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ON SOME FUNCTIONAL EQUATIONS RELATED TO DERIVATIONS AND BICIRCULAR PROJECTIONS IN RINGS

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Abstract. In this paper we prove the following result. Let $n \geq 1$ be some fixed integer and let R be a prime ring with $2n < char(R) \neq 2$. Suppose there exist additive mappings $S, T : R \to R$ satisfying the relations

$$S(x^{2n}) = S(x)x^{2n-1} + xT(x)x^{2n-2} + x^2S(x)x^{2n-3} + \cdots + x^{2n-1}T(x),$$
$$T(x^{2n}) = T(x)x^{2n-1} + xS(x)x^{2n-2} + x^2T(x)x^{2n-3} + \cdots + x^{2n-1}S(x)$$

for all $x \in R$. In this case S and T are of the form $2S(x) = D(x) + \zeta(x)$, $2T(x) = D(x) - \zeta(x)$ for all $x \in R$, where $D : R \to R$ is a derivation and ζ is an additive mapping, which maps R into its extended centroid. Besides, $\zeta(x^{2n}) = 0$ for all $x \in R$. Functional equations related to bicircular projections are also investigated.

This research is a continuation of a recent work of M. Fošner and Vukman ([14]). Throughout the paper, R will represent an associative ring with center $Z(R)$. Given an integer $n > 1$, a ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. An additive mapping $x \mapsto x^*$ on a ring R is called an involution if $(xy)^* = y^*x^*$ and $x^{**} = x$ hold for all pairs $x, y \in R$. A ring equipped with an involution is called a ring with involution or *-ring. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$ and is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by $char(R)$ the characteristic of a prime ring R. An additive mapping $D : R \to R$, where R is an arbitrary ring, is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$

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is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$.

Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein ([15]) asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. Cusack ([8]) generalized Herstein theorem to 2-torsion free semiprime rings.

We denote by Q_{mr}, Q_S and C the maximal right ring of quotients, symmetric Martidale ring of quotients and extended centroid of a semiprime ring R, respectively. For the explanation of Q_{mr}, Q_S and C we refer the reader to [1]. Given some $X \subset R$ we denote $C(X) = \{r \in R \mid [r, X] = 0\}$.

Let X be a complex Banach space and let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on X. A projection $P \in \mathcal{L}(X)$ is bicircular in case all mappings of the form $e^{i\alpha}P + e^{i\beta}(I - P)$, where I denotes the identity operator, are isometric for all pairs of real numbers α, β.

Beidar, Brešar, Chebotar and Martindale ([3]) have proved the following result, which fairly generalizes Herstein theorem.

Theorem 1. *Let $n > 1$ be some fixed integer and let R be a prime ring with $char(R) \neq 2$. Suppose there exists an additive mapping $D : R \to R$ satisfying the relation*

$$D(x^n) = \sum_{i=1}^{n} x^{i-1} D(x) x^{n-i}$$

for all $x \in R$. In this case D is a derivation.

Recently, M. Fošner and Vukman ([14]) proved the following result.

Theorem 2. *Let $n \geq 1$ be some fixed integer and let R be a prime ring with $2n \leq char(R) \neq 2$. Suppose there exists an additive mapping $T : R \to R$ satisfying the relation*

$$T(x^{2n+1}) = \sum_{i=1}^{2n+1} (-1)^{i+1} x^{i-1} T(x) x^{2n+1-i} \tag{1}$$

for all $x \in R$. In this case T is of the form $T(x) = qx + xq$ for all $x \in R$ and some fixed $q \in Q_S$.

It seems natural to ask what can be proved in case we have an even number in the relation (1). More precisely, we are talking about the solution of the functional equation

$$T(x^{2n}) = \sum_{i=1}^{2n} (-1)^{i+1} x^{i-1} T(x) x^{2n-i}.$$

It is our aim in this paper to prove the following result.

THEOREM 3. *Let $n \geq 1$ be some fixed integer and let R be a prime ring with $2n < char(R) \neq 2$. Suppose there exists an additive mapping $T : R \to R$ satisfying*

$$T(x^{2n}) = \sum_{i=1}^{2n} (-1)^{i-1} x^{i-1} T(x) x^{2n-i} \tag{2}$$

for all $x \in R$. In this case T maps R into C and $T(x^{2n}) = 0$ for all $x \in R$.

In the proof of Theorem 3 we use as the main tool the theory of functional identities (Beidar - Brešar - Chebotar theory). The theory of functional identities considers set-theoretic maps on rings that satisfy some identical relations. When treating such relations, one usually concludes that the form of the maps involved can be described, unless the ring is very special. For the full treatment on this theory, we refer the reader to [7].

For the proof of Theorem 3 we need Theorem 4, which might be of independent interest. Let R be an algebra over a commutative ring ξ and let

$$p(x_1, x_2, \dots, x_{2n}) = \sum_{\pi \in S_{2n}} x_{\pi(1)} x_{\pi(2)} \cdots x_{\pi(2n)}$$

be a fixed multilinear polynomial in noncommuting indeterminates $x_1, x_2, \dots,$ x_{2n}. Further, let $\mathcal{L}$ be a subset of R closed under p, which means $p(\bar{x}_{2n}) \in \mathcal{L}$ for all $x_1, x_2, \dots, x_{2n} \in \mathcal{L}$, where $\bar{x}_{2n} = (x_1, x_2, \dots, x_{2n})$. We shall consider a mapping $T : \mathcal{L} \to R$ satisfying

$$T(p(\bar{x}_{2n})) = \sum_{\pi \in S_{2n}} \sum_{i=1}^{2n} (-1)^{i-1} x_{\pi(1)} \cdots x_{\pi(i-1)} T(x_{\pi(i)}) x_{\pi(i+1)} \cdots x_{\pi(2n)} \tag{3}$$

for all $x_1, x_2, \dots, x_{2n} \in \mathcal{L}$. Let us mention that the idea of considering the expression $[p(\bar{x}_{2n}), p(\bar{y}_{2n})]$ in its proof is taken from [2].

