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THE GRAPH OF EQUIVALENCE CLASSES OF ZERO-DIVISORS OF A POSET

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Abstract. In this paper, we give the definition of the graph of equivalence classes of zero-divisors of a poset P. We prove that if $[a]$ has maximal degree in $V(\Gamma_E(P))$, then $\mathrm{ann}(a)$ is maximal in $\mathrm{Anih}(P)$. Also, we give some other properties of the graph $\Gamma_E(P)$. Moreover, we characterize the cut vertices of $\Gamma_E(P)$ and study the cliques of these graphs.

1. Introduction

The concept of zero-divisor graph was first introduced by Beck in [7] to investigate the interplay between ring-theoretic properties and graph-theoretic properties. The concept of zero-divisor graph has also been extended to many algebraic structures such as rings, semigroups, semirings (see [4–11,16]). Halaš and Jukl ([13]) introduced the zero-divisor graph of a poset. Since then, many authors continued to study the zero-divisor graphs of posets, see [1,15,16,20]. Let R be a ring and $r, s \in R$. Define $r \sim s$ if and only if $\mathrm{ann}(r) = \mathrm{ann}(s)$. Write $[r] = \{s \in R \mid r \sim s\}$ and $R_E = \{[r] \mid r \in R\}$. Denote by $\Gamma_E(R)$ the graph of equivalence classes of zero-divisors of R. The set of vertices $V(\Gamma_E(R))$ is $R_E \backslash \{[0], [1]\}$ and two vertices are adjacent if and only if $[r][s] = [0]$, if and only if $rs = 0$. Motivated by ideas in paper [18], Spiroff and Wickham ([19]) studied the graph of equivalence classes of zero-divisors of a commutative Noetherian ring. Anderson and LaGrange ([2]) continued to study these graphs. In [2], the graph is called the compressed zero-divisor graph. In this paper, we will extend the graph of equivalence classes of zero-divisors to a poset P and study the properties of these graphs.

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The paper is constructed as follows: In Section 2, we give some relevant definitions and notations of graphs and posets. In Section 3, we give the definition of the graph of equivalence classes of zero-divisors of a poset P and study the basic properties of these graphs. In Section 4, we investigate the cut vertices and clique number of the graph $\Gamma_E(P)$.

Throughout, all posets P will be a poset with 0 and 1 and all graphs will be simple graphs.

2. Preliminaries

Let $(P, \leq)$ be a partially ordered set (abbreviated as a poset) and $X \subseteq P$. Let $L(X) = \{y \in P \mid y \leq x \text{ for all } x \in X\}$ denote the lower cone of X. Dually, let $U(X) = \{y \in P \mid y \geq x \text{ for all } x \in X\}$ denote the upper cone of X. If $X = \{x_1, \ldots, x_n\}$, we shall write $L(x_1, \ldots, x_n)$ or $U(x_1, \ldots, x_n)$ instead of $L(X)$ or $U(X)$.

Let P be a poset and $\emptyset \neq I \subseteq P$. Then I is called an ideal of P if $x \in I$ and $y \leq x$, then $y \in I$. A proper ideal I of P is called prime if for all $x, y \in P, L(x, y) \subseteq I$ implies $x \in I$ or $y \in I$.

For $x \in P$, the set $\operatorname{ann}(x) = \{y \in P | L(x, y) = \{0\}\}$ is called the annihilator of x.

For $x \in P$, x is called a zero-divisor of P if there exists $0 \neq y \in P$ such that $L(x, y) = \{0\}$. Denote by $Z(P)$ the zero-divisors of P and write $Z(P)^{\times} = Z(P) \backslash \{0\}$.

The zero-divisor graph of P, denoted by $\Gamma(P)$, is as follows: the set of vertices is $V(\Gamma(P)) = Z(P)^{\times}$ and distinct vertices x and y are adjacent if and only if $L(x, y) = \{0\}$ ([1]).

Let G be a graph. For $k \geq 2$, a graph is called a k-partite graph if the vertices of the graph are partitioned into k disjoint sets such that there is no edge between two vertices in the same set. A 2-partite graph is usually called a bipartite graph. It is well known that a graph is bipartite if and only if it contains no cycle of odd length. A complete bipartite graph is a bipartite graph such that every vertex in one set is connected to every vertex in the other set. The complete graph K_n is a graph with n vertices in which each vertex is connected to each of the others. The diameter of a graph G is the largest distance between two vertices in G, denoted by $\operatorname{diam}(G)$. A clique of a graph G is a subset of its vertices such that there exists an edge between each pair of vertices in the subset. The clique number $\operatorname{cl}(G)$ of a graph G is the number of vertices in a maximum clique in G.

