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# On $\varepsilon$-uniform convergence of exponentially fitted methods*

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**Abstract.** A class of methods constructed to numerically approximate the solution of two-point singularly perturbed boundary value problems of the form $\varepsilon u'' + bu' + cu = f$ use exponentials to mimic exponential behavior of the solution in the boundary layer(s). We refer to them as exponentially fitted methods. Such methods are usually exact on polynomials of certain degree and some exponential functions. Shortly, they are exact on exponential sums. It is often possible that consistency of the method follows from the convergence of the interpolating function standing behind the method. Because of that, we consider the interpolation error for exponential sums. The main result of the paper is an error bound for interpolation by the exponential sum to the solution of the singularly perturbed problem that does not depend on perturbation parameter $\varepsilon$ when $\varepsilon$ is small with the respect to mesh width. The numerical experiment implies that the use of a dense mesh in the boundary layer for small meshwidth results in $\varepsilon$-uniform convergence.
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## 1. Introduction

The main objective of this paper is interpolation by the exponential sum of the form

$$s(x) = \sum_{\nu=0}^{k-2} \alpha_i x^i + \alpha_{k-1} \mathrm{e}^{-px/h}.$$

Parameter $p$ is a method defined parameter and it is not fitted to satisfy interpolation conditions. A more general approach to interpolation by exponential sums is considered in [1] and [9]. An interpolatory exponential sum is uniquely defined from $k$ interpolation conditions (cf. [1]).

In interpolation of sufficiently smooth function $u$ at $k$ equidistant points $t_1, \ldots, t_k$ ($t_{i+1} - t_i = h$), the interpolation error may be bounded [9] by

$$\| u - s \|_\infty \le C_1(p) h^{k-1} \left\| \frac{h}{p} u^{(k)} + u^{(k-1)} \right\|_\infty, \quad \text{and} \tag{1}$$

$$\| u - s \|_\infty \le C_2(p) h^{k} \left\| u^{(k)} + \frac{p}{h} u^{(k-1)} \right\|_\infty. \tag{2}$$

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Constants $C_1$ and $C_2$ depend on $p$, and they satisfy

$$\lim_{p\to\pm\infty} C_1(p) = C_1^* < \infty \quad \text{and} \quad \lim_{p\to 0} C_2(p) = C_2^* < \infty.$$

So, bound (1) is appropriate for large values of $p$, while bound (2) is applicable to the cases when $p$ is small or moderate. These results are consistent with interpolation error bounds for polynomials. This is not surprising, since the exponential sum approaches the interpolation polynomial of order $k$ when $p$ tends to 0, and the interpolation polynomial of order $k-1$ when $p$ tends to $\pm\infty$ (satisfying the first or the last $k-1$ interpolation conditions).

When $u$ does not depend on $p$, or when $u^{(k-1)}$ and $u^{(k)}$ are bounded independently on $p$, expression (1) gives a bound independent of $p$ (for $p$ large). But, there are situations where function $u$ depends on $p$. One example is the two-point singularly perturbed boundary value problem

$$\varepsilon u'' + bu' + cu = f \tag{3}$$

$$u(0) = 0, \quad u(1) = 0. \tag{4}$$

Coefficient $\varepsilon$, called a perturbation parameter, is positive and small with respect to $b$ or $c$ ($0 < \varepsilon \ll 1$). Further, we assume that $c$ satisfies $c(x) \le 0$ for all $x \in [0,1]$.

For $b \ne 0$, solution $u$ exhibits steep exponential behavior at the one end of interval $[0,1]$, or at both ends when $b \equiv 0$. This is known as boundary layer phenomena. For $b(x) < 0$, asymptotic behavior of the solution of the singularly perturbed problem (3) - (4) with respect to $\varepsilon$ is described by (cf. [11])

$$|u^{(l)}(x)| \le E_l \left[1 + \varepsilon^{-l} \exp\left(-b_{\min}\frac{1-x}{\varepsilon}\right)\right], \tag{5}$$

for $l = 0, 1, 2, \ldots$, where

$$b_{\min} = \min_{x\in[0,1]} |b(x)| \tag{6}$$

and $E_l$ are constants independent of $\varepsilon$ and $x$. For $b(x) > 0$, we obtain an analogous result by simple substitution $x \mapsto 1 - x$. Solution $u$ may be bounded in a similar way when $b \equiv 0$.

For small perturbation parameter $\varepsilon$ classical methods fail to give satisfactory approximations, unless a mesh size is unrealistically large. This is the reason why special methods are constructed for the singular perturbation problem. Best known are methods based on fitted meshes and exponentially fitted difference schemes (cf. [10, 11]). In this paper, we have in mind application to exponentially fitted methods. Despite a long history of such methods, there are just a few results concerning an approach by the interpolation error. Exponentially fitted difference schemes were introduced by Il'in in 1969 [4]. After that, this approach has been used in many papers to solve the singularly perturbed problem. There are, for example, also exponentially fitted finite elements methods [3, 13] and exponentially fitted splines (cf. [2]). An extensive overview of methods developed for singularly perturbed problems may be found in [5].

Error for interpolation by exponential sums to the solution of problem (3) - (4) is studied in [13, 14]. In [14], Zadorin considered interpolation by the exponential sum at two points. In [13], Stynes and O'Riordan used exponentially fitted linear splines in the finite elements method. The interpolation error for a linear spline is indeed an interpolation error for the exponential sum of order two. Our goal is to find error estimate for an arbitrary order of exponential sum, i.e., for interpolation at arbitrary number of points.