THEOREM 4. *Let $\mathcal{L}$ be a $4n$-free Lie subring of R closed under p. If $T : \mathcal{L} \to R$ is an additive mapping satisfying (3), then T maps $\mathcal{L}$ into C and $T(x^{2n}) = 0$ for all $x \in \mathcal{L}$.*

PROOF. Let us write $k = 2n$ for brevity. Note that for any $a \in R$ and $\bar{x}_k \in \mathcal{L}^k$, we have

$$[p(\bar{x}_k), a] = \sum_{i=1}^{k} p(x_1, x_2, \dots, x_{i-1}, [x_i, a], x_{i+1}, \dots, x_k).$$

Thus

$$T[p(\bar{x}_k), a] = \sum_{i=1}^{k} T(p(x_1, x_2, \dots x_{i-1}, [x_i, a], x_{i+1}, \dots, x_k)).$$

Let $f, g : \mathbb{N} \to \mathbb{N}$ be mappings with properties $f(1) = g(k) = 0$ and $f(i) = 1$ for $i = 2, 3, \ldots, k$ and $g(i) = 1$ for $i = 1, 2, \ldots, k-1$. Therefore

$$\begin{aligned}
&T[p(\bar{x}_k), a] \\
&\quad = \sum_{\pi \in S_k} \sum_{i=1}^{k} (-1)^{i-1} \Big(f(i)[x_{\pi(1)} \ldots x_{\pi(i-1)}, a] T(x_{\pi(i)}) x_{\pi(i+1)} \ldots x_{\pi(k)} \\
&\qquad + x_{\pi(1)} \ldots x_{\pi(i-1)} T[x_{\pi(i)}, a] x_{\pi(i+1)} \ldots x_{\pi(k)} \\
&\qquad + g(i) x_{\pi(1)} \ldots x_{\pi(i-1)} T(x_{\pi(i)}) [x_{\pi(i+1)} \ldots x_{\pi(k)}, a] \Big).
\end{aligned}$$

In particular, we have

$$\begin{aligned}
&T[p(\bar{x}_k), p(\bar{y}_k)] \\
&\quad = \sum_{\pi \in S_k} \sum_{i=1}^{k} (-1)^{i-1} \Big(f(i)[x_{\pi(1)} \ldots x_{\pi(i-1)}, p(\bar{y}_k)] T(x_{\pi(i)}) x_{\pi(i+1)} \ldots x_{\pi(k)} \\
&\qquad + x_{\pi(1)} \ldots x_{\pi(i-1)} T[x_{\pi(i)}, p(\bar{y}_k)] x_{\pi(i+1)} \ldots x_{\pi(k)} \\
&\qquad + g(i) x_{\pi(1)} \ldots x_{\pi(i-1)} T(x_{\pi(i)}) [x_{\pi(i+1)} \ldots x_{\pi(k)}, p(\bar{y}_k)] \Big).
\end{aligned} \tag{4}$$

For $i = 1, 2, \ldots, k$ let us denote $\varphi(x_{\pi(i)}) = T[x_{\pi(i)}, p(\bar{y}_k)]$. We therefore have

$$\begin{aligned}
\varphi(x_{\pi(i)}) = {} & T[x_{\pi(i)}, p(\bar{y}_k)] = -T[p(\bar{y}_k), x_{\pi(i)}] \\
= {} & \sum_{\sigma \in S_k} \sum_{j=1}^{k} (-1)^{j-1} \Big(f(j)[x_{\pi(i)}, y_{\sigma(1)} \ldots y_{\sigma(j-1)}] T(y_{\sigma(j)}) y_{\sigma(j+1)} \ldots y_{\sigma(k)} \\
& + y_{\sigma(1)} \ldots y_{\sigma(j-1)} T[x_{\pi(i)}, y_{\sigma(j)}] y_{\sigma(j+1)} \ldots y_{\sigma(k)} \\
& + g(j) y_{\sigma(1)} \ldots y_{\sigma(j-1)} T(y_{\sigma(j)}) [x_{\pi(i)}, y_{\sigma(j+1)} \ldots y_{\sigma(k)}] \Big).
\end{aligned}$$

Therefore (4) can be written as

$$\begin{aligned}
&T[p(\bar{x}_k), p(\bar{y}_k)] \\
&\quad = \sum_{\pi \in S_k} \sum_{\sigma \in S_k} \sum_{i=1}^{k} (-1)^{i-1} f(i) [x_{\pi(1)} \ldots x_{\pi(i-1)}, y_{\sigma(1)} \ldots y_{\sigma(k)}] \cdot \\
&\qquad\qquad \cdot T(x_{\pi(i)}) x_{\pi(i+1)} \ldots x_{\pi(k)} \\
&\qquad + \sum_{\pi \in S_k} \sum_{\sigma \in S_k} \sum_{i=1}^{k} (-1)^{i-1} x_{\pi(1)} \ldots x_{\pi(i-1)} \varphi(x_{\pi(i)}) x_{\pi(i+1)} \ldots x_{\pi(k)} \\
&\qquad + \sum_{\pi \in S_k} \sum_{\sigma \in S_k} \sum_{i=1}^{k} (-1)^{i-1} g(i) x_{\pi(1)} \ldots x_{\pi(i-1)} T(x_{\pi(i)}) \cdot \\
&\qquad\qquad \cdot [x_{\pi(i+1)} \ldots x_{\pi(k)}, y_{\sigma(1)} \ldots y_{\sigma(k)}].
\end{aligned} \tag{5}$$

If we replace the roles of denotations π and σ, we obtain from (5) that

$$
\begin{aligned}
&T[p(\bar{x}_k), p(\bar{y}_k)]\\
&\quad= \sum_{\pi\in S_k}\sum_{\sigma\in S_k}\sum_{i=1}^{k}(-1)^{i-1}f(i)[x_{\pi(1)}\dots x_{\pi(k)}, y_{\sigma(1)}\dots y_{\sigma(i-1)}]\cdot\\
&\qquad\qquad\cdot T(y_{\sigma(i)})y_{\sigma(i+1)}\dots y_{\sigma(k)}\\
&\qquad+ \sum_{\pi\in S_k}\sum_{\sigma\in S_k}\sum_{i=1}^{k}(-1)^{i-1}y_{\sigma(1)}\dots y_{\sigma(i-1)}\bar{\varphi}(y_{\sigma(i)})y_{\sigma(i+1)}\dots y_{\sigma(k)}\\
&\qquad+ \sum_{\pi\in S_k}\sum_{\sigma\in S_k}\sum_{i=1}^{k}(-1)^{i-1}g(i)y_{\sigma(1)}\dots y_{\sigma(i-1)}T(y_{\sigma(i)})\cdot\\
&\qquad\qquad\cdot[x_{\pi(1)}\dots x_{\pi(k)}, y_{\sigma(i+1)}\dots y_{\sigma(k)}],
\end{aligned}
\tag{6}
$$

where $\bar{\varphi}(y_{\sigma(i)}) = T[p(\bar{x}_k), y_{\sigma(i)}]$. One can easily check that

