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Necessary p-th order optimality conditions for irregular Lagrange problem in calculus of variations

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Abstract. The paper is devoted to singular calculus of variations problems with constraints which are not regular mappings at the solution point, i.e., its derivatives are not surjective. We pursue an approach based on the constructions of the p-regularity theory. For p-regular calculus of the variations problem we present necessary conditions for optimality in a singular case and illustrate our results by a classical example of calculus of the variations problem.

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1. Introduction

We investigate the following Lagrange problem

$$J_0(x) = \int_{t_1}^{t_2} F(t, x(t), x'(t))dt \to \min \tag{1}$$

subject to

$$H(t, x(t), x'(t)) = 0, \ M_1 x(t_1) + M_2 x(t_2) = 0, \tag{2}$$

where $x \in \mathcal{C}_n^2[t_1, t_2]$, $H(t, x(t), x'(t)) = (H_1(t, x(t), x'(t)), \ldots, H_m(t, x(t), x'(t)))^T$,

$$H_i : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \qquad i = 1, \ldots, m,$$
$$F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

$t \in [t_1, t_2]$, M_1, M_2 are $n \times n$ matrices and $\mathcal{C}_n^l([t_1, t_2])$ are Banach spaces of n-dimensional l-times continuously differentiable vector functions with usual norms.

The system of equations (2) can be replaced by the following operator equation

$$G(x(\cdot)) = 0,$$

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where

$$\begin{aligned} G\ &:\ X \to Y, \\ X &= \{x(\cdot) \in \mathcal{C}_n^2[t_1,t_2] : M_1x(t_1) + M_2x(t_2) = 0\}, \\ Y &= \mathcal{C}_m[t_1,t_2] \end{aligned}$$

and

$$G(x)(\cdot) = H(\cdot, x(\cdot), x'(\cdot)).$$

Let us define

$$\begin{aligned} \lambda(t) &= (\lambda_1(t),\dots,\lambda_m(t))^T, \\ \lambda(t)H &= \lambda_1(t)H_1 + \cdots + \lambda_m(t)H_m, \\ \lambda(t)H_x &= \lambda_1(t)H_{1x} + \cdots + \lambda_m(t)H_{mx}, \\ \lambda(t)H_{x'} &= \lambda_1(t)H_{1x'} + \cdots + \lambda_m(t)H_{mx'}. \end{aligned}$$

If $\mathrm{Im}G'(\hat{x}) = Y$, where $\hat{x}(t)$ is a solution to (1) - (2), then necessary conditions of Euler-Lagrange

$$F_x + \lambda(t)H_x - \frac{d}{dt}(F_{x'} + \lambda(t)H_{x'}) = 0$$

hold. Here, F_x, H_x, $F_{x'}$, $H_{x'}$ are partial derivatives of $F(t,x(t),x'(t))$ and $H(t,x(t),x'(t))$ with respect to x and x', respectively.

In a *singular* (*nonregular*) case when $\mathrm{Im}G'(\hat{x}) \neq Y$, we can only guarantee that the following equations

$$\lambda_0F_x + \lambda(t)H_x - \frac{d}{dt}(\lambda_0F_{x'} + \lambda(t)H_{x'}) = 0 \tag{3}$$

hold, where $\lambda_0^2 + \|\lambda(t)\|^2 = 1$, i.e., λ_0 might be equal to 0, and then we have not constructive information of the functional $F(t,x(t),x'(t))$.

Example 1. *Consider the problem*

$$J_0(x) = \int_0^{2\pi} (x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t) + x_5^2(t))dt \to \min \tag{4}$$

subject to

$$\begin{aligned} &H(t,x(t),x'(t)) \\ &\quad= \begin{pmatrix} x_1'(t) - x_2(t) + x_3^p(t)x_1(t) + x_4^p(t)x_2(t) - x_5^p(t)(x_1(t)+x_2(t)) \\ x_2'(t) + x_1(t) + x_3^p(t)x_2(t) - x_4^p(t)x_1(t) - x_5^p(t)(x_2(t)-x_1(t)) \end{pmatrix} = 0, \end{aligned} \tag{5}$$

$x_i(0) - x_i(2\pi) = 0$, $i = 1,\dots,5$, $p \geq 2$. *Here*

$$F(t,x(t),x'(t)) = x_1^2(t) + x_2^2(t) + x_3^2(t) + x_4^2(t) + x_5^2(t), \quad M_1 = -M_2 = I_5,$$

where I_5 *is the unit matrix of size* 5 *and*

$$G(x) = \begin{pmatrix} x_1'(\cdot) - x_2(\cdot) + x_3^p(\cdot)x_1(\cdot) + x_4^p(\cdot)x_2(\cdot) - x_5^p(\cdot)(x_1(\cdot)+x_2(\cdot)) \\ x_2'(\cdot) + x_1(\cdot) + x_3^p(\cdot)x_2(\cdot) - x_4^p(\cdot)x_1(\cdot) - x_5^p(\cdot)(x_2(\cdot)-x_1(\cdot)) \end{pmatrix} = 0.$$