3. Basic properties of the graph $\Gamma_E(P)$

In this section, we will define the graph of equivalence classes of zero-divisors of a poset P and investigate the properties of this graph.

An element $0 \neq p$ of a poset P is called an atom if there exists no element $x \in P$ such that $0 < x < p$. The set of atoms of P is denoted by $\mathrm{Atom}(P)$. If $p \in P$, set $\mathrm{atom}(p) = \{a \in \mathrm{Atom}(P) \mid a \leq p\}$.

For any elements $a, b \in P$, define a relation on P by $a \sim b$ if and only if $\mathrm{ann}(a) = \mathrm{ann}(b)$. Then $\sim$ is an equivalence relation on P.

For any $a \in P$, let $[a] = \{r \in P \mid r \sim a\}$. It is easy to get the following statements.

Lemma 3.1. *Let P be a poset. Then:*

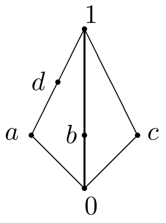
1) *$ann(1) = \{0\}$ and $ann(0) = P$. Moreover, if $a \neq 0$, then $[a] \neq [0]$.*
2) *$[a] \subseteq Z(P)$, for all $a \in P \setminus \{0, 1\}$.*

Let $\overline{P} = \{[a] \mid a \in P\}$. Define a partial order relation on $\overline{P}$ by $[a] \leq^{'} [b]$ if and only if $\mathrm{ann}(b) \subseteq \mathrm{ann}(a)$. It is clear that this partial order relation is well-defined and $(\overline{P}, \leq^{'})$ is a poset. $[0]$ is the least element in $\overline{P}$ and $[1]$ is the largest element in $\overline{P}$. Without causing confusion, we will let $\leq$ represent the partial order relation on both P and $\overline{P}$ in the following.

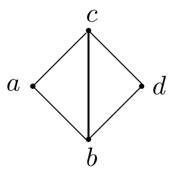
Now, we give the definition of the graph of equivalence classes of zero-divisors of a poset P.

Definition 3.2. *The graph of equivalence classes of zero-divisors of a poset P is the graph $\Gamma_E(P) = \Gamma(\overline{P})$ whose vertices are the elements in $\overline{P} \setminus \{[0], [1]\}$, such that two distinct vertices $[a]$ and $[b]$ are adjacent if and only if $L([a], [b]) = \{[0]\}$.*

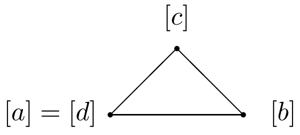
Let P be a poset as below. Then one can check that $\mathrm{diam}(\Gamma(P)) = 2$ while $\mathrm{diam}(\Gamma_E(P)) = 1$. The properties of the graph $\Gamma_E(P)$ and the graph $\Gamma(P)$ are different.



P



$\Gamma(P)$



$\Gamma_E(P)$

Lemma 3.3. *Let P be a poset and $a, b \in P$. Then*

1) *If $a \leq b$, then $ann(b) \subseteq ann(a)$ and $[a] \leq [b]$ in $\overline{P}$.*
2) *If $[a] \neq [1]$, $[b] \neq [1]$, and $L([a], [b]) = \{[0]\}$, then $L(a, b) = \{0\}$.*
3) *If $L(a, b) = \{0\}$, then $L([a], [b]) = \{[0]\}$.*

Proof. 1) Obvious.

2) Suppose $x \in L(a, b)$. Then $x \leq a$ and $x \leq b$. It follows that $\mathrm{ann}(a) \subseteq \mathrm{ann}(x)$ and $\mathrm{ann}(b) \subseteq \mathrm{ann}(x)$. Hence, $[x] \leq [a]$ and $[x] \leq [b]$. Therefore, we have $[x] = [0]$, and so $x = 0$ by 1) in Lemma 3.1. Hence, $L(a, b) = \{0\}$.

3) Suppose $[c] \in L([a],[b])$. Then $[c] \leq [a]$ and $[c] \leq [b]$. Hence, $\mathrm{ann}(a) \subseteq \mathrm{ann}(c)$ and $\mathrm{ann}(b) \subseteq \mathrm{ann}(c)$. By $L(a,b)=\{0\}$, we have $b \in \mathrm{ann}(a) \subseteq \mathrm{ann}(c)$, and so $L(b,c)=\{0\}$. Thus $c \in \mathrm{ann}(b) \subseteq \mathrm{ann}(c)$. It follows that $c=0$, and so $[c]=[0]$. Therefore, $L([a],[b])=\{[0]\}$. ☐

Proposition 3.4. *Let P be a poset. If $[x]=[x_1]$ and $[y]=[y_1]$, then $L(x,y)=\{0\}$ if and only if $L(x_1,y_1)=\{0\}$.*

Proof. $\Rightarrow$: Suppose $[x]=[x_1]$ and $[y]=[y_1]$. Then $\mathrm{ann}(x)=\mathrm{ann}(x_1)$ and $\mathrm{ann}(y)=\mathrm{ann}(y_1)$. Since $L(x,y)=\{0\}$, we have $y \in \mathrm{ann}(x)=\mathrm{ann}(x_1)$, and hence $L(x_1,y)=\{0\}$. That is, $x_1 \in \mathrm{ann}(y)=\mathrm{ann}(y_1)$. Thus $L(x_1,y_1)=\{0\}$.