$$\bar{\varphi}(x_{\pi(i)}) = -\varphi(x_{\pi(i)})$$

for all $i = 1, 2, \dots, k$. Comparing (5) and (6) we obtain the identity

$$
\begin{aligned}
&0 =\\
&\quad= \sum_{\pi\in S_k}\sum_{\sigma\in S_k}\sum_{i=1}^{k}(-1)^{i-1}f(i)[x_{\pi(1)}\dots x_{\pi(i-1)}, y_{\sigma(1)}\dots y_{\sigma(k)}]\cdot\\
&\qquad\qquad\cdot T(x_{\pi(i)})x_{\pi(i+1)}\dots x_{\pi(k)}\\
&\qquad+ \sum_{\pi\in S_k}\sum_{\sigma\in S_k}\sum_{i=1}^{k}(-1)^{i-1}x_{\pi(1)}\dots x_{\pi(i-1)}\varphi(x_{\pi(i)})x_{\pi(i+1)}\dots x_{\pi(k)}\\
&\qquad+ \sum_{\pi\in S_k}\sum_{\sigma\in S_k}\sum_{i=1}^{k}(-1)^{i-1}g(i)x_{\pi(1)}\dots x_{\pi(i-1)}T(x_{\pi(i)})\cdot\\
&\qquad\qquad\cdot[x_{\pi(i+1)}\dots x_{\pi(k)}, y_{\sigma(1)}\dots y_{\sigma(k)}]\\
&\qquad- \sum_{\pi\in S_k}\sum_{\sigma\in S_k}\sum_{i=1}^{k}(-1)^{i-1}f(i)[x_{\pi(1)}\dots x_{\pi(k)}, y_{\sigma(1)}\dots y_{\sigma(i-1)}]\cdot\\
&\qquad\qquad\cdot T(y_{\sigma(i)})y_{\sigma(i+1)}\dots y_{\sigma(k)}\\
&\qquad+ \sum_{\pi\in S_k}\sum_{\sigma\in S_k}\sum_{i=1}^{k}(-1)^{i-1}y_{\sigma(1)}\dots y_{\sigma(i-1)}\varphi(y_{\sigma(i)})y_{\sigma(i+1)}\dots y_{\sigma(k)}\\
&\qquad- \sum_{\pi\in S_k}\sum_{\sigma\in S_k}\sum_{i=1}^{k}(-1)^{i-1}g(i)y_{\sigma(1)}\dots y_{\sigma(i-1)}T(y_{\sigma(i)})\cdot\\
&\qquad\qquad\cdot[x_{\pi(1)}\dots x_{\pi(k)}, y_{\sigma(i+1)}\dots y_{\sigma(k)}]
\end{aligned}
$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathcal{L}$. The last relation can be written as

$$
\begin{aligned}
0 = &\sum_{\pi \in S_k} \sum_{\sigma \in S_k} \Big(T(y_{\sigma(1)}) y_{\sigma(2)} \ldots y_{\sigma(k)} x_{\pi(1)} \ldots x_{\pi(k-1)} \\
&+ \varphi(x_{\pi(1)}) x_{\pi(2)} \ldots x_{\pi(k-1)} \\
&- T(x_{\pi(1)}) y_{\sigma(1)} \ldots y_{\sigma(k)} x_{\pi(2)} \ldots x_{\pi(k-1)} \\
&+ \sum_{i=2}^{k-1} (-1)^{i-1} x_{\pi(1)} \ldots x_{\pi(i-1)} \varphi(x_{\pi(i)}) x_{\pi(i+1)} \ldots x_{\pi(k-1)} \\
&+ \sum_{i=2}^{k-1} (-1)^{i-1} [x_{\pi(1)} \ldots x_{\pi(i-1)}, y_{\sigma(1)} \ldots y_{\sigma(k)}] \cdot \\
&\qquad \cdot T(x_{\pi(i)}) x_{\pi(i+1)} \ldots x_{\pi(k-1)} \Big) x_{\pi(k)} \\
&+ \sum_{\pi \in S_k} \sum_{\sigma \in S_k} \Big(T(x_{\pi(1)}) x_{\pi(2)} \ldots x_{\pi(k)} y_{\sigma(1)} \ldots y_{\sigma(k-1)} \\
&+ \varphi(y_{\sigma(1)}) y_{\sigma(2)} \ldots y_{\sigma(k-1)} \\
&- T(y_{\sigma(1)}) x_{\pi(1)} \ldots x_{\pi(k)} y_{\sigma(2)} \ldots y_{\sigma(k-1)} \\
&+ \sum_{i=2}^{k-1} (-1)^{i-1} y_{\sigma(1)} \ldots y_{\sigma(i-1)} \varphi(y_{\sigma(i)}) y_{\sigma(i+1)} \ldots y_{\sigma(k-1)} \\
&+ \sum_{i=2}^{k-1} (-1)^{i-1} [y_{\sigma(1)} \ldots y_{\sigma(i-1)}, x_{\pi(1)} \ldots x_{\pi(k)}] \cdot \\
&\qquad \cdot T(y_{\sigma(i)}) y_{\sigma(i+1)} \ldots y_{\sigma(k-1)} \Big) y_{\sigma(k)} \\
&+ \sum_{\pi \in S_k} \sum_{\sigma \in S_k} x_{\pi(1)} \Big(- x_{\pi(2)} \ldots x_{\pi(k-1)} y_{\sigma(1)} \ldots y_{\sigma(k)} T(x_{\pi(k)}) \\
&+ x_{\pi(2)} \ldots x_{\pi(k)} y_{\sigma(1)} \ldots y_{\sigma(k-1)} T(y_{\sigma(k)}) \\
&- x_{\pi(2)} \ldots x_{\pi(k-1)} \varphi(x_{\pi(k)}) \\
&+ \sum_{i=2}^{k-1} (-1)^{i-1} x_{\pi(2)} \ldots x_{\pi(i-1)} T(x_{\pi(i)}) \cdot \\
&\qquad \cdot [x_{\pi(i+1)} \ldots x_{\pi(k)}, y_{\sigma(1)} \ldots y_{\sigma(k)}] \Big) \\
&+ \sum_{\pi \in S_k} \sum_{\sigma \in S_k} y_{\sigma(1)} \Big(- y_{\sigma(2)} \ldots y_{\sigma(k-1)} x_{\pi(1)} \ldots x_{\pi(k)} T(y_{\sigma(k)}) \\
&+ y_{\sigma(2)} \ldots y_{\sigma(k)} x_{\pi(1)} \ldots x_{\pi(k-1)} T(x_{\pi(k)}) \\
&- y_{\sigma(2)} \ldots y_{\sigma(k-1)} \varphi(y_{\sigma(k)}) \\
&+ \sum_{i=2}^{k-1} (-1)^{i-1} y_{\sigma(2)} \ldots y_{\sigma(i-1)} T(y_{\sigma(i)}) \cdot \\
&\qquad \cdot [y_{\sigma(i+1)} \ldots y_{\sigma(k)}, x_{\pi(1)} \ldots x_{\pi(k)}] \Big)
\end{aligned} \tag{7}
$$

for all $x_1,\dots,x_k,y_1,\dots,y_k\in\mathcal{L}$. Let us define mappings $E,F:\mathcal{L}^{2k-1}\to R$ by the rule

$$\begin{aligned}E(u_1,u_2,u_3,\dots,u_{2k-1}) &= T(u_k)u_{k+1}\dots u_{2k-1}u_1\dots u_{k-1}\\ &+\varphi(u_1)u_2\dots u_{k-1}-T(u_1)u_k\dots u_{2k-1}u_2u_3\dots u_{k-1}\\ &+\sum_{i=2}^{k-1}(-1)^{i-1}u_1\dots u_{i-1}\varphi(u_i)u_{i+1}\dots u_{k-1}\\ &+\sum_{i=2}^{k-1}(-1)^{i-1}[u_1\dots u_{i-1},u_k\dots u_{2k-1}]T(u_i)u_{i+1}\dots u_{k-1}\end{aligned}$$