The solution of (1) - (2) *is* $\hat{x}(t) = 0$ *and* $G'(0)$ *is singular.*

The corresponding Euler-Lagrange system of equations (see (3)*) in this case is as follows:*

$$
\begin{aligned}
2\lambda_0 x_1 + \lambda_2 - \lambda_1' + \lambda_1 x_3^p - \lambda_1 x_5^p + \lambda_2 x_5^p - \lambda_2 x_4^p &= 0\\
2\lambda_0 x_2 - \lambda_1 - \lambda_2' + \lambda_1 x_4^p + \lambda_2 x_3^p - \lambda_1 x_5^p - \lambda_2 x_5^p &= 0\\
2\lambda_0 x_3 + p\lambda_1 x_1 x_3^{p-1} + p\lambda_2 x_2 x_3^{p-1} &= 0\\
2\lambda_0 x_4 + p\lambda_1 x_2 x_4^{p-1} - p\lambda_2 x_1 x_4^{p-1} &= 0\\
2\lambda_0 x_5 - p\lambda_1 x_5^{p-1} x_1 - p\lambda_1 x_2 x_5^{p-1} - p\lambda_2 x_2 x_5^{p-1} + p\lambda_2 x_1 x_5^{p-1} &= 0\\
\lambda_i(0) - \lambda_i(2\pi) = 0, \qquad i = 1, 2.
\end{aligned} \tag{6}
$$

(to simplify formulas we omit dependence of t here and further in the paper).

If $\lambda_0 = 0$*, we obtain the series of spurious solutions to the system* (4) - (5)*:*

$$x_1 = a\sin t, \quad x_2 = a\cos t, \quad x_3 = x_4 = x_5 = 0,$$

$$\lambda_1 = b\sin t, \quad \lambda_2 = b\cos t, \quad a, b \in \mathbb{R}.$$

2. Elements of p-regularity theory

Let us consider the equation

$$f(x) = 0, \tag{7}$$

where $f : X \to Y$ and X, Y are Banach spaces, $f \in \mathcal{C}^{p+1}(X)$. Moreover, let us assume that $f'(\hat{x})$ is singular, where $\hat{x}$ is a solution of (7).

We describe the basic constructions of p-regularity theory (see e.g. [6]) which are used for the investigation of singular problems.

Suppose that the space Y is decomposed into a direct sum

$$Y = Y_1 \oplus \ldots \oplus Y_p, \tag{8}$$

where $Y_1 = \overline{\mathrm{Im} f'(\hat{x})}$, $Z_1 = Y$. Let Z_2 be a closed complementary subspace to Y_1 (we assume that such closed complement exists), and let $P_{Z_2} : Y \to Z_2$ be the projection operator onto Z_2 along Y_1. By Y_2 we mean the closed linear span of the image of the quadratic map $P_{Z_2} f^{(2)}(\hat{x})[\cdot]^2$. More generally, define inductively

$$Y_i = \overline{\mathrm{span}\ \mathrm{Im} P_{Z_i} f^{(i)}(\hat{x})[\cdot]^i} \subseteq Z_i, \qquad i = 2, \ldots, p-1,$$

where Z_i is a chosen closed complementary subspace for $(Y_1 \oplus \ldots \oplus Y_{i-1})$ with respect to Y, $i = 2, \ldots, p$ and $P_{Z_i} : Y \to Z_i$ is the projection operator onto Z_i along $(Y_1 \oplus \ldots \oplus Y_{i-1})$ with respect to Y, $i = 2, \ldots, p$. Finally, $Y_p = Z_p$. The order p is chosen as the minimum number for which (8) holds. Let us define the following mappings

$$f_i(x) = P_i f(x), \quad f_i : X \to Y_i \qquad i = 1, \ldots, p,$$

where $P_i := P_{Y_i} : Y \to Y_i$ is the projection operator onto Y_i along $(Y_1 \oplus \ldots \oplus Y_{i-1} \oplus Y_{i+1} \oplus \ldots \oplus Y_p)$ with respect to Y, $i = 1, \ldots, p$.

Definition 1. *The linear operator* $\Psi_p(\hat{x},h) \in \mathcal{L}(X, Y_1 \oplus \ldots \oplus Y_p)$*,* $h \in X$*,* $h \neq 0$

$$\Psi_p(\hat{x},h) = f_1'(\hat{x}) + f_2''(\hat{x})h + \ldots + f_p^{(p)}(\hat{x})[h]^{p-1},$$

is called the p-factor operator.