$\Leftarrow$: The proof is similar to that of "$\Rightarrow$". ☐

Remark 3.5. By Definition 3.2, Lemma 3.3, and Proposition 3.4, we know that the graph $\Gamma_E(P)$ is isomorphic to a subgraph of $\Gamma(P)$.

Let a be a vertex of a graph G. The degree of a is the number of edge ends at a, denoted by $\deg(a)$. Let $N(a)$ be the set of vertices which are adjacent to a, then $|N(a)|=\deg(a)$. For any two vertices u and v of a graph G, define $u \approx v$ if and only if $N(u)=N(v)$. Let $\Gamma(P)$ be the zero-divisor graph of a poset P and $u,v \in P$. Note that $N(u)=\mathrm{ann}(u)\backslash\{0\}$. Then $u \approx v$ if and only if $\mathrm{ann}(u)=\mathrm{ann}(v)$, if and only if $[u]=[v]$. Let $\bar{u}=\{r \in G \mid r \approx u\}$ and $G/_{\approx}=\{\bar{u} \mid u \in G\}$. Then $G/_{\approx}$ becomes a graph in the natural way with $[u]$ and $[v]$ are adjacent in $G/_{\approx}$ if and only if u and v are adjacent in G. Using Lemma 3.3, we get the following analog of [2, Theorem 2.4].

Theorem 3.6. *Let P be a poset. Then $\Gamma_E(P) \cong \Gamma(P)/_{\approx}$.*

Proof. Suppose $a \in P$. Define a map $\varphi : \Gamma_E(P) \to \Gamma(P)/_{\approx}$ by $[a] \mapsto \bar{a}$. By the above comments, the map φ is well-defined. One can easily check that φ is also bijective. If $[a]-[b]$ is an edge in $\Gamma_E(P)$, then $L([a],[b])=\{[0]\}$, and hence $L(a,b)=\{0\}$ by Lemma 3.3. Therefore, $\bar{a}-\bar{b}$ is an edge in $\Gamma(P)/_{\approx}$.

Conversely, if $\bar{a}-\bar{b}$ is an edge in $\Gamma(P)/_{\approx}$, then a and b are adjacent in $\Gamma(P)$, and hence $L(a,b)=\{0\}$. By Lemma 3.3, we get $L([a],[b])=\{[0]\}$. Therefore, $[a]-[b]$ is an edge in $\Gamma_E(P)$. ☐

The diameter of the graph $\Gamma_E(R)$ is less or equal to 3, where R is a commutative ring with identity (Proposition 1.4 in [19]). The following statement gives a similar result for the graph $\Gamma_E(P)$, where P is a poset.

Theorem 3.7. *Let P be a poset. Then $\Gamma_E(P)$ satisfies the following conditions.*

1) *$\Gamma_E(P)$ is connected.*
2) *$diam(\Gamma_E(P)) \leq 3$.*

Proof. By the definition of $\Gamma_E(P)$, we know that it is also a zero-divisor graph of the poset $\overline{P}$. Using [1, Theorem 3.3], we have that $\Gamma_E(P)$ is connected and $\operatorname{diam}(\Gamma_E(P)) \leq 3$. ∎

In [19], Spiroff and Wickham investigated infinite graphs of equivalence classes of zero-divisors of a ring R and associated primes of R, where R is a commutative Noetherian ring with identity. We shall study the corresponding problems in poset settings.

Proposition 3.8. *Let P be a poset and $a, b \in P$. Then $ann([a]) = ann([b])$ if and only if $[a] = [b]$.*

Proof. $\Rightarrow$: Let $a, b \in P$ and $\operatorname{ann}([a]) = \operatorname{ann}([b])$. Suppose $z \in \operatorname{ann}(a)$. By Lemma 3.3, we have $[z] \in \operatorname{ann}([a]) = \operatorname{ann}([b])$, and so $L([z], [b]) = \{[0]\}$. Using Lemma 3.3 again, we have $L(z, b) = \{0\}$. This proves that $z \in \operatorname{ann}(b)$, and hence $\operatorname{ann}(a) \subseteq \operatorname{ann}(b)$. Similarly, one can prove that $\operatorname{ann}(b) \subseteq \operatorname{ann}(a)$. Therefore, $[a] = [b]$.