and

$$\begin{aligned}F(u_1,u_2,\,u_3,\dots,u_{2k-1}) &= -u_1u_2\dots u_{k-2}u_ku_{k+1}\dots u_{2k-1}T(u_{k-1})\\ &+u_1u_2\dots u_{k-1}u_ku_{k+1}\dots u_{2k-2}T(u_{2k-1})-u_1u_2\dots u_{k-2}\varphi(u_{k-1})\\ &+\sum_{i=1}^{k-2}(-1)^{i-1}u_1u_2\dots u_{i-1}T(u_i)[u_{i+1}\dots u_{k-1},u_k\dots u_{2k-1}]\end{aligned}$$

for all $\bar{u}_{2k-1}\in\mathcal{L}^{2k-1}$. Accordingly, (7) can be rewritten as

$$\begin{aligned}0 &= \sum_{\pi\in S_k}\sum_{\sigma\in S_k}E\big(x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(k-1)},y_{\sigma(1)},y_{\sigma(2)},\dots,y_{\sigma(k)}\big)x_{\pi(k)}\\ &+\sum_{\pi\in S_k}\sum_{\sigma\in S_k}E\big(y_{\sigma(1)},y_{\sigma(2)},\dots,y_{\sigma(k-1)},x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(k)}\big)y_{\sigma(k)}\\ &+\sum_{\pi\in S_k}\sum_{\sigma\in S_k}x_{\pi(1)}F\big(x_{\pi(2)},x_{\pi(3)},\dots,x_{\pi(k)},y_{\sigma(1)},y_{\sigma(2)},\dots,y_{\sigma(k)}\big)\\ &+\sum_{\pi\in S_k}\sum_{\sigma\in S_k}y_{\sigma(1)}F\big(y_{\sigma(2)},y_{\sigma(3)},\dots,y_{\sigma(k)},x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(k)}\big)\end{aligned}$$

and hence

$$\begin{aligned}0 &= \sum_{i=1}^{k}\left(\sum_{\substack{\pi\in S_k\\ \pi(k)=i}}\sum_{\sigma\in S_k}E\big(x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(k-1)},y_{\sigma(1)},y_{\sigma(2)},\dots,y_{\sigma(k)}\big)\right)x_i\\ &+\sum_{i=k+1}^{2k}\left(\sum_{\pi\in S_k}\sum_{\substack{\sigma\in S_k\\ \sigma(k)=i}}E\big(y_{\sigma(1)},y_{\sigma(2)},\dots,y_{\sigma(k-1)},x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(k)}\big)\right)y_i\\ &+\sum_{j=1}^{k}x_j\left(\sum_{\substack{\pi\in S_k\\ \pi(1)=j}}\sum_{\sigma\in S_k}F\big(x_{\pi(2)},x_{\pi(3)},\dots,x_{\pi(k)},y_{\sigma(1)},y_{\sigma(2)},\dots,y_{\sigma(k)}\big)\right)\end{aligned}$$

$$+\sum_{j=k+1}^{2k} y_j\left(\sum_{\pi\in S_k}\sum_{\substack{\sigma\in S_k\\ \sigma(1)=j}} F\big(y_{\sigma(2)},y_{\sigma(3)},\dots,y_{\sigma(k)},x_{\pi(1)},x_{\pi(2)},\dots,x_{\pi(k)}\big)\right).$$

Let $s:\mathbb{Z}\to\mathbb{Z}$ be a mapping defined by $s(i)=i-k$. For each $\sigma\in S_k$ the mapping $s^{-1}\sigma s:\{k+1,\dots,2k\}\to\{k+1,\dots,2k\}$ will be denoted by $\bar{\sigma}$. Writting x_{k+i} instead of $y_i, i=1,2,\dots,k$ in the above identity, we can express this relation as

$$\sum_{i=1}^{2k} E_i(\bar{x}_{2k}^i)x_i+\sum_{j=1}^{2k} x_jF_j(\bar{x}_{2k}^j)=0.$$

According to the theory of functional identities (see [7, Chapter 3.2]) there exist mappings $p_{k,i}:\mathcal{L}^{k-2}\to R$, $i=1,\dots,k-1$ and $\lambda_k:\mathcal{L}^{k-1}\to C(\mathcal{L})$ such that

$$\begin{aligned}\sum_{\substack{\pi\in S_k\\ \pi(1)=1}}\sum_{\sigma\in S_k} &F\big(x_{\pi(2)},x_{\pi(3)},\dots x_{\pi(k)},x_{\bar{\sigma}(k+1)},x_{\bar{\sigma}(k+2)},\dots,x_{\bar{\sigma}(2k)}\big)\\ &=\sum_{i=1}^{k-1} p_{k,i}(\bar{x}_{k-1}^i)x_i+\lambda_k(\bar{x}_{k-1})\end{aligned}$$

for all $\bar{x}_{k-1}\in\mathcal{L}^{k-1}$. Recalling the definition of a mapping F leads to

$$\begin{aligned}\sum_{\substack{\pi\in S_k\\ \pi(1)=1}}\sum_{\sigma\in S_k} \quad &x_{\pi(2)}\Big(-x_{\pi(3)}\dots x_{\pi(k-1)}y_{\sigma(1)}\dots y_{\sigma(k)}T(x_{\pi(k)})\\ &+x_{\pi(3)}\dots x_{\pi(k)}y_{\sigma(1)}\dots y_{\sigma(k-1)}T(y_{\sigma(k)})\\ &-x_{\pi(3)}\dots x_{\pi(k-1)}\varphi(x_{\pi(k)})\Big)\\ +\sum_{\substack{\pi\in S_k\\ \pi(1)=1}}\sum_{\sigma\in S_k}&\Big(\sum_{i=2}^{k-1}(-1)^{i-1}x_{\pi(2)}\dots x_{\pi(i-1)}\cdot\\ &\cdot T(x_{\pi(i)})x_{\pi(i+1)}\dots x_{\pi(k)}y_{\sigma(1)}\dots y_{\sigma(k-1)}\Big)y_{\sigma(k)}\\ -\sum_{\substack{\pi\in S_k\\ \pi(1)=1}}\sum_{\sigma\in S_k}&\Big(\sum_{i=2}^{k-1}(-1)^{i-1}x_{\pi(2)}\dots x_{\pi(i-1)}\cdot\\ &\cdot T(x_{\pi(i)})y_{\sigma(1)}\dots y_{\sigma(k)}x_{\pi(i+1)}\dots x_{\pi(k-1)}\Big)x_{\pi(k)}\\ -\sum_{i=1}^{k-1}&p_{k,i}(\bar{x}_{k-1}^i)x_i \quad \in C(\mathcal{L})\end{aligned}$$