Definition 2. *We say that the mapping* f *is p-regular at* $\hat{x}$ *along an element* h *if* $\mathrm{Im}\Psi_p(\hat{x},h) = Y$*.*

Remark 1. *The condition of p-regularity of the mapping* $f(x)$ *at the point* $\hat{x}$ *along* h *is equivalent to*

$$\mathrm{Im} f_p^{(p)}(\hat{x})[h]^{p-1} \circ \mathrm{Ker}\ \Psi_{p-1}(\hat{x},h) = Y_p,$$

where

$$\Psi_{p-1}(\hat{x},h) = f_1'(\hat{x}) + f_2''(\hat{x})h + \ldots + f_{p-1}^{(p-1)}(\hat{x})[h]^{p-2}.$$

Definition 3. *We say that the mapping* f *is p-regular at* $\hat{x}$ *if it is p-regular along any* h *from the set*

$$H_p(\hat{x}) = \bigcap_{k=1}^{p} \mathrm{Ker}^k f_k^{(k)}(\hat{x}) \setminus \{\mathbf{0}\},$$

where

$$\mathrm{Ker}^k f_k^{(k)}(\hat{x}) = \{\xi \in X : f_k^{(k)}(\hat{x})[\xi]^k = 0\}$$

is the k-kernel of the k-order mapping $f_k^{(k)}(\hat{x})[\xi]^k$*.*

For a linear surjective operator $\Psi_p(\hat{x},h) : X \mapsto Y$ between Banach spaces, by $\{\Psi_p(\hat{x},h)\}^{-1}$ we denote its *right inverse*. Therefore,

$$\{\Psi_p(\hat{x},h)\}^{-1} : Y \mapsto 2^X$$

and we have

$$\{\Psi_p(\hat{x},h)\}^{-1}(y) = \{x \in X \colon \Psi_p(\hat{x},h)x = y\}.$$

We define the *norm* of $\{\Psi_p(\hat{x},h)\}^{-1}$ via the formula

$$\|\{\Psi_p(\hat{x},h)\}^{-1}\| = \sup_{\|y\|=1} \inf\{\|x\| : x \in \{\Psi_p(\hat{x},h)\}^{-1}(y)\}.$$

We say that $\{\Psi_p(\hat{x},h)\}^{-1}$ is *bounded* if $\|\{\Psi_p(\hat{x},h)\}^{-1}\| < \infty$.

Definition 4. *The mapping* f *is called strongly p-regular at the point* $\hat{x}$ *if there exists* $\gamma > 0$ *such that*

$$\sup_{h \in H_\gamma} \left\| \{\Psi_p(\hat{x},h)\}^{-1} \right\| < \infty,$$

where

$$H_\gamma = \left\{ h \in X : \left\| f_k^{(k)}(\hat{x})[h]^k \right\|_{Y_k} \leq \gamma, k = 1, \ldots, p, \|h\| = 1 \right\}.$$

3. Optimality conditions for p-regular optimization problems

We recall the p-order necessary conditions for singular optimization problems (see [2]-[5]) of the form:

$$\min \varphi(x) \tag{9}$$

subject to

$$f(x) = 0, \tag{10}$$

where $f : X \to Y$, $f \in \mathcal{C}^{p+1}(X)$, $\varphi : X \to \mathbb{R}$, $\varphi \in \mathcal{C}^2(X)$ and X, Y are Banach spaces. We assume that $\hat{x}$ is a solution of (9) - (10) and $\mathrm{Im} f'(\hat{x}) \neq Y$.

Let us define the p-factor Lagrange function

$$\mathcal{L}p(x, \lambda, h) = \varphi(x) + \left\langle \sum_{k=1}^{p} f_k^{(k-1)}(x)[h]^{k-1}, \lambda \right\rangle,$$

where $\lambda \in Y^*$, $f_1^{(0)}(x) = f(x)$ and

$$\bar{\mathcal{L}}p(x, \lambda, h) = \varphi(x) + \left\langle \sum_{k=1}^{p} \frac{2}{k(k+1)} f_k^{(k-1)}(x)[h]^{k-1}, \lambda \right\rangle.$$

The following basic theorems on optimality conditions in a nonregular case were formulated and proved in [2].

Theorem 1 (Necessary and sufficient conditions for optimality)**.** *Let X and Y be Banach spaces,*

$$\varphi \in \mathcal{C}^2(X), \quad f \in \mathcal{C}^{p+1}(X), \quad f : X \to Y, \quad \varphi : X \to \mathbb{R}.$$

Suppose that $h \in H_p(\hat{x})$ and f is p-regular along h at the point $\hat{x}$. If $\hat{x}$ is a local solution to problem (9) - (10)*, then there exist multipliers $\hat{\lambda}(h) \in Y^*$ such that*

$$\mathcal{L}p'_x(\hat{x}, \hat{\lambda}(h), h) = 0 \Leftrightarrow \varphi'(\hat{x}) + \left(f_1'(\hat{x}) + \cdots + f_p^{(p)}(\hat{x})[h]^{(p-1)} \right)^* \hat{\lambda}(h) = 0. \tag{11}$$

Moreover, if f is strongly p-regular at $\hat{x}$, there exist $\alpha > 0$ and multipliers $\hat{\lambda}(h)$ such that (11) *is fulfilled and*

$$\bar{\mathcal{L}}p_{xx}(\hat{x}, \hat{\lambda}(h), h)[h]^2 \geq \alpha \|h\|^2,$$

for every $h \in H_p(\hat{x})$, then $\hat{x}$ is a strict local minimizer to problem (9) - (10)*.*

The next theorem also gives necessary and sufficient conditions for optimality but it is more convenient for application (see [1]).