When $b(x) \neq 0$, parameter $p$ in the exponential part of approximation (i.e., in $\exp(-px/h)$) is chosen according to asymptotic expansion of the solution in the boundary layer:

$$p_i = h\frac{b(x_i)}{\varepsilon},$$

for some given point $x_i$. There are other possibilities. For example, in collocation by tension splines [8] parameter $p$ is defined as

$$p_i = h\frac{b(x_i) + \operatorname{sgn} b(x_i)\sqrt{b(x_i)^2 - 4\varepsilon c(x_i)}}{2\varepsilon}.$$

For self-adjoint problem ($b \equiv 0$), $p_i$ is defined by

$$p_i = \pm h\sqrt{\frac{-c(x_i)}{\varepsilon}},$$

and it is positive for $x_i \in [0, 1/2)$ and negative for $x_i \in [1/2, 1]$

Hence, parameter $p$ is proportional to $\varepsilon^{-1}$ or to $\sqrt{\varepsilon^{-1}}$ in a self-adjoint case. Since for $x = 1$ (5) gives that $|u^{(l)}(1)| \leq C(1+\varepsilon^{-l})$, error estimate (1) is not $\varepsilon$ independent or bounded for small $\varepsilon$.

An important property of methods for singularly perturbed problems is eventual independence of the convergence of perturbation parameter $\varepsilon$. Such convergence is called $\varepsilon$-uniform convergence. In other words, if we assume that a method is defined on a mesh $(x_i)_i$ with mesh width $h$, then a method is $\varepsilon$-uniform convergent if there exist constants $C$ and $m$, independent of $\varepsilon$, such that solution $u$ of problem (3) - (4) and its approximation at mesh points $u_i$ satisfy

$$\max_{\varepsilon \in [0,1]} |u(x_i) - u_i| \leq Ch^m$$

for all $i$. In this paper, we are going to prove weaker property of interpolation by a exponential sum. We will show that exponential sum $s$ that interpolates the solution of problem (3) - (4) at $k$ points satisfies

$$|u^{(r)}(x) - s^{(r)}(x)| \leq Ch^{k-1-r},$$

when mesh width $h$ satisfies $h \geq 4(k-1)\varepsilon \ln(1/\varepsilon)$. A numerical example will illustrate that the application of a dense mesh on interval $[0, 4(k-1)\varepsilon \ln(1/\varepsilon)]$ results in the same bound for small $h$.

## 2. Interpolation error bound

Now, we revisit results on the interpolation error bound from [9] and apply them to the solution of singularly perturbed boundary value problem (3) - (4).

**Theorem 1.** *Let $u$ be a solution of singularly perturbed boundary value problem (3) - (4), and let $(x_i)_0^n$ be an equidistant sequence of points ($0 = x_0 < x_1 < \ldots < x_n = 1$, $h = x_i - x_{i-1}$) for arbitrary chosen integer $n$ satisfying*

$$h \geq 4(k-1)\varepsilon \ln(1/\varepsilon)/b_{\min}, \tag{7}$$

*where constant $b_{\min}$ is defined by (6). Further, assume that*

1. *Functions $b$, $c$ and $f$ are sufficiently smooth such that $u \in C^k(0,1)$;*
2. *$b(x) \neq 0$ and $c(x) \leq 0$ for all $x \in [0,1]$;*
3. *Parameter $p$ from the exponential part of the exponential sum is of the same sign as function $b$ and satisfies $K \leq h/(|p|\varepsilon) \leq b_{\min}^{-1}$ for some positive constant $K$.*

*Then, the exponential sum $s$ of order $k$ ($k \geq 2$) that interpolates solution $u$ at $k$ consecutive mesh points (denoted by $(t_i)_1^k$) satisfies*

$$|u^{(r)}(x) - s^{(r)}(x)| \leq R h^{k-1-r}, \qquad r = 0, 1, \ldots, k-1,$$

*for all $x \in [t_1, t_{k-1}]$ when $b(x) < 0$ or all $x \in [t_2, t_k]$ when $b(x) > 0$. Constant $R$ is independent of $h$ and $\varepsilon$.*

*Further,*

$$\lim_{\varepsilon \to 0} |u^{(r)}(x) - s^{(r)}(x)| \leq R h^{k-1-r}, \qquad r = 0, 1, \ldots, k-1,$$

*for all $x \in [t_1, t_k)$ when $b(x) < 0$ or all $x \in (t_1, t_k]$ when $b(x) > 0$.*

**Proof**. Let $u$ be a solution of boundary value problem (3) - (4) and let $s$ be the exponential sum that interpolates $u$ at $k$ mesh points $t_1, \ldots, t_k$. Then, by $e$ we denote the error function

$$e := u - s.$$

To shorten the notation, we define a differential operator

$$(\mathcal{L}_k u)(x) := u^{(k)}(x) + \frac{p}{h} u^{(k-1)}(x).$$

Now, error function $e$ is the solution of the multipoint problem for ODE:

$$\mathcal{L}_k e = \mathcal{L}_k u, \tag{8}$$

$$e(t_j) = 0, \quad j = 1, \ldots, k. \tag{9}$$

In the proof, we consider a case when $b(x) < 0$. Substitution $1 - x \mapsto x$ simply extends results to the case when $b$ is positive. Hence, we consider the case with a boundary layer on the right-hand side of segment $[0,1]$ and negative parameter $p$.