for all $\bar{x}_{k-1} \in \mathcal{L}^{k-1}$. Applying the theory of functional identities it follows that

$$\begin{aligned}\sum_{\substack{\pi\in S_k\\ \pi(1)=1\\ \pi(2)=2}} \sum_{\sigma\in S_k} & -x_{\pi(3)}\dots x_{\pi(k-1)}y_{\sigma(1)}\dots y_{\sigma(k)}T(x_{\pi(k)})\\ &+ x_{\pi(3)}\dots x_{\pi(k)}y_{\sigma(1)}\dots y_{\sigma(k-1)}T(y_{\sigma(k)})\\ &- x_{\pi(3)}\dots x_{\pi(k-1)}\varphi(x_{\pi(k)})\\ -\sum_{i=1}^{k-2} & p_{k,i}(\bar{x}^i_{k-2})x_i \quad \in C(\mathcal{L})\end{aligned}$$

for all $\bar{x}_{k-1} \in \mathcal{L}^{k-1}$. Recalling the definition of a mapping $\varphi(x_{\pi(k)})$ leads to

$$\begin{aligned}\sum_{\substack{\pi\in S_k\\ \pi(1)=1\\ \pi(2)=2}} \sum_{\sigma\in S_k} & x_{\pi(3)}\Big(- x_{\pi(4)}\dots x_{\pi(k-1)}y_{\sigma(1)}\dots y_{\sigma(k)}T(x_{\pi(k)})\\ &+ x_{\pi(4)}\dots x_{\pi(k)}y_{\sigma(1)}\dots y_{\sigma(k-1)}T(y_{\sigma(k)})\\ &+ x_{\pi(4)}\dots x_{\pi(k)}y_{\sigma(1)}\dots y_{\sigma(k-1)}T(y_{\sigma(k)})\\ &- x_{\pi(4)}\dots x_{\pi(k-1)}y_{\sigma(1)}\dots y_{\sigma(k-1)}x_{\pi(k)}T(y_{\sigma(k)})\\ &+ x_{\pi(4)}\dots x_{\pi(k-1)}y_{\sigma(1)}\dots y_{\sigma(k-1)}T[x_{\pi(k)}, y_{\sigma(k)}]\Big)\\ +\sum_{\substack{\pi\in S_k\\ \pi(1)=1\\ \pi(2)=2}} \sum_{\sigma\in S_k}\sum_{j=1}^{k-1} & \Big(x_{\pi(3)}\dots x_{\pi(k-1)}y_{\sigma(1)}\dots y_{\sigma(j-1)}\cdot\\ &\cdot T(y_{\sigma(j)})y_{\sigma(j+1)}\dots y_{\sigma(k)}\Big)x_{\pi(k)}\\ +\sum_{\substack{\pi\in S_k\\ \pi(1)=1\\ \pi(2)=2}} \sum_{\sigma\in S_k}\sum_{j=1}^{k-1} & \Big(- f(j)x_{\pi(3)}\dots x_{\pi(k-1)}[x_{\pi(i)}, y_{\sigma(1)}\dots y_{\sigma(j-1)}]\cdot\\ &\cdot T(y_{\sigma(j)})y_{\sigma(j+1)}\dots y_{\sigma(k-1)}\\ &- x_{\pi(3)}\dots x_{\pi(k-1)}y_{\sigma(1)}\dots y_{\sigma(j-1)}T[x_{\pi(k)}, y_{\sigma(j)}]y_{\sigma(j+1)}\dots y_{\sigma(k-1)}\\ &- x_{\pi(3)}\dots x_{\pi(k-1)}y_{\sigma(1)}\dots y_{\sigma(j-1)}T(y_{\sigma(j)})x_{\pi(k)}y_{\sigma(j+1)}\dots y_{\sigma(k-1)}\Big)y_{\sigma(k)}\\ -\sum_{i=1}^{k-2} & p_{k,i}(\bar{x}^i_{k-2})x_i \quad \in C(\mathcal{L}).\end{aligned}$$

Applying the theory of functional identities gives

$$\begin{aligned}\sum_{\substack{\pi\in S_k\\ \pi(1)=1\\ \pi(2)=2\\ \pi(3)=3}}\sum_{\sigma\in S_k} \quad & - x_{\pi(4)}\dots x_{\pi(k-1)}y_{\sigma(1)}\dots y_{\sigma(k)}T(x_{\pi(k)})\\ & + x_{\pi(4)}\dots x_{\pi(k)}y_{\sigma(1)}\dots y_{\sigma(k-1)}T(y_{\sigma(k)})\\ & + x_{\pi(4)}\dots x_{\pi(k)}y_{\sigma(1)}\dots y_{\sigma(k-1)}T(y_{\sigma(k)})\\ & - x_{\pi(4)}\dots x_{\pi(k-1)}y_{\sigma(1)}\dots y_{\sigma(k-1)}x_{\pi(k)}T(y_{\sigma(k)})\\ & + x_{\pi(4)}\dots x_{\pi(k-1)}y_{\sigma(1)}\dots y_{\sigma(k-1)}T[x_{\pi(k)}, y_{\sigma(k)}]\\ -\sum_{i=1}^{k-3} p_{k,i}(\bar{x}^i_{k-3})x_i \quad & \in C(\mathcal{L}).\end{aligned}$$

After finite number of steps we arrive at

$$2T(x) = px + \mu(x) \tag{8}$$

for all $x \in \mathcal{L}$, where $p \in \mathcal{L}$ and $\mu : \mathcal{L} \to C(\mathcal{L})$. Putting x^k for x in the above relation leads to

$$2T(x^k) = px^k + \mu(x^k). \tag{9}$$

Combining the above relation and (2) we obtain

$$0 = px^k + \mu(x^k) - \sum_{i=1}^{k}(-1)^{i-1}x^{i-1}(px + \mu(x))x^{k-i}$$

and few calculations lead to

$$0 = \mu(x^k) - \mu(x)x^{k-1} - \sum_{i=2}^{k}(-1)^{i-1}x^{i-1}(px + \mu(x))x^{k-i}.$$

Since $\mu(x) \in C(\mathcal{L})$ for all $x \in \mathcal{L}$, the above relation reduces to

$$0 = \mu(x^k) - \sum_{i=2}^{k}(-1)^{i-1}x^{i-1}px^{k+1-i}. \tag{10}$$

After a complete linearization of the last relation we obtain

$$\begin{aligned}0 = \mu(x_{\pi(1)}\cdots x_{\pi(k)}) + \sum_{\pi\in S_k} x_{\pi(1)}\Big(& px_{\pi(2)}\dots x_{\pi(k)} - x_{\pi(2)}px_{\pi(3)}\dots x_{\pi(k)}\\ & + \dots - x_{\pi(2)}\cdots x_{\pi(k-2)}px_{\pi(k-1)}x_{\pi(k)} + x_{\pi(2)}\cdots x_{\pi(k-1)}px_{\pi(k)}\Big).\end{aligned}$$