$\Leftarrow$: Obvious. ∎

A poset P is atomic if for all $0 < b \in P$, there exists an atom $a \in P$ such that $0 < a \leq b$. Let P be a poset. Let $\operatorname{Anih}(P) = \{\operatorname{ann}(a) \mid a \in P,\ \operatorname{ann}(a) \neq P\}$. If $a \in P$ and $\operatorname{ann}(a)$ is maximal among $\operatorname{Anih}(P)$, then $\operatorname{ann}(a)$ is a prime ideal of P ([13], Lemma 2.2).

Proposition 3.9. *Let P be a poset. If a is an atom of P, then $ann(a)$ is maximal in $Anih(P)$. Moreover, $ann(a)$ is prime. Conversely, if P is atomic and $ann(b)$ is maximal in $Anih(P)$, then there exists an atom a such that $ann(a) = ann(b)$.*

Proof. Suppose there exists an element $0 \neq c \in P$ with $\operatorname{ann}(a) \subset \operatorname{ann}(c)$. Then there exists $x \in \operatorname{ann}(c) \backslash \operatorname{ann}(a)$, that is, $L(x, c) = \{0\}$, but $L(x, a) \neq \{0\}$. Assume $0 \neq z \in L(x, a)$. Since a is an atom, we must have $z = a$. Hence $a \leq x$. Thus $L(a, c) = \{0\}$, and so $c \in \operatorname{ann}(a)$. Therefore $c \in \operatorname{ann}(c)$. This is impossible. Thus $\operatorname{ann}(a)$ is maximal. By Lemma 2.2 in [13], it follows that $\operatorname{ann}(a)$ is prime.

Conversely, suppose $\operatorname{ann}(b)$ is maximal in $\operatorname{Anih}(P)$ and a is an atom such that $0 < a \leq b$. We have $\operatorname{ann}(b) \subseteq \operatorname{ann}(a)$, and so $\operatorname{ann}(b) = \operatorname{ann}(a)$ by the maximality of $\operatorname{ann}(b)$. ∎

The following proposition is similar to Proposition 2.2 in [19].

Proposition 3.10. *Let P be a poset and $|Atom(P)| < \infty$. Then $|V(\Gamma_E(P))| = \infty$ if and only if there exists $x \in P$ such that $ann(x)$ is maximal in $Anih(P)$ and $deg([x]) = \infty$.*

Proof. $\Rightarrow$: Suppose $\operatorname{Atom}(P) = \{a_1, a_2, \ldots, a_n\}$. By Proposition 3.9, we know that $\operatorname{ann}(a_1)$, $\operatorname{ann}(a_2)$, $\ldots$, $\operatorname{ann}(a_n)$ are maximal in $\operatorname{Anih}(P)$. If

$\deg([a_1]) < \infty$, there exist infinitely many vertices $[x]$ such that $L([x],[a_1]) \neq \{[0]\}$. By Lemma 3.3, we have $L(x,a_1) \neq \{0\}$. If $[v] \neq [x]$ and $L([x],[v]) = \{[0]\}$, then $L(x,v) = \{0\} \subseteq \operatorname{ann}(a_1)$. Since $\operatorname{ann}(a_1)$ is prime and $x \notin \operatorname{ann}(a_1)$, we have $v \in \operatorname{ann}(a_1)$, and so $[v]$ is adjacent to $[a_1]$. If there exist infinitely many distinct vertices $[v]$ which are adjacent to $[a_1]$, then $\deg([a_1]) = \infty$. This is a contradiction. Hence, the set of $[v]$'s is finite. Note that $[x]$ is adjacent to $[v]$ and the set of $[x]$'s is infinite. We have $\deg([v]) = \infty$ for some v. If $\operatorname{ann}(v)$ is maximal, we get the desired result. If $\operatorname{ann}(v) \subseteq \operatorname{ann}(a_i)$ for some $i \neq 1$, we have $\deg([a_i]) = \infty$, and we also get the desired result.