Theorem 2. *Let X and Y be Banach spaces,*

$$\varphi \in \mathcal{C}^2(X), \quad f \in \mathcal{C}^{p+1}(X), \quad f : X \to Y, \quad \varphi : X \to \mathbb{R}, \quad h \in H_p(\hat{x}),$$

and let f be p-regular along h at the point $\hat{x}$. If $\hat{x}$ is a solution to problem (9) - (10), *then there exist multipliers $\bar{\lambda}_i(h) \in Y_i^*$, $i = 1, \dots, p$ such that*

$$\varphi'(\hat{x}) + \left(f'(\hat{x})\right)^* \bar{\lambda}_1(h) + \ldots + \left(f^{(p)}(\hat{x})[h]^{(p-1)}\right)^* \bar{\lambda}_p(h) = 0, \qquad (12)$$

and

$$\left(f^{(k)}(\hat{x})[h]^{(k-1)}\right)^* \bar{\lambda}_i(h) = 0,\ k = 1, \dots, i-1,\ i = 2, \dots, p. \qquad (13)$$

Moreover, if f is strongly p-regular at $\hat{x}$, there exist $\alpha > 0$ and multipliers $\bar{\lambda}_i(h)$, $i = 1, \dots, p$ such that (12) - (13) *hold, and*

$$\left(\varphi''(\hat{x}) + \frac{1}{3} f''(\hat{x})\bar{\lambda}_1(h) + \ldots + \frac{2}{p(p+1)} f^{(p+1)}(\hat{x})[h]^{p-1}\bar{\lambda}_p(h)\right)[h]^2 \geq \alpha\|h\|^2,$$

for every $h \in H_p(\hat{x})$, then $\hat{x}$ is a strict local minimizer to problem (9) - (10).

Proof. We need to prove only formula (13). From (11) we obtain

$$\varphi'(\hat{x}) + \left(P_1 f'(\hat{x}) + \cdots + P_p f^{(p)}(\hat{x})[h]^{(p-1)}\right)^* \hat{\lambda}(h) = 0.$$

This expression can be transformed as follows

$$\varphi'(\hat{x}) + f'(\hat{x})^* P_1^* \hat{\lambda}(h) + \cdots + \left(f^{(p)}(\hat{x})[h]^{(p-1)}\right)^* P_p^* \hat{\lambda}(h) = 0.$$

Let $\bar{\lambda}_k(h) := P_k^* \hat{\lambda}(h)$, $k = 1, \dots, p$. Then, for $i < k$, $k = 1, \dots, p$,

$$\begin{aligned}\left(f^{(k)}(\hat{x})[h]^{(k-1)}\right)^* \bar{\lambda}(h) &= \left(f^{(k)}(\hat{x})[h]^{(k-1)}\right)^* P_i^* \hat{\lambda}_i(h) \\ &= \left(P_i f^{(k)}(\hat{x})[h]^{(k-1)}\right)^* \hat{\lambda}_i(h) = 0,\end{aligned}$$

which proves (13). □

To apply the previous theorem to singular calculus of variations problems let us define the p-factor Euler-Lagrange function

$$\begin{aligned}S(x) &= F(x) + \left\langle \lambda(t), \left(g_1(x) + g_2'(x)[h] + \ldots + g_p^{(p-1)}(x)[h]^{p-1}\right)\right\rangle \\ &= F(x) + \lambda(t) G^{(p-1)}(x)[h]^{p-1},\end{aligned}$$

where

$$\begin{aligned}G^{(p-1)}(x)[h]^{p-1} &= g_1(x) + g_2'(x)[h] + \cdots + g_p^{(p-1)}(x)[h]^{p-1}, \\ \lambda(t) &= (\lambda_1(t), \dots, \lambda_m(t))^T\end{aligned}$$

and $g_k(x)$, for $k = 1, \dots, p$ are determined for the mapping $G(x)$ similarly to $f_k(x)$, $k = 1, \dots, p$ for the mapping $f(x)$ in the construction of the p-factor operator, i.e., $g_k(x) = P_{Y_k} G(x)$, $k = 1, \dots, p$. Let us define

$$g_k^{(k-1)}(x)[h]^{k-1} = \sum_{i+j=k-1} C_{k-1}^i g^{(k-1)}_{kx^i(x')^j}(x) h^i (h')^j,\ k = 1, \dots, p,$$

where

$$g^{(k-1)}_{kx^i(x')^j}(x) = g^{(k-1)}_{k\underbrace{x\ldots x}_{i}\underbrace{x'\ldots x'}_{j}}(x).$$