The theory of functional identities implies

$$0 = \sum_{\substack{\pi\in S_k\\ \pi(1)=1}} \Big(px_{\pi(2)}\dots x_{\pi(k-1)}\Big)x_{\pi(k)}$$

$$+\sum_{\substack{\pi\in S_k\\ \pi(1)=1}} x_{\pi(2)}\Big(-px_{\pi(3)}\dots x_{\pi(k)}+\dots-x_{\pi(3)}\dots x_{\pi(k-2)}px_{\pi(k-1)}x_{\pi(k)}$$
$$+x_{\pi(3)}\dots x_{\pi(k-1)}px_{\pi(k)}\Big).$$

After a finite number of steps we obtain

$$px = xq + \lambda(x) \tag{11}$$

for all $x \in \mathcal{L}$, where $q \in \mathcal{L}$ and $\lambda : \mathcal{L} \to C(\mathcal{L})$. Right multiplication of the relation (11) by $y \in \mathcal{L}$ gives $pxy = xqy + \lambda(x)y$. Putting xy for x in the relation (11) leads to $pxy = xyq + \lambda(xy)$. Comparing the last two relations we obtain

$$0 = x[q, y] + \lambda(x)y - \lambda(xy),$$

which implies $[q, y] = 0$ for all $y \in \mathcal{L}$. Note that (11) can now be rewritten as $(p - q)x = \lambda(x)$. From the last relation we obtain $p = q$ and $\lambda(x) = 0$. Considering that $p \in C(\mathcal{L})$ in the relation (10) gives

$$0 = px^k + \mu(x^k).$$

Considering the above relation in (9) gives $T(x^k) = 0$ for all $x \in \mathcal{L}$. After a complete linearization of the above relation and using the theory of functional identities, we obtain $p = 0$. The relation (8) now implies that $2T(x) = \mu(x)$, which means that $T(x) \in C(\mathcal{L})$. Thereby the proof is complete. ∎

We are now in the position to prove Theorem 3.

Proof of Theorem 3. The complete linearization of (2) gives (3). Assume first that R is not a PI ring. According to Theorem 4 we have $T(R) \in C$. Now suppose that R is a PI ring. It is well-known that in this case R has a nonzero center (see [21]). Let c be a nonzero central element and let us write k for $2n$ for brevity. Putting $x_1 = x_2 = \ldots = x_k = c$ in (3) leads to

$$T(c^k) = 0.$$

Pick any $x \in R$ and set $x_1 = cx^2$ and $x_2 = x_3 = \ldots = x_k = c$ in (3). Hence we obtain

$$\begin{aligned}
k!T(c^kx^2) &= (k-1)!T(cx^2)c^{k-1} + (k-1)(k-1)!T(c)x^2c^{k-1}\\
&- (k-1)!T(cx^2)c^{k-1} - (k-1)!x^2T(c)c^{k-1} - (k-2)(k-1)!T(c)x^2c^{k-1}\\
&+ (k-1)!T(cx^2)c^{k-1} + 2(k-1)!x^2T(c)c^{k-1} + (k-3)(k-1)!T(c)x^2c^{k-1}\\
&- \cdots\\
&+ (k-1)!T(cx^2)c^{k-1} + (k-2)(k-1)!x^2T(c)c^{k-1} + (k-1)!T(c)x^2c^{k-1}\\
&- (k-1)!T(cx^2)c^{k-1} - (k-1)(k-1)!x^2T(c)c^{k-1}.
\end{aligned}$$

The above relation reduces to

$$2k!T(c^k x^2) = k(k-1)!c^{k-1}\Big(T(c)x^2 - x^2T(c)\Big). \tag{12}$$

Putting $x_1 = x_2 = cx$ and $x_3 = x_4 = \ldots = x_k = c$ in (3) gives, after some calculations,

$$\begin{aligned} 2k!T(c^k x^2) = k(k-2)!c^{k-1}\Big(&2T(cx)x - 2xT(cx) \\ &+ (k-2)\big(T(c)x^2 - x^2T(c)\big)\Big). \end{aligned}$$

Comparing the last two relations leads to

$$2T(cx)x - 2xT(cx) = T(c)x^2 - x^2T(c). \tag{13}$$

Putting $x_1 = x$ and $x_2 = c^{k-1}$ in the complete linearization of the above relation we obtain

$$T(c^k)x - xT(c^k) = T(c)xc^{k-1} - xT(c)c^{k-1}.$$

Since $T(c^k) = 0$, it follows from the above relation that

$$T(c)x = xT(c). \tag{14}$$

Considering the above relation in (13) gives

$$T(cx)x = xT(cx). \tag{15}$$

Putting $x_1 = x$, $x_2 = c^2x$ and $x_3 = x_4 = \ldots = x_k = c$ in (3) and also considering (14) and (15) gives

$$2k!T(c^k x^2) = k(k-2)!c^{k-2}\Big(T(x)xc^2 - xT(x)c^2\Big).$$

Since $T(c^k x^2) = 0$ by relations (12) and (14), the above relation reduces to

$$[T(x), x] = 0 \tag{16}$$

for all $x \in R$. Considering the above relation in (2) leads to

$$T(x^k) = 0 \tag{17}$$

for all $x \in R$. The relation (16) implies the existence of such $\lambda \in R$ and $\mu : R \to C$ that

$$T(x) = \lambda x + \mu(x) \tag{18}$$

for all $x \in R$. By (17) we also have

$$0 = \lambda x^k + \mu(x^k)$$

for all $x \in R$. Therefore $\lambda x^k \in C$, whence it follows that

$$[\lambda x^k, y] = 0 \tag{19}$$

for all $x, y \in R$. Putting $x_1 = x_2 = \ldots = x_k = c$ in the complete linearization of the above relation gives $[\lambda, y] = 0$ for all $y \in R$, which implies that $\lambda \in Z(R)$. Now putting $x_1 = x$ and $x_2 = x_3 = \ldots = x_k = c$ in the complete linearization of the relation (19) leads to $[\lambda x, y] = 0$. Left multiplication of the last relation by $z \in R$ and considering $\lambda \in Z(R)$ gives $\lambda z[x, y] = 0$ for all $x, y, z \in R$. Since R is prime, it follows that $\lambda = 0$ or $[x, y] = 0$ for all $x, y \in R$. If $\lambda = 0$ the relation (18) gives $T(x) = \mu(x)$ and therefore $T(x) \in C$ for all $x \in R$. In case $[x, y] = 0$ it is obvious that $[T(x), y] = 0$ for all $x, y \in R$, which means that $T(x) \in Z(R)$. The proof of the theorem is therefore complete. ☐

We proceed with the following result.