First, we consider the case when $t_k < 1$. Function

$$e_P(t) := \int_{t_k}^{t} \int_{t_k}^{z_1} \ldots \int_{t_k}^{z_{k-2}} \int_{t_k}^{z_{k-1}} \mathrm{e}^{-p(z_{k-1} - z_k)/h} (\mathcal{L}_k u)(z_k) dz_k \ldots dz_1 \tag{10}$$

satisfies

$$\mathcal{L}_k e_P = \mathcal{L}_k u,$$

i.e., it is a particular integral for problem (8) - (9). The $r$-th derivative ($r = 0, \ldots, k-1$) of particular integral (10) is given by

$$e_P^{(r)}(t) = \int_{t_k}^{t} \int_{t_k}^{z_{r+1}} \cdots \int_{t_k}^{z_{k-2}} \int_{t_k}^{z_{k-1}} \mathrm{e}^{-p(z_{k-1}-z_k)/h} \left(\mathcal{L}_k u\right)(z_k) dz_k \ldots dz_{r+1}.$$

We start with the bound for $|\mathcal{L}_k u|$. Note that condition (7) is equivalent to

$$\mathrm{e}^{-b_{\min} h/\varepsilon} \le \varepsilon^{4(k-1)}.$$

Applying this to bound (5), we obtain

$$\exp\left(-b_{\min}\frac{1-z}{\varepsilon}\right) \le \exp\left(-b_{\min}\frac{1-t_k}{\varepsilon}\right) \le \exp\left(-b_{\min}\frac{h}{\varepsilon}\right) \le \varepsilon^{4(k-1)}, \qquad (11)$$

because of $z \le t_k < 1$, i.e., $z - 1 \le t_k - 1 \le -h$. Now, for $l = k-1, k$ and $C = \max\{E_{k-1}, E_k\}$, bound (5) reads

$$|u^{(l)}(z)| \le C(1 + \varepsilon^{-l+4(k-1)}) \le 2C, \qquad (12)$$

and

$$\left|\frac{h}{p}\left(\mathcal{L}_k u\right)(z)\right| = \left|\frac{h}{p} u^{(k)}(z) + u^{(k-1)}(z)\right| \le 2C\frac{h}{|p|} + 2C \le 2C(b_{\min}^{-1}\varepsilon + 1) \le \tilde{C}, \quad (13)$$

for some constant $\tilde{C}$ independent of $\varepsilon$.

For $t \le t_k$, $z_{k-1} \le t_k$ holds and

$$\left|\int_{t_k}^{z_{k-1}} \frac{p}{h} \mathrm{e}^{-p(z_{k-1}-z_k)/h} dz_k\right| = 1 - \mathrm{e}^{-p(z_{k-1}-t_k)/h} \le 1.$$

So, since $t_k < 1$, (13) implies

$$\left|\int_{t_k}^{z_{k-1}} \mathrm{e}^{-p(z_{k-1}-z_k)/h} \left(\mathcal{L}_k u\right)(z_k) dz_k\right| \le \tilde{C}$$

and

$$\begin{aligned} \left|e_P^{(r)}(t)\right| &\le \tilde{C} \left|\int_{t_k}^{t} \int_{t_k}^{z_{r+1}} \cdots \int_{t_k}^{z_{k-2}} dz_{k-1} \ldots dz_{r+1}\right| \\ &\le \tilde{C} \frac{|t-t_k|^{k-1-r}}{(k-1-r)!} \le C_r h^{k-1-r}, \end{aligned} \qquad (14)$$

where $C_r$ is a constant independent of $h$.

In the next step, we consider the homogeneous problem

$$\begin{aligned} \mathcal{L}_k e_H &= 0 \\ e_H(t_j) &= -e_P(t_j), \qquad j = 1, \ldots, k. \end{aligned} \qquad (15)$$

Solution $e_H$ is an exponential sum that interpolates $-e_P$, so it can be written as

$$e_H(t) = \sum_{\nu=1}^{k-1} \alpha_\nu \frac{1}{(\nu-1)!} \left(\frac{t-t_1}{h}\right)^{\nu-1} + \alpha_k \mathrm{e}^{-p(t-t_k)/h}. \tag{16}$$

Coefficients $\alpha_\nu$ are determined by interpolation conditions (15). Therefore, they are the solution of equation

$$A\alpha = \mathbf{b}, \tag{17}$$

where $a_{j\,\nu} = (j-1)^{\nu-1}/(\nu-1)!$, $\nu = 1,\dots,k-1$, $a_{j\,k} = \exp(-p(t_j - t_k)/h)$ and $b_j = -e_P(t_j)$ for $j = 1,\dots,k$.

Limit $\lim_{\varepsilon\to 0} A = \lim_{p\to-\infty} A$ exists and

$$\bar{A} := \lim_{p\to-\infty} A = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{c}^T & 1 \end{bmatrix}.$$

$B$ is a Vandermonde matrix defined by points $0,\dots,k-2$, while

$$c_\nu = \frac{(k-1)^{\nu-1}}{(\nu-1)!}.$$

Therefore, $\bar{A}$ is regular:

$$\bar{A}^{-1} = \begin{bmatrix} B^{-1} & \mathbf{0} \\ -\mathbf{c}^T B^{-1} & 1 \end{bmatrix}.$$

Since $A$ is regular for all $\varepsilon \in [0,1]$, matrix $A^{-1}$ is bounded independently of $\varepsilon$ ($\| A^{-1} \|_\infty \le K_1$). The bound does not depend on $h$, because matrix $A$ does not depend on $h$ either.