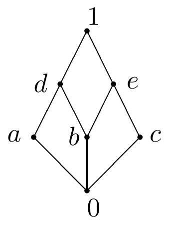
$\Leftarrow$: It is obvious. ◻

THEOREM 3.11. *Let P be a poset and $a \in P$. If $[a]$ has maximal degree in $V(\Gamma_E(P))$, then ann(a) is maximal in Anih(P).*

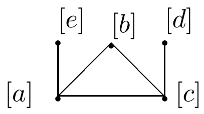
PROOF. Suppose $\operatorname{ann}(a) \subseteq \operatorname{ann}(b)$. It is easy to show $N([a]) \subseteq N([b])$. By the maximality of the degree of $[a]$, we have $N([a]) = N([b])$. If there exists $z \in \operatorname{ann}(b) \backslash \operatorname{ann}(a)$, by Lemma 3.3 we get $[z]$ is adjacent to $[b]$, but not adjacent to $[a]$. That is, $[z] \in N([b])$, but $[z] \notin N([a])$. This is a contradiction. Therefore, $\operatorname{ann}(a) = \operatorname{ann}(b)$. ◻

The following example proves that the converse of the preceding theorem is not true.

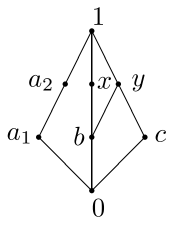
EXAMPLE 3.12. Let P_A be the poset in Figure (A). Then ann(b) is maximal in Anilh(P_A). One can check that $\deg([b]) = 2$ and $\deg([a]) = 3$. Hence, the degree of $[b]$ is not maximal.



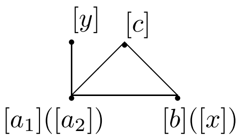
(A)



$\Gamma_E(P_A)$



(B)



$\Gamma_E(P_B)$

4. CUT VERTICES, CLIQUES OF THE GRAPH $\Gamma_E(P)$

In this section, we will give a characterization of the cut vertices of the graph $\Gamma_E(P)$ and also study the cliques of these graphs.

Let G be a graph. A vertex a is called a cut vertex of G if the removal of a along with edges through a leads to more components than G. That is, a vertex a is called a cut vertex if there exist distinct vertices b and c such that a is in every $b-c$ path, where both b and c are different from a. Axtell et al. ([6]) studied cut vertices in zero-divisor graphs of commutative rings with identity and proved that if x is a cut vertex of the graph $\Gamma(R)$, then the annihilator of x is properly maximal (see [6, Proposition 2.7]). In the following, we investigate cut vertices in the graph $\Gamma_E(P)$.

Proposition 4.1. *Let P be a poset. If $[a]$ is a cut vertex in $\Gamma_E(P)$, then $[a]$ is an atom in $\overline{P}$.*

Proof. Suppose $[x]-[a]$ is an edge in $\Gamma_E(P)$ and $[0] \neq [b] < [a]$. Then $[x]-[b]$ is also an edge in $\Gamma_E(P)$. Using this fact, one can prove that if $[a]$ is not an atom in $\overline{P}$, then $[a]$ is not a cut vertex in $\Gamma_E(P)$. $\square$

Let P be a poset and $0 \neq x, 0 \neq y \in P$. By Lemma 3.3, $[x]-[y]$ is an edge in $\Gamma_E(P)$ if and only if $x-y$ is an edge in $\Gamma(P)$. Hence, we have the following lemma.

Lemma 4.2. *Let P be a poset and $a \in P$. If a is a cut vertex in $\Gamma(P)$, then $[a]$ is also a cut vertex in $\Gamma_E(P)$.*

The following example shows that the converse of Lemma 4.2 is not true.

Example 4.3. Let P_B be the poset in Figure (B). In $\Gamma_E(P_B)$, $[a_1]=[a_2]$ is a cut vertex, since $[b]-[a_1]-[y]$ is the only path from $[b]$ to $[y]$. While, both $b-a_1-y$ and $b-a_2-y$ are paths from b to y in $\Gamma(P_B)$. Hence, a_1 is not a cut vertex.

Proposition 4.4. *Let P be a poset and $a \in P$. If $[a]$ is a cut vertex in $\Gamma_E(P)$, then $[a] \cup \{0\}$ is an ideal of P.*

Proof. Suppose $b \in [a]$ and $y < b$. We have to show that $y \in [a]$. Since $y < b$, we have that $\mathrm{ann}(b)$ is contained in $\mathrm{ann}(y)$. So $N([a]) = N([b])$ is contained in $N([y])$. On the other hand, since $[a]$ is a cut vertex, there exists no vertex $[x]$ distinct from $[a]$ with $N([a])$ containing $N([x])$. Hence, $[y]=[a]$. $\square$

Let P be a poset. For $x, y \in P$, if x and y are incomparable, we denote by $y||x$. For $a \in \mathrm{Atom}(P)$, we define

$$\widetilde{U}(\mathrm{Atom}(P)\backslash\{a\}) = \{y \in P \mid y||a \text{ and } \forall b \in \mathrm{Atom}(P), \text{ if } b \neq a, \text{ then } y \geq b\}.$$

Proposition 4.5. *Let P be a poset and $a \in P$. Then a is an atom in P if and only if $[a]$ is an atom in $\overline{P}$ and a is a minimal element in $[a]$.*

Proof. $\Rightarrow$: Suppose $0 \neq [b] \in \overline{P}$ and $[b] \leq [a]$. Then we have $\mathrm{ann}(a) \subseteq \mathrm{ann}(b)$. By Proposition 3.9, $\mathrm{ann}(a)$ is maximal in $\mathrm{Anih}(P)$. So we have $\mathrm{ann}(a) = \mathrm{ann}(b)$. That is, $[a]=[b]$. Thus $[a]$ is an atom in $\overline{P}$. Obviously, a is a minimal element in $[a]$.