THEOREM 5. *Let $n \geq 1$ be some fixed integer and let R be a prime ring with $2n < char(R) \neq 2$. Suppose there exist additive mappings $S, T : R \to R$ satisfying the relations*

$$\begin{aligned} S(x^{2n}) &= S(x)x^{2n-1} + xT(x)x^{2n-2} + x^2S(x)x^{2n-3} + \cdots + x^{2n-1}T(x), \\ T(x^{2n}) &= T(x)x^{2n-1} + xS(x)x^{2n-2} + x^2T(x)x^{2n-3} + \cdots + x^{2n-1}S(x). \end{aligned} \tag{20}$$

In this case S and T are of the form

$$\begin{aligned} 2S(x) &= D(x) + \zeta(x), \\ 2T(x) &= D(x) - \zeta(x) \end{aligned}$$

for all $x \in R$, where $D : R \to R$ is a derivation and $\zeta : R \to C$ is an additive mapping such that $\zeta(x^{2n}) = 0$ for all $x \in R$.

PROOF. Combining relations (20) we obtain

$$D(x^{2n}) = D(x)x^{2n-1} + xD(x)x^{2n-2} + x^2D(x)x^{2n-3} + \cdots + x^{2n-1}D(x),$$

where D stands for $S + T$. According to the above relation and Theorem 1, D is a derivation. Subtracting relations (20) we obtain

$$\zeta(x^{2n}) = \zeta(x)x^{2n-1} - x\zeta(x)x^{2n-2} + x^2\zeta(x)x^{2n-3} + \cdots - x^{2n-1}\zeta(x), \tag{21}$$

where ζ denotes $S - T$. From (21) and Theorem 3 it follows that ζ maps R into C and $\zeta(x^{2n}) = 0$ for all $x \in R$. We therefore have $S + T = D$ and $S - T = \zeta$, whence it follows that $2S(x) = D(x) + \zeta(x)$, $2T(x) = D(x) - \zeta(x)$ for all $x \in R$, which completes the proof of the theorem. ☐

Let us point out that in Theorem 3 we have not assumed that the ring has an identity element. In case the ring has an identity element, the proof is considerably simpler and one can prove the result below even in case R is an arbitrary ring with some torsion restrictions.

THEOREM 6. *Let $n \geq 1$ be some fixed integer and let R be a $2n!$-free semiprime ring with the identity element. Suppose there exists an additive mapping $T : R \to R$ satisfying*

$$T(x^{2n}) = \sum_{i=1}^{2n} (-1)^{i-1} x^{i-1} T(x) x^{2n-i} \tag{22}$$

for all $x \in R$. In this case $T(x) = 0$ for all $x \in R$.

PROOF. Let e be the identity element. Putting e for x in the relation (22) gives

$$T(e) = 0. \tag{23}$$

Let y be any element of $Z(R)$. Linearization in the relation (22) leads to

$$\begin{aligned}
\sum_{i=0}^{2n} \binom{2n}{i} T(x^{2n-i}y^i) = {} & T(x+y)\left(\sum_{i=0}^{2n-1} \binom{2n-1}{i} x^{2n-1-i}y^i\right) \\
& - (x+y)T(x+y)\left(\sum_{i=0}^{2n-2} \binom{2n-2}{i} x^{2n-2-i}y^i\right) \\
& + (x^2+2xy+y^2)T(x+y)\left(\sum_{i=0}^{n-3} \binom{n-3}{i} x^{n-3-i}y^i\right) \\
& - \cdots \\
& - \left(\sum_{i=0}^{n-3} \binom{n-3}{i} x^{n-3-i}y^i\right)T(x+y)(x^2+2xy+y^2) \\
& + \left(\sum_{i=0}^{2n-2} \binom{2n-2}{i} x^{2n-2-i}y^i\right)T(x+y)(x+y) \\
& - \left(\sum_{i=0}^{2n-1} \binom{2n-1}{i} x^{2n-1-i}y^i\right)T(x+y).
\end{aligned}$$

Using (22) and rearranging the above relation in sense of collecting together terms involving equal number of factors of y, we obtain

$$\sum_{i=1}^{2n-1} f_i(x,y) = 0,$$

where $f_i(x,y)$ stands for the expression of terms involving i factors of y. Replacing x by $x+2y$, $x+3y$, $\ldots$, $x+(2n-1)y$ in turn in the relation (22) and expressing the resulting system of $2n-1$ homogeneous equations of variables $f_i(x,y)$, $i = 1, 2, \ldots, 2n-1$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{2n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 2n-1 & (2n-1)^2 & \cdots & (2n-1)^{2n-1} \end{bmatrix}.$$

Since the determinant of this matrix is different from zero, it follows that the system has only a trivial solution. In particular,

$$\begin{aligned}
f_{2n-1}(x,e) = &-\tbinom{2n}{2n-1}T(x)\\
&+T(x)+\tbinom{2n-1}{2n-2}T(e)x\\
&-T(x)-xT(e)-\tbinom{2n-2}{2n-3}T(e)x\\
&+T(x)+2xT(e)+\tbinom{2n-3}{2n-4}T(e)x\\
&-\cdots\\
&-T(x)-\tbinom{2n-3}{2n-4}xT(e)-2T(e)x\\
&+T(x)+\tbinom{2n-2}{2n-3}xT(e)+T(e)x\\
&-T(x)-\tbinom{2n-1}{2n-2}xT(e).
\end{aligned}$$

The relation (23) reduces the above relation to

$$2nT(x)=0$$

and since R is $2n!$-torsion free, we obtain $T(x)=0$ for all $x\in R$. The proof of the theorem is therefore complete. ∎

Kosi-Ulbl and Vukman ([19]) have proved the following result.

Theorem 7. *Let $n>1$ be some fixed integer and let R be a $n!$-torsion free semiprime ring with the identity element. Suppose there exists an additive mapping $D: R\to R$ satisfying the relation*

$$D(x^n)=\sum_{i=1}^{n} x^{i-1}D(x)x^{n-i}$$

for all $x\in R$. In this case D is a derivation.

Applying Theorem 6 and Theorem 7 we obtain the result below.

Theorem 8. *Let $n\geq 1$ be some fixed integer and let R be a $2n!$-torsion free semiprime ring with the identity element. Suppose there exist additive mappings $S,T: R\to R$ satisfying the relations*

$$S(x^{2n})=S(x)x^{2n-1}+xT(x)x^{2n-2}+x^2S(x)x^{2n-3}+\cdots+x^{2n-1}T(x),$$
$$T(x^{2n})=T(x)x^{2n-1}+xS(x)x^{2n-2}+x^2T(x)x^{2n-3}+\cdots+x^{2n-1}S(x)$$

for all $x\in R$. In this case S and T are derivations and $S=T$.