Differentiation of (16) yields

$$|e_H^{(r)}(t)| \le \|\alpha\|_\infty \frac{1}{h^r} \left( \sum_{\nu=1}^{k-1-r} \frac{[(t-t_1)/h]^{\nu-1}}{(\nu-1)!} + |p|^r \mathrm{e}^{-p(t-t_k)/h} \right). \tag{18}$$

For $t \le t_{k-1}$, $0 \ge t - t_k \ge t_{k-1} - t_k = -h$ holds. Now, bounds on $p$ give

$$\frac{hb_{\min}}{\varepsilon} \le |p| = -p \le \frac{h}{K\varepsilon}.$$

Therefore, for $r = 0,1,\dots,k-1$

$$|p|^r \mathrm{e}^{-p(t-t_k)/h} \le \frac{h^r}{K^r}\varepsilon^{-r}\mathrm{e}^{-b_{\min}h/\varepsilon} \le \frac{h^r}{K^r}\varepsilon^{4(k-1)-r} \le \frac{1}{K^r},$$

because of $4(k-1) - r \ge 3(k-1) \ge 0$.

Hence, from

$$\|\alpha\|_\infty \le \|A^{-1}\|_\infty \|\mathbf{b}\|_\infty$$

and

$$\|\mathbf{b}\|_\infty = \max_i |e_P(t_i)| \le C_0 h^{k-1},$$

we obtain that

$$|e_H^{(r)}(t)| \le C_0 K_1 \left( \sum_{\nu=1}^{k-1-r} \frac{(k-1)^{\nu-1}}{(\nu-1)!} + \frac{1}{K^r} \right) h^{k-1-r}, \tag{19}$$

for $t \le t_{k-1}$.

The bound for the error function follows from $e = e_P + e_H$.

Note that the above result also covers the limiting case when $\varepsilon \to 0$ for $t \in [t_1, t_{k-1}]$. When $t \in (t_{k-1}, t_k)$, the bound for $e_P^{(r)}$ does not depend on $\varepsilon$, so the same bound holds for $\varepsilon \to 0$. Since $\|\alpha\|_\infty$ is also bounded independently of $\varepsilon$, from (18) it follows that $e_H^{(r)}$ is bounded by

$$\lim_{\varepsilon \to 0} |e_H^{(r)}(t)| \le C_0 K_1 \left( \sum_{\nu=1}^{k-1-r} \frac{(k-1)^{\nu-1}}{(\nu-1)!} \right) h^{k-1-r}, \tag{20}$$

for $t < t_k$.

When $s$ interpolates $u$ at $t_k = 1$, we first consider behavior of the error in interval $[t_1, t_{k-1}]$. Let us define $X = (t_{k-1} + t_k)/2$. We define a particular integral as

$$e_P(t) := \int_X^t \int_X^{z_1} \dots \int_X^{z_{k-2}} \int_X^{z_{k-1}} \mathrm{e}^{-p(z_{k-1}-z_k)/h} \left(\mathcal{L}_k u\right)(z_k) dz_k \dots dz_1. \tag{21}$$

As in (11), taking into account that $1 - X = h/2$, we obtain

$$\exp\left(-b_{\min} \frac{1-z}{\varepsilon}\right) \le \exp\left(-b_{\min} \frac{1-X}{\varepsilon}\right) = \exp\left(-b_{\min} \frac{h}{2\varepsilon}\right) \le \varepsilon^{2(k-1)}.$$

Now bounds (12) and, consequently, (13) hold. Since $t \le z_{k-1} \le X$, we obtain that

$$\left| \int_X^{z_{k-1}} \frac{p}{h} \mathrm{e}^{-p(z_{k-1}-z_k)/h} dz_k \right| = 1 - \mathrm{e}^{-p(z_{k-1}-X)/h} \le 1.$$

Similar argumentation as in (14) yields

$$\left| e_P^{(r)}(t) \right| \le C_r h^{k-1-r},$$

when $t \in [t_1, t_{k-1}]$.

When $t = t_k = 1$, for $z \in [X, 1]$ we obtain

$$|u^{(l)}(z)| \le C \left[ 1 + \varepsilon^{-l} \exp\left(-b_{\min} \frac{1-z}{\varepsilon}\right) \right] \le C\left(1 + \varepsilon^{-l}\right),$$

for $l = k-1, k$ and $C = \max\{E_{k-1}, E_k\}$. Since $h/(|p|\varepsilon)$ is bounded by $b_{\min}^{-1}$ and $\varepsilon \le 1$, the bound for $\mathcal{L}_k u$ is given by

$$\begin{aligned}
\left| \frac{h}{p} \left(\mathcal{L}_k u\right)(z) \right| &= \left| \frac{h}{p} u^{(k)}(z) + u^{(k-1)}(z) \right| \\
&\le C \left[ \frac{h}{p} + \frac{h}{p} \varepsilon^{-k} + 1 + \varepsilon^{-k+1} \right] \\
&\le C \left[ b_{\min}^{-1} \varepsilon + b_{\min}^{-1} \varepsilon\, \varepsilon^{-k} + 1 + \varepsilon^{-k+1} \right] \\
&\le 2C\left(b_{\min}^{-1} + 1\right) \varepsilon^{-k+1} =: \bar{C} \varepsilon^{-k+1}.
\end{aligned}$$