Definition 5. *We say that problem* (1) - (2) *is p-regular at $\hat{x}$ along*

$$h \in \bigcap_{k=1}^{p} \mathrm{Ker}^k g_k^{(k)}(\hat{x}), \|h\| \neq 0$$

if

$$\mathrm{Im}\left(g_1'(\hat{x}) + \ldots + g_p^{(p)}(\hat{x})[h]^{p-1}\right) = \mathcal{C}_m[t_1, t_2].$$

We consider the following theorem

Theorem 3. *Let $\hat{x}(t)$ be a solution of problem* (1) - (2) *and assume that the problem is p-regular at $\hat{x}$ along*

$$h \in \bigcap_{k=1}^{p} \mathrm{Ker}^k g_k^{(k)}(\hat{x}).$$

Then there exists a multiplier $\hat{\lambda}(t) = (\hat{\lambda}_1(t), \ldots, \hat{\lambda}_m(t))^T$ such that the following p-factor Euler-Lagrange equation

$$\begin{aligned} S_x(\hat{x}) - \frac{d}{dt} S_{x'}(\hat{x}) &= F_x(\hat{x}) + \left\langle \hat{\lambda}, \sum_{k=1}^{p} \sum_{i+j=k-1} C_{k-1}^i g^{(k-1)}_{x^i(x')^j}(\hat{x}) h^i (h')^j \right\rangle_x \\ &\quad - \frac{d}{dt}\left[F_{x'}(\hat{x}) + \left\langle \hat{\lambda}(t), \sum_{k=1}^{p} \sum_{i+j=k-1} C_{k-1}^i g^{(k-1)}_{x^i(x')^j}(\hat{x}) h^i (h')^j \right\rangle_{x'} \right] \\ &= 0 \end{aligned} \tag{14}$$

holds.

The proof of this theorem is very similar to the one of the analogous result for the singular isoperimetric problem, as in [1] or [4].

In problem (4) - (5) of Example 1, the mapping G is singular at $\bar{x} = (a \sin t, a \cos t, 0, 0, 0)^T$, $a \in \mathbb{R}$. Indeed,

$$G'(\bar{x})(\cdot) = \begin{pmatrix} (\cdot)_1' - (\cdot)_2 \\ (\cdot)_2' + (\cdot)_1 \end{pmatrix},$$

where

$$G'(\bar{x})x(t) = \begin{pmatrix} x_1'(t) - x_2(t) \\ x_2'(t) + x_1(t) \end{pmatrix}.$$

If we replace $\begin{pmatrix} x_1' - x_2 \\ x_2' + x_1 \end{pmatrix}$ by $x' + Lx$, where

$$L = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

then $G'(\bar{x})(\cdot) = (\cdot)' + L(\cdot)$ and

$$\begin{aligned}\mathrm{Ker}G'(\bar{x}) &= Span\left\{(\Phi_1(t),0,0,0)^T,(\Phi_2(t),0,0,0)^T\right\}\\ &\qquad \oplus \left\{(0,0,x_3(t),x_4(t),x_5(t))^T,\ x_i\in\mathcal{C}^2[0,2\pi],\ i=3,4,5\right\},\end{aligned}$$

where $\Phi_1(t) = (\sin t, \cos t)^T$, $\Phi_2(t) = (\cos t, -\sin t)^T$, and moreover,

$$\begin{aligned}\mathrm{Im}G'(\bar{x}) &= (\mathrm{Ker}(G'(\bar{x})^*)^\perp = \left(\mathrm{Ker}(-\frac{d}{dt}(\cdot)' + L^T(\cdot))\right)^\perp\\ &= \left\{\xi\in\mathcal{C}_2[0,2\pi] : \langle\xi,\psi_i\rangle = 0, i=1,2, \psi_1(t) = (\sin t,\cos t)^T,\right.\\ &\qquad \left.\psi_2(t) = (\cos t, -\sin t)^T\right\}\\ &\neq \mathcal{C}_2[0,2\pi].\end{aligned}$$

It means that the mapping $G(x)$ is non-regular at the points $\bar{x}$. We obtain that $Y_2 = \{0\}$, …, $Y_{p-1} = \{0\}$ and $Y_p = (\mathrm{Im}G^{(p-1)}(\bar{x}))^\perp = Span\{\psi_1,\psi_2\}$, where $\psi_1' = \psi_2$, $\psi_2' = -\psi_1$ and $\langle\Phi_i,\psi_j\rangle = \delta_{ij}$, $\langle\zeta,\eta\rangle = \int_0^{2\pi}\zeta(\tau)\eta(\tau)d\tau$.