$\Leftarrow$: Suppose $0 \neq b \in P$ such that $b \leq a$. We have $\mathrm{ann}(a) \subseteq \mathrm{ann}(b)$, and so $[b] \leq [a]$. Since $[a]$ is an atom in $\overline{P}$, this proves that $[b]=[a]$ or $[b]=[0]$. If $[b]=[0]$, then $b=0$. This is a contradiction. Therefore, we have $[b]=[a]$. Since a is the minimal element in $[a]$, we have $b=a$, and so a is also an atom in P. $\square$

Using Proposition 4.5, we have the following theorem characterizing the cut vertices of $\Gamma_E(P)$.

THEOREM 4.6. *Let P be a poset. If $[a] \in Atom(\overline{P})$ and a is a minimal element in $[a]$, then $[a]$ is a cut vertex in $\Gamma_E(P)$ if and only if $\widetilde{U}(Atom(P)\backslash\{a\}) \neq \emptyset$.*

PROOF. $\Rightarrow$: Without loss of generality, let $[x] - [a] - [y]$ be a path of shortest length from $[x]$ to $[y]$. By Lemma 3.3, we have that $x - a - y$ is a path in $\Gamma(P)$. This concludes that $x||a$ and $y||a$. If $\widetilde{U}(\mathrm{Atom}(P)\backslash\{a\}) = \emptyset$, then we have $u, v \in \mathrm{Atom}(P)$ with $x||u$ and $y||v$. If $u \neq v$, then $x - u - v - y$ is a path in $\Gamma(P)$. Using Lemma 3.3 again, we have that $[x] - [u] - [v] - [y]$ is a path in $\Gamma_E(P)$. If $u = v$, then $[x] - [u] - [y]$ is a path in $\Gamma_E(P)$. In either case, we have a contradiction.

$\Leftarrow$: If $x \in \widetilde{U}(\mathrm{Atom}(P)\backslash\{a\})$, then $[a]$ is the unique vertex which is adjacent to $[x]$. This proves that $[a]$ is a cut vertex. ☐

In paper [12], Estaji and Khashyarmanesh proved that the clique number of the graph $\Gamma(L)$ is equal to the number of atoms in L, where $\Gamma(L)$ is the zero-divisor graph of a lattice L (Theorem 5.13). The following theorem shows that the clique number of the graph $\Gamma_E(P)$ is also equal to the number of atoms in P.

THEOREM 4.7. *Let P be a poset. Then $cl(\Gamma_E(P)) = |Atom(P)|$.*

PROOF. By Proposition 4.5, we have $|\mathrm{Atom}(P)| = |\mathrm{Atom}(\overline{P})|$. Since any two atoms in $\overline{P}$ are adjacent, we have $\mathrm{cl}(\Gamma_E(P)) \geq |\mathrm{Atom}(P)|$. Suppose $|\mathrm{cl}(\Gamma_E(P))| > |\mathrm{Atom}(P)|$. Let $\mathrm{cl}(\Gamma_E(P)) = m$ and $|\mathrm{Atom}(P)| = n$. Then $\Gamma_E(P)$ has a complete subgraph with vertices $\{[p_1], [p_2], \ldots, [p_m]\}$. Since $[p_i]$ and $[p_j]$ are adjacent in $\Gamma_E(P)$, then $\mathrm{atom}(p_i) \cap \mathrm{atom}(p_j) = \emptyset$, for all $i \neq j$. This is impossible, since $m > n$. Hence, $\mathrm{cl}(\Gamma_E(P)) = |\mathrm{Atom}(P)|$. ☐

Let G be a graph and $a, b \in V(G)$. Two vertices a and b are called complements in G if a is connected to b, and no vertex in G is connected to both a and b, denoted by $a \perp b$. We say that a graph G is complemented if each vertex in G has a complement. The set of all complements in G induces a subgraph of G, denoted by G^c. It is easy to see that G is complemented if and only if $G = G^c$. Complements were studied for the zero-divisor graph $\Gamma(R)$ in [3] and for $\Gamma_E(R)$ in [2]. The next result is the analog of [2, Theorem 4.3].