Taking into account that

$$\left|\int_X^{z_{k-1}} \frac{p}{h} \mathrm{e}^{-p(z_{k-1}-z_k)/h} dz_k\right| = \mathrm{e}^{-p(z_{k-1}-X)/h} - 1 \le \mathrm{e}^{-p(1-X)/h} = \mathrm{e}^{-p/2},$$

we obtain that

$$\begin{aligned}|e_P^{(r)}(t_k)| &\le \bar{C}\varepsilon^{-k+1}\mathrm{e}^{-p/2}\int_X^t\int_X^{z_{r+1}}\cdots\int_X^{z_{k-2}} dz_{k-1}\dots dz_{r+1}\\ &\le \bar{C}_r\varepsilon^{-k+1}\mathrm{e}^{-p/2}h^{k-1-r}.\end{aligned}$$

For $t_k = 1$ we modify function $e_H$ given by (16):

$$e_H(t) = \sum_{\nu=1}^{k-1}\alpha_\nu \frac{1}{(\nu-1)!}\left(\frac{t-t_1}{h}\right)^\nu + \alpha_k\varepsilon^{-k+1}\mathrm{e}^{-p/2}\mathrm{e}^{-p(t-t_k)/h}. \tag{22}$$

Interpolation conditions (4) give a system similar to (17) with the only difference in the last equation due to a different definition of exponential sum $e_H$. But, after dividing the last equation by $\varepsilon^{-k+1}\mathrm{e}^{-p/2}$, we obtain

$$A_1\alpha = \mathbf{c},$$

where matrix $A_1$ is similar to $A$ from (17). The difference is in the last column:

$$a_{j\,k}^{(1)} = \varepsilon^{-k+1}\mathrm{e}^{-p/2}\mathrm{e}^{-p(t_j-t_k)/h},$$

for $j = 1,\dots,k-1$. Because of the normalization, $a_{k\,k}^{(1)} = 1$, vector $\mathbf{c}$ also differs from vector $\mathbf{b}$ in the last entry:

$$c_k = -e_P(t_k)/(\varepsilon^{-k+1}\mathrm{e}^{-p/2}).$$

Note that

$$|c_k| \le \max\{\bar{C}_0, C_0\}h^{k-1-r}.$$

Matrix

$$\bar{A}_1 := \lim_{p\to-\infty} A_1 = \begin{bmatrix} B & \mathbf{0}\\ \mathbf{0}^T & 1\end{bmatrix}$$

is regular. As in the case for $t_k < 1$, we conclude that there exists constant $K_2$ independent of $\varepsilon$ such that $\| \bar{A} \|_\infty \le K_2$.

Taking into account that

$$\begin{aligned}\left|\frac{d^r}{dt^r}\varepsilon^{-k+1}\mathrm{e}^{-p/2}\mathrm{e}^{-p(t-t_k)/h}\right| &= |\alpha_k|\,\varepsilon^{-k+1}\left|\frac{p}{h}\right|^r \mathrm{e}^{-p/2}\mathrm{e}^{-p(t-t_k)/h}\\ &\le |\alpha_k|\,\varepsilon^{-k+1}K^{-r}\varepsilon^{-r}\mathrm{e}^{-p/2}\mathrm{e}^{-p(t_{k-1}-t_k)/h}\\ &= |\alpha_k|\,K^{-r}\varepsilon^{-k+1-r}\mathrm{e}^{p/2}\\ &\le |\alpha_k|\,K^{-r}\varepsilon^{-k+1-r}\mathrm{e}^{-hb_{min}/(2\varepsilon)}\\ &\le |\alpha_k|\,K^{-r}\varepsilon^{-k+1-r+2(k-1)}\\ &\le |\alpha_k|\,K^{-r},\end{aligned}$$

for $t \leq t_{k-1}$, we easily obtain that bound (19) is valid in this case, too. The bound for $e^{(r)}$ follows from $e = e_P + e_H$, again.

Limit behavior of the error on the interval $(t_{k-1}, t_k)$ for $t_k = 1$ is analyzed in a similar way. Instead of $X = (t_{k-1} + t_k)/2$, for given $t \in (t_{k-1}, t_k)$ we choose an arbitrary $X$ satisfying $t_{k-1} < X < t_k = 1$. With $X$ defined in this way, we use particular integral (21). Since for $t < z < X$

$$\exp\left(-b_{\min}\frac{1-z}{\varepsilon}\right) \leq \exp\left(-b_{\min}\frac{1-X}{\varepsilon}\right),$$

it follows that

$$\lim_{\varepsilon\to 0}\left|\frac{h}{p}u^{(k)}(z)\right| = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0}\left|u^{(k-1)}(z)\right| \leq C.$$

Therefore,

$$\lim_{\varepsilon\to 0}\left|\frac{h}{p}\left(\mathcal{L}_k u\right)(z)\right| \leq C.$$

Now, it can be easily verified that

$$\lim_{\varepsilon\to 0}\left|e_P^{(r)}(t)|\right| \leq C_r h^{k-1-r}.$$

For $t > X$ we obtain

$$\left|\int_X^{z_{k-1}} \frac{p}{h}\mathrm{e}^{-p(z_{k-1}-z_k)/h} dz_k\right| \leq \mathrm{e}^{-p(1-X)/h},$$

and

$$|e_P^{(r)}(t)| \leq \tilde{C}\varepsilon^{-k+1}\mathrm{e}^{-p(1-X)/h}h^{k-1-r}.$$