The projection operator P_{Y_p} is defined as

$$P_p\begin{pmatrix} y_1\\ y_2\end{pmatrix} = P_p y = \bar{y}_1\psi_1 + \bar{y}_2\psi_2,$$

where $y = (y_1, y_2)^T$ and

$$\langle y - (\bar{y}_1\psi_1 + \bar{y}_2\psi_2), \psi_1\rangle = 0,$$

$$\langle y - (\bar{y}_1\psi_1 + \bar{y}_2\psi_2), \psi_2\rangle = 0,$$

i.e.,

$$\frac{1}{2\pi}\langle y,\psi_1\rangle = \bar{y}_1, \frac{1}{2\pi}\langle y,\psi_2\rangle = \bar{y}_2.$$

Let us point out that $P_p(x_1,\psi_1 + x_2\psi_2) = x_1\psi_1 + x_2\psi_2$.

Based on Remark 1, we can verify surjectivity of $P_pG^{(p)}(\bar{x})[h]^{p-1}$ only on $\mathrm{Ker}G^{(p-1)}(\bar{x})$ for

$$\begin{aligned}h &\in \mathrm{Ker}G'(\bar{x})\cap\cdots\cap\mathrm{Ker}^p P_pG^{(p)}(\bar{x}),\\ h &= (a\sin t, a\cos t, 1,1,1)^T.\end{aligned}$$

In order to calculate $P_pG^{(p)}(\bar{x})[h]^{p-1}$ let us determine $G^{(p)}(\bar{x})$ and

$$G^{(p)}(\bar{x})[h]^{p-1} = p!a\begin{pmatrix} 0 & 0 & h_3^{p-1}\sin t & h_4^{p-1}\cos t & -h_5^{p-1}(\cos t + \sin t)\\ 0 & 0 & h_3^{p-1}\cos t & -h_4^{p-1}\sin t & h_5^{p-1}(\sin t - \cos t)\end{pmatrix}.$$

It is obvious that $h = (a\sin t, a\cos t, 1,1,1)^T$ belongs to $\mathrm{Ker}G'(\bar{x})\cap\cdots\cap\mathrm{Ker}^pG^{(p)}(\bar{x})$ and consequently to $\mathrm{Ker}G'(\bar{x})\cap\cdots\cap\mathrm{Ker}^pP_pG^{(p)}(\bar{x})$. We have

$$G^{(p)}(\bar{x})[h^{p-1},x] = p!a(x_3 - x_5)\begin{pmatrix}\sin t\\ \cos t\end{pmatrix} + p!a(x_4 - x_5)\begin{pmatrix}\cos t\\ -\sin t\end{pmatrix}.$$

It means that

$$P_pG^{(p)}(\bar{x})[h^{p-1}, x] = G^{(p)}(\bar{x})[h^{p-1}, x]$$

and

$$G^{(p)}(\bar{x})[h]^p \circ \mathrm{Ker} G'(\bar{x}) = \mathrm{Span}\ \{\Phi_1, \Phi_2\} = Y_p.$$

Therefore, $G^{(p)}(\bar{x})[h]^{p-1}$ is surjection. Hence, $G(x)$ is p-regular along h at the points $\bar{x} = (a \sin t, a \cos t, 0, 0, 0)^T$. Finally, we can apply Theorem 3 with $\lambda_0 = 1$. We have constructed an operator

$$G'(\bar{x}) + P_pG^{(p)}(\bar{x})[h]^{p-1} = \begin{pmatrix} (\cdot)'_1 \\ (\cdot)'_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 & p!a\sin t & p!a\cos t & -p!a(\cos t + \sin t) \\ 1 & 0 & p!a\cos t & -p!a\sin t & p!a(\sin t - \cos t) \end{pmatrix}$$

which corresponds to the following system ($F_{x'} = 0$):

$$F_x(\bar{x}) + (G'(\bar{x}) + P_pG^{(p)}(\bar{x})[h]^{p-1})^*\lambda = 0 \Leftrightarrow$$
$$F_x(\bar{x}) + G'(\bar{x})^T\lambda + \left(P_pG^{(p)}(\bar{x})[h]^{p-1}\right)^T \lambda = 0.$$

It leads to the system of equations

$$\begin{cases} 2\bar{x}_1 + \lambda'_1 - \lambda_2 = 0 \\ 2\bar{x}_2 + \lambda'_2 + \lambda_1 = 0 \\ 2\bar{x}_3 + p!\lambda_1 a\sin t + p!\lambda_2 a\cos t = 0 \\ 2\bar{x}_4 + p!\lambda_1 a\cos t - p!\lambda_2 a\sin t = 0 \\ 2\bar{x}_5 - p!\lambda_1 a(\cos t + \sin t) + p!\lambda_2 a(\sin t - \cos t) = 0 \\ \lambda_i(0) - \lambda_i(2\pi) = 0,\ i = 1, 2. \end{cases} \tag{15}$$

or

$$\begin{cases} 2a\sin t + \lambda'_1 - \lambda_2 = 0 \\ 2a\cos t + \lambda'_2 + \lambda_1 = 0 \\ \lambda_1 \sin t + \lambda_2 \cos t = 0 \\ \lambda_1 \cos t - \lambda_2 \sin t = 0 \\ -\lambda_1(\cos t + \sin t) + \lambda_2(\sin t - \cos t) = 0, \\ \lambda_i(0) - \lambda_i(2\pi) = 0,\ i = 1, 2. \end{cases}$$

One can verify that the false solutions of (6)

$$x_1 = a\sin t,\ x_2 = a\cos t,\ x_3 = x_4 = x_5 = 0$$

do not satisfy system (15) for $a \neq 0$. It means that $x_1 = a\sin t$, $x_2 = a\cos t$, $x_3 = x_4 = x_5$ do not satisfy the 2-factor Euler-Lagrange equation (14).