PROPOSITION 4.8. *Let P be a poset. Then the following statements are equivalent.*

1) $\Gamma_E(P) = \Gamma_E(P)^c$.
2) $\Gamma_E(P)$ *is complemented.*
3) $\Gamma(P)$ *is complemented.*

Proof. 1) $\Leftrightarrow$ 2) is obvious.

2) $\Rightarrow$ 3) Suppose $a \in P$ and $[a]$ has a complement $[b]$. Then $[a] \neq [b]$, $[a] \neq [0]$, $[b] \neq [0]$ and $L([a],[b]) = \{[0]\}$. Therefore, $a \neq b, a \neq 0, b \neq 0$ and $L(a,b) = \{0\}$ by Lemma 3.3. If there exists a $c \in P$ such that $L(c,a) = L(c,b) = \{0\}$, then $L([c],[a]) = L([c],[b]) = \{[0]\}$ by Lemma 3.3 and $[c] \notin \{[a],[b]\}$. That is, $[c]$ is adjacent to both $[a]$ and $[b]$. This is a contradiction. Hence b is a complement of a in $\Gamma(P)$.

3) $\Rightarrow$ 2) Suppose $[a] \in V(\Gamma_E(P))$ and $a \perp b$. Then we have $L([a],[b]) = \{[0]\}$. If there exists $[c] \in V(\Gamma_E(P))$ such that $L([c],[a]) = L([c],[b]) = \{[0]\}$, then $L(c,a) = L(c,b) = \{0\}$ and $c \notin \{a,b\}$. This is a contradiction. Hence $[a]$ has a complement $[b]$. □

Proposition 4.9. *Let P be a poset and Atom$(P) = \{a_1, a_2, \dots, a_n\}$. Then*

1) *$\Gamma(P)$ is an n-partite graph.*
2) *$\Gamma_E(P)$ is an n-partite graph.*

Proof. 1) Define

$$V_i = \{x \mid x \geq a_i \text{ and if } j < i, \text{there exists no } a_j \text{ such that } x \geq a_j\}.$$

Then $V_1, \dots, V_n$ are disjoint sets and $P \backslash \{0\} = \bigcup_{i=1}^{n} V_i$. Suppose $x, y \in V_i$, for all $i = 1, 2, \dots, n$. Since $x \geq a_i$ and $y \geq a_i$, there is no edge between x and y. Hence, we get the desired result.

2) Let $\overline{V_i} = \{[x] \mid x \in V_i\}$. If $[x],[y] \in \overline{V_i}$, for all $i = 1, 2, \dots, n$, it is easy to see that there is no edge between $[x]$ and $[y]$. So $\Gamma_E(P)$ is an n-partite graph. □

Remark 4.10. Proposition 4.9 can also be obtained directly from [13, Theorem 4.7 and Theorem 2.9].

Theorem 4.11. *Let P be a poset. Then $\Gamma_E(P)$ is a complete bipartie graph if and only if $|Atom(P)| = 2$.*

Proof. $\Rightarrow$: Suppose $\Gamma_E(P)$ is a complete bipartite graph. If P has only one atom, then $\Gamma_E(P)$ is the null graph. Hence, $|\text{Atom}(P)| \geq 2$. If there exist three atoms $a, b, c \in \text{Atom}(P)$, we obviously have a triangle $[a]-[b]-[c]-[a]$. This is impossible, since a complete bipartite graph has no cycle of odd length.

$\Leftarrow$: Suppose $\text{Atom}(P) = \{a, b\}$. Then $\Gamma_E(P)$ is a bipartite graph by Proposition 4.9.

1) If $x \in P$ such that $x \geq a$ and $x||b$, then $\text{ann}(x) = \text{ann}(a)$, i.e., $[x] = [a]$.

2) Similarly, if $x \in P$ such that $x \geq b$ and $x||a$, then $[x] = [b]$.

3) If $x \in P$ such that $x \geq a$ and $x \geq b$, then $\text{ann}(x) = \{0\}$, i.e., $[x] = [1]$.

In all cases, $\Gamma_E(P)$ has two vertices $\{[a],[b]\}$ and so we have $\Gamma_E(P) = K_2$. □

By the proof of Theorem 4.11, we get the following corollary.

Corollary 4.12. *Let P be a poset. Then $\Gamma_E(P) = K_2$ if and only if $|Atom(P)| = 2$.*

Estaji and Khashyarmanesh ([12]) showed that two vertices a and b are adjacent in a zero-divisor graph of a lattice if and only if $\mathrm{atom}(a) \cap \mathrm{atom}(b) = \emptyset$ (Theorem 5.8). The following statement is similar to Theorem 5.8 in [12].