The solution of homogenous problem $e_H$ is modified according to asymptotic behavior, similarly to (22):

$$e_H(t) = \sum_{\nu=1}^{k-1}\alpha_\nu\frac{1}{(\nu-1)!}\left(\frac{t-t_1}{h}\right)^\nu + \alpha_k\varepsilon^{-k+1}\mathrm{e}^{-p(1-X)/h}\mathrm{e}^{-p(t-t_k)/h}.$$

Applying the same argumentation as before, we obtain

$$\lim_{\varepsilon\to 0}\left|e_H^{(r)}(t)|\right| \leq \tilde{\tilde{C}}h^{k-1-r}$$

for $t < x$. Since $t$ is arbitrarily chosen, the result is valid for all $t < t_k = 1$. $\square$

**Remark 1.** *The result of Theorem 1 can be easily extended to a non-equidistant mesh. One should define $h_i = x_{i+1} - x_i$ and $h = \max_i h_i$. Instead of $h$ in (7), there would be $h_i$. Also, in assumption 3, from Theorem, $h_i$ should be used, while all bounds stand with term $h$. A proof of this more general case is essentially the same, but with more technical details because of a more complex structure of the mesh.*

An assumption on $p$ in Theorem 1 may also be weakened. Namely, there is no need to impose restriction 3 when all interpolation points lie outside the boundary layer. But, this would lead to a more extensive proof without practical benefit since outside the boundary layer exponential fitting is used as stated in the theorem or it is not used at all, i.e., polynomials are used. Therefore, in the following corollary we comment interpolation by polynomials outside the boundary layer.

**Corollary 1.** *Let functions $b$, $c$ and $f$ satisfy conditions of Theorem 1 and let $u$ be a solution of singularly perturbed boundary value problem (3) - (4). Further, for given sequence of points $(t_i)_1^k$, ($0 \leq t_1 < \ldots < t_k \leq 1$, $h_i = t_{i+1} - t_i$, $h = \max_i h_i$) satisfying*

$$1 - t_k \geq k\varepsilon \ln(1/\varepsilon)/b_{\min} \tag{23}$$

*when $b(x) < 0$ or*

$$t_1 \geq k\varepsilon \ln(1/\varepsilon)/b_{\min} \tag{24}$$

*when $b(x) > 0$, let $P$ be a polynomial of order $k$ ($k \geq 2$) that interpolates solution $u$ at $(t_i)_1^k$. Then there exists some constant $R$, independent of $h$ and $\varepsilon$, that the following estimate holds*

$$|u^{(r)}(x) - p^{(r)}(x)| \leq Rh^{k-r} \qquad r = 0, 1, \ldots, k-1,$$

*for all $x \in [t_1, t_k]$.*

**Proof**. Just note that condition (23) applied to (5) implies $|u^{(k)}(z)| \leq 2E_k$, for $z \in [t_1, t_k]$. Now, standard results for the interpolation error for polynomials give the assertion of the corollary. □

**Remark 2.** *If $p$ is chosen as $p = b(0)h/\varepsilon$ for $b(x) > 0$ or $p = b(1)h/\varepsilon$ for $b(x) < 0$, then condition (7) may be substituted by $h \geq 4(k-2)\varepsilon \ln(1/\varepsilon)/b_{\min}$. Namely, Kellogg and Tsan [6] proved that solution $u$ is of the form $u(x) = z(x) + \gamma \exp(b(1)(1-x)/\varepsilon)$ when $b(x) < 0$. Function $z$ satisfies*

$$|z^{(l)}(x)| \leq E_l \left[1 + \varepsilon^{1-l} \exp\left(-b_{\min} \frac{1-x}{\varepsilon}\right)\right].$$

*Since $(\mathcal{L}_k u) = (\mathcal{L}_k z)$, the above bound can be used in the proof instead of bound (5). Note that for $k = 2$ there is no restriction on $h$. This is in agreement with results from [14] where $\varepsilon$-uniform bound is obtained for interpolation in two points. The exponentially fitted finite elements method considered in [13] is also $\varepsilon$-uniform convergent.*

## 3. On $\varepsilon$-uniform convergence

To illustrate results from the paper, we choose the solution of singularly perturbed problem

$$\varepsilon u'' + 2(1-x)u' - 2u = (\varepsilon - 2x)\mathrm{e}^x,$$

with boundary conditions

$$u(0) = 2 \quad \text{and} \quad u(1) = \mathrm{e}^{-1/\varepsilon} + \mathrm{e}.$$

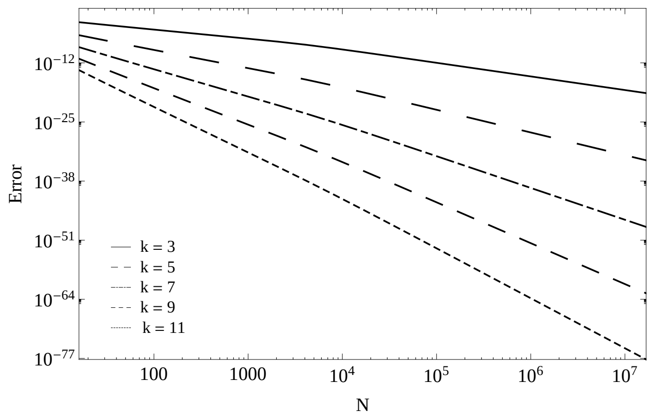


Figure 1: *Interpolation error to the solution of singularly perturbed problem (25) for exponential sums of different order k in dependence on mesh size $h = 1/N$ ($x_1 = 0.5$, perturbation parameter $\varepsilon = 10^{-4}$)*

Function

$$u(x) = \mathrm{e}^{-(2x-x^2)/\varepsilon} + \mathrm{e}^x \tag{25}$$

is a solution of the boundary value problem. This solution has a boundary layer near point $x = 0$.