Let us consider the same problem with higher derivatives $x'(t), \ldots, x^{(r)}$, $r \geq 2$,

$$J(x) = \int_{t_1}^{t_2} F(t, x(t), x'(t), \ldots, x^{(r)}(t))dt \to \min, \quad x(t) \in C_n^{2r}[t_1, t_2],$$

subject to a subsidiary differential relation

$$H(t, x(t), x'(t), \ldots, x^{(r)}(t)) = \begin{pmatrix} H_1(t, x(t), x'(t), \ldots, x^{(r)}(t)) \\ \cdots \\ H_m(t, x(t), x'(t), \ldots, x^{(r)}(t)) \end{pmatrix} = \begin{pmatrix} 0 \\ \ldots \\ 0 \end{pmatrix},$$

$A_k x^{(k)}(t_1) + B_k x^{(k)}(t_2) = 0$, where A_k, B_k are $n \times n$ matrices, $k = 1, \ldots, r$. Let

$$G(x) = H(\cdot, x(\cdot), \ldots, x^{(r)}(\cdot)), G : X \to Y,$$

where $Y = \mathcal{C}_m([t_1, t_2])$ and

$$X = \{x(\cdot) \in \mathcal{C}_n^{2r}[t_1, t_2] : A_k x^{(k)}(t_1) + B_k x^{(k)}(t_2) = 0, k = 1, \ldots, r\}.$$

Moreover,

$$g_k^{(k-1)}(x)[h]^{k-1} = \sum_{i_1+\cdots+i_r=k-1} g^{(k-1)}_{x^{i_1}\cdots(x^{(r)})^{i_r}}[h + h' + \cdots + h^{(r)}]^{k-1},\ k = 1, \ldots, p,$$

and introduce the so - called p-factor Euler-Poisson function

$$K(x) = F(x) + \left\langle \lambda(t), \left(g_1(x) + g_2'(x)[h] + \ldots + g_p^{(p-1)}(x)[h]^{p-1}\right)\right\rangle.$$

Theorem 4. *Let $\hat{x}(t)$ be a solution of problem* (1) - (2) *and assume that this problem is p-regular at $\hat{x}$ along*

$$h \in \bigcap_{k=1}^{p} \mathrm{Ker}^k g_k^{(k)}(\hat{x}).$$

Then there exists a multiplier $\hat{\lambda}(t) = (\hat{\lambda}_1(t), \ldots, \hat{\lambda}_m(t))^T$ such that the following p-factor Euler-Poisson equation

$$\begin{aligned}
&K_x(\hat{x}) - \frac{d}{dt}K_{x'}(\hat{x} + \frac{d^2}{dt^2}K_{x''}(\hat{x}) - ... + (-1)^r K_{x^{(r)}}(\hat{x}) \\
&\quad = F_x(\hat{x}) + \left\langle \hat{\lambda}(t), \sum_{k=1}^{p} g_k^{(k-1)}(\hat{x})[h]^{k-1} \right\rangle_x \\
&\qquad - \frac{d}{dt}\left[F_{x'}(\hat{x}) + \left\langle \hat{\lambda}(t), \sum_{k=1}^{p} g_k^{(k-1)}(\hat{x})[h]^{k-1} \right\rangle_{x'} \right] \\
&\qquad + \ldots + (-1)^{(r)} \frac{d^r}{dt^r}\left[F_{x^{(r)}}(\hat{x}) + \left\langle \hat{\lambda}(t), \sum_{k=1}^{p} g_k^{(k-1)}(\hat{x})[h]^{k-1} \right\rangle_{x^{(r)}} \right] = 0
\end{aligned}$$

holds.

The proof of Theorem 4 is similar to the one the reader can find in [4] for an isoperimetric problem.

Example 2. *Consider the following problem*

$$J(x) = \int_0^{\pi} (x_1^2(t) + x_2^2(t) + x_3^2(t))dt \to \min \tag{16}$$

subject to

$$H(t, x(t), x'(t), x''(t)) = x_1''(t) + x_1(t) + x_2^p(t)x_1(t) - x_3^p(t)x_1(t) = 0, \tag{17}$$

$x_i(0)-x_i(\pi)=0, x_i'(0)+x_i'(\pi)=0, i=1,2,3, \quad p\geq 2, p=2k, k=1,2,\dots.$

Here $A_1=-B_1=I_3$, $A_2=B_2=I_3$, *where* I_3 *means the unit matrix of size* 3.

