,ds = \lambda_1(\alpha_j)\int_\Omega u_{1,\alpha_j} v\,dx. \qquad (51)$$

Now, we can choose a subsequence $u_{1,\alpha_j} \rightharpoonup u$ weakly in $H^1(\Omega)$ and $\|u_{1,\alpha_j} - u\|_{L_2(\Omega)} \to 0$, $\|u_{1,\alpha_j} - u\|_{L_2(\Gamma)} \to 0$. It means that $u \geq 0$ and $\|u\|_{L_2(\Omega)} = 1$. Due to (51), u satisfies the integral identity

$$\int_\Omega (\nabla u, \nabla v)\, dx + \alpha \int_\Gamma \sigma uv\, ds = \lambda_1(\alpha) \int_\Omega uv\, dx. \tag{52}$$

Hence, by the uniqueness of the first positive normalized eigenfunction $u = u_{1,\alpha}$ and

$$\|u_{1,\alpha_j} - u_{1,\alpha}\|_{L_2(\Omega)} \to 0, \quad j \to \infty. \tag{53}$$

Now, we have

$$\begin{aligned} \int_\Omega |\nabla(u_{1,\alpha_j} - u_{1,\alpha})|^2 dx &+ \alpha \int_\Gamma \sigma(u_{1,\alpha_j} - u_{1,\alpha})^2 ds \\ &= \lambda_1(\alpha) \int_\Omega (u_{1,\alpha_j} - u_{1,\alpha})^2 dx \\ &\quad + (\lambda_1(\alpha_j) - \lambda_1(\alpha)) \int_\Omega u_{1,\alpha_j}(u_{1,\alpha_j} - u_{1,\alpha}) dx \\ &\quad - (\alpha_j - \alpha) \int_\Gamma \sigma u_{1,\alpha_j}(u_{1,\alpha_j} - u_{1,\alpha}) ds. \end{aligned} \tag{54}$$

It follows from (54) that

$$\begin{aligned} \|u_{1,\alpha_j} - u_{1,\alpha}\|^2_{H^1(\Omega)} \leq C_9 \Big(&|\alpha| \|u_{1,\alpha_j} - u_{1,\alpha}\|^2_{L_2(\Gamma)} \\ &+ (|\lambda_1(\alpha)| + 1) \|u_{1,\alpha_j} - u_{1,\alpha}\|^2_{L_2(\Omega)} \\ &+ |\lambda_1(\alpha_j) - \lambda_1(\alpha)|\, \|u_{1,\alpha_j} - u_{1,\alpha}\|_{L_2(\Omega)} \|u_{1,\alpha_j}\|_{L_2(\Omega)} \\ &+ |\alpha_j - \alpha|\, \|u_{1,\alpha_j} - u_{1,\alpha}\|_{L_2(\Gamma)} \|u_{1,\alpha_j}\|_{L_2(\Gamma)} \Big). \end{aligned} \tag{55}$$

Applying (50) and (16) with sufficiently small ε we obtain

$$\|u_{1,\alpha_j} - u_{1,\alpha}\|^2_{H^1(\Omega)} \leq C_{10} \left(\|u_{1,\alpha_j} - u_{1,\alpha}\|^2_{L_2(\Omega)} + (\alpha_j - \alpha)^2 \|u_{1,\alpha_j}\|^2_{H^1(\Omega)} \right). \tag{56}$$

Due to (16), (53) and (56) we get

$$\|u_{1,\alpha_j} - u_{1,\alpha}\|_{L_2(\Gamma)} \leq C_{11} \|u_{1,\alpha_j} - u_{1,\alpha}\|_{H^1(\Omega)} \to 0, \quad j \to \infty.$$

Therefore,

$$\int_\Gamma \sigma u^2_{1,\alpha_j} ds \to \int_\Gamma \sigma u^2_{1,\alpha} ds, \quad j \to \infty. \tag{57}$$