The basic idea of the experiment is to divide interval $[0, 1]$ into $N$ equal parts and interpolate function $u$ at $k$ consecutive points. We choose one set of $k$ points outside the boundary layer, in the middle of considered interval: $x_1 = 0.5$. Another set of points, more interesting, is in the boundary layer: $x_1 = 0$. To determine the interpolation error we use a dense mesh on each of subintervals $[t_i, t_{i+1}]$.

We start with an analysis of the interpolation error outside the boundary layer, so we fix $x_1 = 0.5$. The interpolation error of the exponential sum for different mesh sizes $h = 1/N$ is shown in Figure 1. We present errors for methods given by $k = 3, 5, 7, 9, 11$. Initial behavior of the error is in accordance with Theorem 1, i.e., the error is of size $h^{k-1}$ when $h > \varepsilon$. For $h < \varepsilon$, order of convergence increases, as expected from (2). This may be clearly seen from Figure 2, where numerical orders of convergence are shown. The error is bounded by term $Ch^{k-1}$ independently of $\varepsilon$. This may be seen from Figure 3. There is shown error for the three-point method ($k = 3$) and for different values of perturbation parameter $\varepsilon$. All error curves are bounded by the curve for smallest value of $\varepsilon$ ($\varepsilon = 10^{-9}$).

Behavior of the interpolation error in the boundary layer is totally different, as may be seen in Figure 4. Behavior for large values of mesh width $h$ is described by Theorem 1. But, at the certain point, the error starts to increase. When $h < \varepsilon$, the error decreases with the order of convergence given by (2).

Such behavior is not unexpected. In [12], Shishkin proved that difference schemes on equidistant meshes can not be $\varepsilon$-uniform convergent. To resolve this problem, we computed, for different values of $\varepsilon$, transition point $\sigma$, the value of $h$ when the error starts to increase. This is done for a different order of exponential sums. The result is shown in Figure 5. It is evident that $\sigma = D_k \varepsilon \ln(1/\varepsilon)$ with constant $D_k$

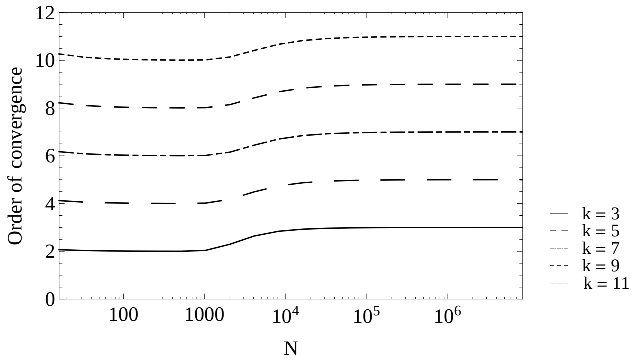


Figure 2: *Numerical order of convergence for interpolation by exponential sums of order $k$ ($x_1 = 0.5$, perturbation parameter $\varepsilon = 10^{-4}$)*

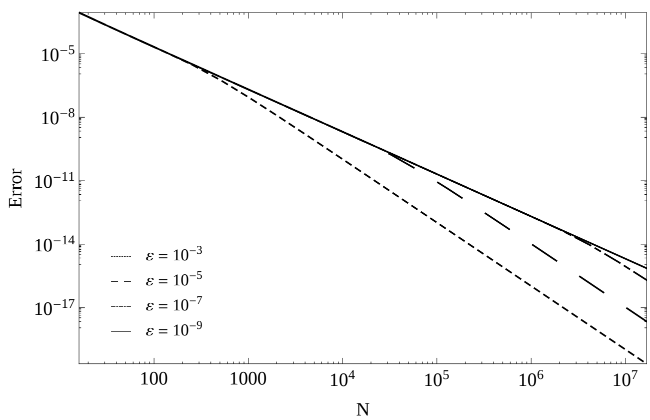


Figure 3: *Interpolation error to the solution of singularly perturbed problem (25) for exponential sum of different order 3 for different values of perturbation parameter $\varepsilon$ ($x_1 = 0.5$)*

independent of $\varepsilon$. Actually, in our example, $\sigma \approx 0.36(k-1)\varepsilon \ln(1/\varepsilon)$. This finding approves that the condition on mesh-width $h$ from Theorem 1 is quite sharp. Further investigation shows that anomaly in convergence is present only on interval $[0, \sigma^*]$, where

$$\sigma^* = (k-1)\varepsilon \ln(1/\varepsilon)/b_{\min}.$$

Since we consider interval $[0, (k-1)h]$, for the value of $b_{\min}$ we actually used $|b(0)|$. A similar definition of the transition point is used in the exponentially fitted finite elements method [13] with $\sigma^* = 2\varepsilon \ln(1/\varepsilon)/b_{\min}$. For definition of Shishkin mesh, $\sigma^* = C_p \varepsilon \ln(1/h)/b_{\min}$ is used, where constant $C_p$ satisfies $C_p \geq p$ for the methods of order $p$ (cf. [11, 7]). In our case, $p = k-1$. This definition of the transition point is also similar to our definition.