The solution of (16) - (17) *is* $\hat{x}(t)=0$. *The Euler-Poisson equation in this case has the following form*

$$\lambda_0 F_x+\lambda(t)H_x-\frac{d}{dt}(\lambda(t)H_{x'})+\frac{d^2}{dt^2}(\lambda(t)H_{x''})=0$$

or

$$\begin{aligned} 2\lambda_0 x_1+\lambda+\lambda x_2^p-\lambda x_3^p+\lambda'' &= 0\\ 2\lambda_0 x_2+p\lambda x_2^{p-1}x_1 &= 0\\ 2\lambda_0 x_3-p\lambda x_3^{p-1}x_1 &= 0,\\ \lambda(0)-\lambda(\pi)=0,\ \lambda'(0)+\lambda'(\pi) &= 0 \end{aligned}$$

and gives the series of spurious solutions $x_1=a\sin t$, $x_2=0$, $x_3=0$, $\lambda=b\sin t$, $\lambda_0=0$, $a\in\mathbb{R}$. *The mapping* $G(x)$ *is singular at these points and* $G'(a\sin t,0,0)$ *is not surjective. But* $G(x)$ *is* p*-regular at* $\bar{x}=(a\sin t,0,0)$ *along* $h=(\sin t,1,-1)^T$.

Indeed, $Y_1=\{\sin t\}^{\perp}$, $Y_2=\{0\}$, …, $Y_{p-1}=\{0\}$, $Y_p=Span\{\sin t\}$,

$$\begin{aligned} G'(\bar{x})h+P_{Y_p}G^{(p)}(\bar{x})[h]^p &= h_1''+h_1+\frac{2p!}{\pi}\sin t\int_0^{\pi}(h_2^p\bar{x}_1-h_3^p\bar{x}_1)\sin\tau d\tau\\ &= h_1''+h_1+\frac{2p!a}{\pi}\sin t\int_0^{\pi}(h_2^p-h_3^p)\sin^2\tau d\tau=0. \end{aligned}$$

It means that $h\in \mathrm{Ker}G'(\bar{x})\cap P_{Y_p}Ker^pG^{(p)}(\bar{x})$ *and*

$$P_{Y_p}G^{(p)}(\bar{x})[h]^{p-1}(\cdot)=2\frac{p!a}{\pi}\sin t\int_0^{\pi}(1^{p-1}(\cdot)_2-(-1)^{p-1}(\cdot)_3)\sin^2\tau d\tau.$$

We have

$$P_{Y_p}G^{(p)}(\bar{x})[h]^{p-1}\begin{pmatrix} b\sin t\\ m\\ n\end{pmatrix}=2\frac{p!ab}{\pi}\sin t\int_0^{\pi}(m+n)\sin^2\tau d\tau=Y_p, b\in\mathbb{R},\ m\neq -n,$$

i.e., G *is* p*-regular at* $\bar{x}=(a\sin t,0,0)$ *along* h *and at these points* $\bar{x}$ *we can guarantee that* $\lambda_0=1$ *in the* p*-factor Euler-Poisson equation*

$$\begin{aligned} 2a\sin t+\lambda''+\lambda &= 0\\ 2\frac{p!a}{\pi}\sin t\int_0^{\pi}\sin^2\tau\lambda(\tau)d\tau &= 0\\ -\frac{p!a}{\pi}\sin t\int_0^{\pi}\sin^2\tau\lambda(\tau)d\tau &= 0\\ \lambda(0)-\lambda(\pi)=0,\ \lambda(0)+\lambda(\pi) &= 0. \end{aligned}$$

The first equation has no solutions for $a\neq 0$ *that satisfy the fourth equation.*

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References

[1] K. N. Belash, A. A. Tret'yakov, *Methods for solving degenerate problems*, USSR Comput. Math and Math. Phys. **28**(1988), 90–94.

[2] O. A. Brezhneva, A. A. Tret'yakov, *Optimality conditions for degenerate extremum problems with equality constraints*, SIAM J. Contr. Optim. **12**(2003), 729–745.

[3] O. A. Brezhneva, A. A. Tret'yakov, *Corrigendum: Optimality Conditions for Degenerate Extremum Problems with Equality Constraints*, SIAM J. Contr. Optim. **48**(2010), 3670–3673.

[4] I. T. Korneva, A. A. Tret'yakov, *Application of the factor-analysis to the calculus of variations*, in: *Proceedings of Simulation and Analysis in Problems of Decision-Making Theory*, Computing Center of Russian Academy of Sciences, Moscow, 2002, 144–162, in Russian.

[5] A. Prusińska, A. A. Tret'yakov, *P-order Necessary and Sufficient Conditions for Optimality in Singular Calculus of Variations*, Discuss. Math. Differ. Incl. Control Optim. **30**(2010), 269–279.

[6] E. Szczepanik, A. A. Tret'yakov, *p-factor methods for nonregular inequality-constrained optimization problems*, Nonlinear Analysis: Theory Methods and Applications **69**(2008), 4241-4251.