Now, to obtain (9) we use the inequalities

$$\begin{aligned} \lambda_1(\alpha_j) - \lambda_1(\alpha) &= \lambda_1(\alpha_j) - \inf_{v \in H^1(\Omega)} \frac{\int_\Omega |\nabla v|^2\, dx + \alpha \int_\Gamma \sigma v^2\, ds}{\int_\Omega v^2\, dx} \\ &\geq \lambda_1(\alpha_j) - \frac{\int_\Omega |\nabla u_{1,\alpha_j}|^2\, dx + \alpha \int_\Gamma \sigma u^2_{1,\alpha_j}\, ds}{\int_\Omega u^2_{1,\alpha_j}\, dx} = (\alpha_j - \alpha) \frac{\int_\Gamma \sigma u^2_{1,\alpha_j}\, ds}{\int_\Omega u^2_{1,\alpha_j}\, dx} \end{aligned}$$

and

$$\lambda_1(\alpha_j) - \lambda_1(\alpha) = \inf_{v \in H^1(\Omega)} \frac{\int_\Omega |\nabla v|^2\,dx + \alpha_j \int_\Gamma \sigma v^2\,ds}{\int_\Omega v^2\,dx} - \lambda_1(\alpha)$$
$$\le \frac{\int_\Omega |\nabla u_{1,\alpha}|^2\,dx + \alpha_j \int_\Gamma \sigma u_{1,\alpha}^2\,ds}{\int_\Omega u_{1,\alpha}^2\,dx} - \lambda_1(\alpha) = (\alpha_j - \alpha)\frac{\int_\Gamma \sigma u_{1,\alpha}^2\,ds}{\int_\Omega u_{1,\alpha}^2\,dx}\,.$$

Therefore, for $\alpha_j > \alpha$

$$\frac{\int_\Gamma \sigma u_{1,\alpha_j}^2\,ds}{\int_\Omega u_{1,\alpha_j}^2\,dx} \le \frac{\lambda_1(\alpha_j) - \lambda_1(\alpha)}{\alpha_j - \alpha} \le \frac{\int_\Gamma \sigma u_{1,\alpha}^2\,ds}{\int_\Omega u_{1,\alpha}^2\,dx}\,. \tag{58}$$

Finally, it follows from (50), (57) and (58) that

$$\lambda_1'(\alpha) = \frac{\int_\Gamma \sigma u_{1,\alpha}^2\,ds}{\int_\Omega u_{1,\alpha}^2\,dx}\,.$$

By ([11, Ch. 4, Par. 2, Th. 4]), $u_{1,\alpha} \in H^2(\Omega)$ and it satisfies equation (1) almost everywhere and the boundary condition in the sense of trace (the so-called strong solution). In the case $\int_\Gamma \sigma u_{1,\alpha}^2 ds = 0$, by (2) we have:

$$u_{1,\alpha} = \frac{\partial u_{1,\alpha}}{\partial \nu} = 0 \quad \text{on} \quad \Gamma.$$

Applying the uniqueness theorem to the Cauchy problem for second-order elliptic equations ([8, Ch. 1, Par. 3, Th. 1.46]), we get $u_{1,\alpha} = 0$ in Ω. This contradiction proves that $\lambda_1'(\alpha) > 0$ for all α. Taking into account (9), we have the inequality $\lambda_1(\alpha) < \lambda_1^D$.

By combining the result from [10] with (9) we obtain the relations

$$\alpha\lambda_1'(\alpha) = \frac{\alpha \int_\Gamma \sigma u_{1,\alpha}^2\,ds}{\int_\Omega u_{1,\alpha}^2\,dx} \le \frac{\int_\Omega |\nabla u_{1,\alpha}|^2\,dx + \alpha \int_\Gamma \sigma u_{1,\alpha}^2\,ds}{\int_\Omega u_{1,\alpha}^2\,dx}$$
$$= \lambda_1(\alpha) = -\alpha^2\sigma_1^2(1 + \varrho(\alpha)), \qquad \varrho(\alpha) \to 0, \quad \alpha \to -\infty.$$

Hence,

$$\frac{\lambda_1'(\alpha)}{-\alpha} \ge \sigma_1^2(1 + \varrho(\alpha)), \quad \alpha < 0,$$

and inequality (10) is proved.

This completes the proof of Theorem 2. □

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