In order to resolve a problem with convergence, we start from the observation

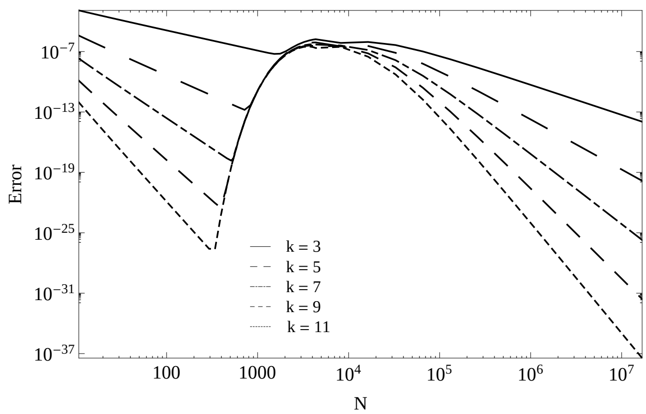


Figure 4: *Interpolation error to the solution of singularly perturbed problem (25) for the exponential sums of different order $k$ in dependence of mesh size $h = 1/N$ ($x_1 = 0$, perturbation parameter $\varepsilon = 10^{-4}$)*

that the error decreases for $h < \varepsilon$ and that the error is approximately the same for $h = \sigma$ and $h = \sigma^2$. So, we have to avoid usage of $h \in [\sigma^2, \sigma]$ in the boundary layer. A simple solution is to use a dense mesh on interval $[0, \sigma]$ for $h \leq \sigma$. A choice of an equidistant mesh of size $\mathcal{O}(N)$ will guarantee required properties. The obtained mesh is a Shiskin type mesh with a different transition point and a different mesh size in the boundary layer. For exponential sums of order 3, 5, 9, 7 and 11, we experimentally determined that good behavior of error is obtained by using of $1/16N$, $7/4N$, $13/2N$, $29/2N$ and $26N$ mesh points in the interval $[0, \sigma]$. The interpolation error for this strategy is shown in Figure 6.

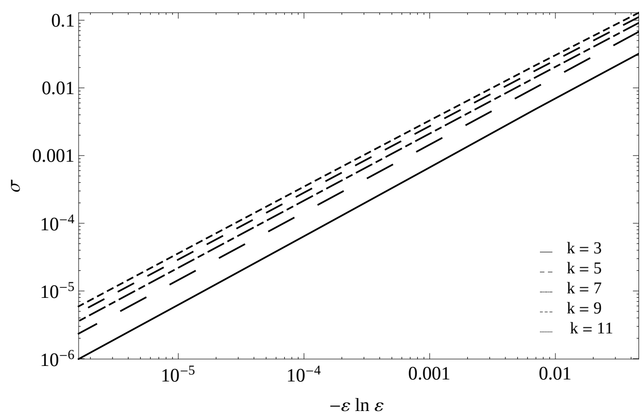


Figure 5: *Dependence of transition point $\sigma$ on perturbation parameter $\varepsilon$*

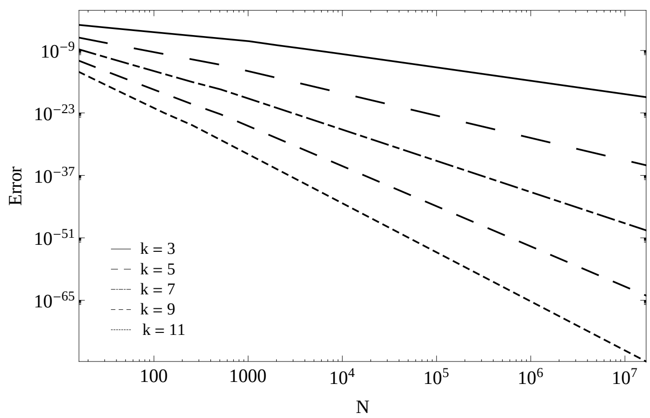


Figure 6: *Interpolation error to the solution of singularly perturbed problem (25) with a dense mesh in the boundary layer for exponential sums of different order $k$ in dependence on mesh size $h = 1/N$ ($x_1 = 0$, perturbation parameter $\varepsilon = 10^{-4}$)*

## 4. Conclusion

Among numerical methods for singularly perturbed problem (3) - (4), a special advantage is given to those that are $\varepsilon$-uniform convergent. For example, a lot of methods based on adapted meshes (such as a Shishkin mesh) are $\varepsilon$-uniform convergent. Exponentially fitted methods that are proved to be $\varepsilon$-uniform convergent are rare, although sometimes numerical evidence suggests the existence of $\varepsilon$-uniform convergence.

If the method is exact on polynomials of certain degree and some exponential functions, it is exact on exponential sums. In such case, it could be possible that the consistency of the method is connected to the convergence of the interpolating function standing behind the method. We hope that the main finding of this paper opens a possibility that much more exponentially fitted methods are $\varepsilon$-uniform convergent.

We prove that exponential sums give $\mathcal{O}(h^{k-1})$ approximation to the solution of the singularly perturbed boundary value problem when $h$ si relatively large ($h \geq 4(k-1)\varepsilon \ln(1/\varepsilon)$). A numerical experiment illustrates that this bound is sharp since the interpolation error starts to increase for smaller $h$. An application of the mesh that is dense in the boundary layer leads to the same rate of convergence for small $h$. A method defined in such way would be $\varepsilon$-uniform convergent. It is notable that the interpolation error is already small for $h = (k-1)\varepsilon \ln(1/\varepsilon)$. In our example, it ranges from $5.8 \cdot 10^{-8}$ for $k = 3$ to $2.4 \cdot 10^{-27}$ for $k = 11$. An application of small $h$ and the dense mesh is needed only if higher accuracy is required.

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