Theorem 4.13. *Let P be a poset. Then*

1) *x and y are adjacent in $\Gamma(P)$ if and only if $atom(x) \cap atom(y) = \emptyset$.*
2) *x and y are not adjacent in $\Gamma(P)$ if and only if $atom(x) \cap atom(y) \neq \emptyset$.*

Proof. 1) $\Rightarrow$: If there exists $a \in \mathrm{Atom}(P)$ such that $a \in \mathrm{atom}(x) \cap \mathrm{atom}(y)$, then $a \leq x$ and $a \leq y$. This contradicts the fact that $L(x, y) = \{0\}$.

$\Leftarrow$: Suppose $z \in L(x, y)$. If $z \neq 0$, then there exists an $a \in \mathrm{Atom}(P)$ such that $a \leq z$. Hence, $a \in \mathrm{atom}(x) \cap \mathrm{atom}(y)$. This is a contradiction.

2) By 1), we obviously get 2). □

By Theorem 4.13 and Proposition 3.4, we have the following theorem.

Theorem 4.14. *Let P be a poset. Then*

1) *$[x]$ and $[y]$ are adjacent in $\Gamma_E(P)$ if and only if for all $x' \in [x]$ and $y' \in [y]$, we have $atom(x') \cap atom(y') = \emptyset$.*
2) *$[x]$ and $[y]$ are not adjacent in $\Gamma_E(P)$ if and only if for all $x' \in [x]$ and $y' \in [y]$, we have $atom(x') \cap atom(y') \neq \emptyset$.*

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References

[1] M. Alizadeh, A. K. Das, H. R. Maimani, M. R. Pournaki and S. Yassemi, *On the diameter and girth of zero-divisor graphs of posets,* Discrete Appl. Math. **160** (2012), 1319–1324.

[2] D. F. Anderson and J. D. LaGrange, *Commutative Boolean monoids, reduced rings, and the compressed zero-divisor graph,* J. Pure Appl. Algebra **216** (2012), 1626–1636.

[3] D. F. Anderson, R. Levy and J. Shapiro, *Zero-divisor graphs, von Neumann regular rings, and Boolean algebras,* J. Pure Appl. Algebra **180** (2003), 221–241.

[4] D. D. Anderson and M. Naseer, *Beck's coloring of a commutative ring,* J. Algebra **159** (1993), 500–514.

[5] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring,* J. Algebra **217** (1999), 434–447.

[6] M. Axtell, N. Baeth and J. Stickles, *Cut vertices in zero-divisor graphs of finite commutative rings,* Comm. Algebra **39** (2011), 2179–2188.

[7] I. Beck, *Coloring of commutative rings,* J. Algebra **116** (1988), 208–226.

[8] F. R. DeMeyer and L. DeMeyer, *Zero-divisor graphs of semigroups,* J. Algebra **283** (2005), 190–198.

[9] F. R. DeMeyer, T. McKenzie and K. Schneider, *The zero-divisor graph of a commutative semigroup,* Semigroup Forum **65** (2002), 206–214.

[10] D. Dolžan and P. Oblak, *The zero-divisor graph of rings and semirings,* Internat. J. Algebra Comput. **22** (2012), 1250033-1–1250033-20.

[11] S. Ebrahimi Atani, *An ideal-based zero-divisor graph of a commutative semiring,* Glas. Mat. Ser. III **44(64)** (2009), 141–153.

[12] E. Estaji and K. Khashyarmanesh, *The zero-divisor graph of a lattice,* Results Math. **61** (2012), 1–11.

[13] R. Halaš and M. Jukl, *On Beck's coloring of posets*, Discrete Math. **309** (2009), 4584–4589.

[14] R. Halaš and H. Länger, *The zerodivisor graph of a qoset*, Order **27** (2010), 343–351.

[15] V. Joshi, *Zero divisor graph of a poset with respect to an ideal,* Order **29** (2012), 499–506.

[16] D. Lu and T. Wu, *The zero-divisor graphs of posets and an application to semigroups*, Graphs Combin. **26** (2010), 793–804.

[17] H. R. Maimani, *Median and center of zero-divisor graph of commutative semigroups,* Iran. J. Math. Sci. Inform. **3** (2008), 69–76.

[18] S. B. Mulay, *Cycles and symmetries of zero-divisors,* Comm. Algebra **30** (2002), 3533–3558.

[19] S. Spiroff and C. Wickham, *A zero divisor graph determined by equivalence classes of zero-divisors*, Comm. Algebra **39** (2011), 2338–2348.

[20] Z. Xue and S. Liu, *Zero-divisor graphs of partially ordered sets*, Appl. Math. Lett. **23** (2010), 449–452.

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