Stachó and Zalar ([22,23]) investigated bicircular projections on C^*–algebra $\mathcal{L}(H)$, where H is a complex Hilbert space. According to [22, Proposition 3.4] every bicircular projection $P:\mathcal{L}(H)\to\mathcal{L}(H)$ satisfies the functional equation

$$P(ABA)=P(A)BA-AP(B^*)^*A+ABP(A) \tag{24}$$

for all pairs $A, B \in \mathcal{L}(H)$, where B^* stands for the adjoint operator of $B \in \mathcal{L}(H)$. Fošner and Ilišević ([11]) investigated the above functional equation on 2-torsion free semiprime *-ring. They expressed the solution of the equation (24) in terms of derivation and so-called double centralizers. Bicircular projections and related functional equations have been extensively investigated during the last few years (see [5,6,9–14,16–18,20,24]). M. Fošner and Vukman ([13]) investigated the following system of functional equations on prime *-rings with $char(R) \neq 2$.

$$P(x^3) = P(x)x^2 + xQ(x^*)^*x + x^2P(x),$$
$$Q(x^3) = Q(x)x^2 + xP(x^*)^*x + x^2Q(x).$$

Recently, in [14] they considered the following much more general situation

$$P(x^{2n+1}) = P(x)x^{2n} + xQ(x^*)^*x^{2n-1} + x^2P(x)x^{2n} + \cdots + x^{2n}P(x),$$
$$Q(x^{2n+1}) = Q(x)x^{2n} + xP(x^*)^*x^{2n-1} + x^2Q(x)x^{2n} + \cdots + x^{2n}Q(x).$$

In this paper we prove the following theorem.

THEOREM 9. *Let $n \geq 1$ be some fixed integer and let R be a prime *-ring with $2n < char(R) \neq 2$. Suppose there exist additive mappings $P, Q : R \to R$ satisfying the relations*

$$P(x^{2n}) = P(x)x^{2n-1} + xQ(x^*)^*x^{2n-2} + \cdots + x^{2n-1}Q(x^*)^*, \tag{25}$$
$$Q(x^{2n}) = Q(x)x^{2n-1} + xP(x^*)^*x^{2n-2} + \cdots + x^{2n-1}P(x^*)^* \tag{26}$$

for all $x \in R$. In this case P and Q are of the form

$$4P(x) = D(x) + G(x) + \zeta(x) + \theta(x),$$
$$4Q(x) = D(x) - G(x) + \zeta(x) - \theta(x)$$

for all $x \in R$, where $D, G : R \to R$ are derivations with properties $D(x^*)^* = D(x)$, $G(x^*)^* = -G(x)$ and $\zeta, \theta : R \to C$ are additive mappings with properties $\zeta(x^*)^* = -\zeta(x)$, $\theta(x^*)^* = \theta(x)$, $\zeta(x^{2n}) = \theta(x^{2n}) = 0$ for all $x \in R$.

PROOF. The proof goes through in three steps.

FIRST STEP. Let us first assume that $Q = P$. In this case, we have the relation

$$P(x^{2n}) = P(x)x^{2n-1} + xP(x^*)^*x^{2n-2} + \cdots + x^{2n-1}P(x^*)^* \tag{27}$$

for all $x \in R$. It is our aim to prove that P is of the form

$$2P(x) = D(x) + \zeta(x)$$

for all $x \in R$, where $D : R \to R$ is a derivation and $\zeta : R \to C$ is an additive mapping such that $\zeta(x^{2n}) = 0$ for all $x \in R$. Besides, $D(x^*)^* = D(x)$ and $\zeta(x^*)^* = -\zeta(x)$ for all $x \in R$. Let us introduce mappings $D, \zeta : R \to R$ by

$$D(x) = P(x) + P(x^*)^*, \tag{28}$$
$$\zeta(x) = P(x) - P(x^*)^* \tag{29}$$

for all $x \in R$. We therefore have $D(x^*)^* = (P(x^*) + P(x)^*)^* = P(x) + P(x^*)^* = D(x)$. Hence,

$$D(x^*)^* = D(x)$$

for all $x \in R$ and similarly we obtain

$$\zeta(x^*)^* = -\zeta(x)$$

for all $x \in R$. From (27) one can easily obtain that

$$D(x^{2n}) = \sum_{i=1}^{2n} x^{i-1} D(x) x^{2n-i} \tag{30}$$

and

$$\zeta(x^{2n}) = \sum_{i=1}^{2n} (-1)^{i+1} x^{i-1} \zeta(x) x^{2n-i} \tag{31}$$

for all $x \in R$. Now it follows from (30) and Theorem 1 that D is a derivation. On the other hand, one can conclude from (31) and Theorem 3 that ζ maps R into C and

$$\zeta(x^{2n}) = 0$$

for all $x \in R$. Combining (28) and (29) gives

$$2P(x) = D(x) + \zeta(x)$$

for all $x \in R$, which completes the proof of the first step.

Second step. Let us now assume that $Q = -P$, which according to (25) and (26) gives

$$Q(x^{2n}) = Q(x)x^{2n-1} - xQ(x^*)^* x^{2n-2} + \cdots - x^{2n-1} Q(x^*)^* \tag{32}$$

for all $x \in R$. In this case, Q is of the form

$$2Q(x) = G(x) + \theta(x)$$

for all $x \in R$, where $G : R \to R$ is a derivation and $\theta : R \to C$ is an additive mapping such that $\theta(x^{2n}) = 0$ for all $x \in R$. Besides, $G(x^*)^* = -G(x)$ and $\theta(x^*)^* = \theta(x)$ for all $x \in R$. The proof of the second step goes through by using the same arguments as in the first step and will therefore be omitted.

Third step. We are now in the position to prove the theorem in its full generality. We have the relations

$$P(x^{2n}) = P(x)x^{2n-1} + xQ(x^*)^* x^{2n-2} + \cdots + x^{2n-1} Q(x^*)^*,$$
$$Q(x^{2n}) = Q(x)x^{2n-1} + xP(x^*)^* x^{2n-2} + \cdots + x^{2n-1} P(x^*)^*$$

for all $x \in R$. Adding (subtracting) the above relations gives, respectively,

$$F(x^{2n}) = F(x)x^{2n-1} + xF(x^*)^* x^{2n-2} + \cdots + x^{2n-1} F(x^*)^*,$$
$$H(x^{2n}) = H(x)x^{2n-1} - xH(x^*)^* x^{2n-2} + \cdots + -x^{2n-1} H(x^*)^*$$

for all $x \in R$, where F denotes $P+Q$ and H stands for $P-Q$. Now according to the results regarding (27) and (32) in first and second step, we obtain from the above relations that

$$2P(x) + 2Q(x) = D(x) + \zeta(x),$$
$$2P(x) - 2Q(x) = G(x) + \theta(x)$$

for all $x \in R$, where $D, G : R \to R$ are derivations with properties $D(x^*)^* = D(x)$, $G(x^*)^* = -G(x)$ and $\zeta, \theta : R \to C$ are additive mappings with properties $\zeta(x^*)^* = -\zeta(x)$, $\theta(x^*)^* = \theta(x)$, $\zeta(x^{2n}) = \theta(x^{2n}) = 0$ for all $x \in R$. The last two relations imply

$$4P(x) = D(x) + G(x) + \zeta(x) + \theta(x),$$
$$4Q(x) = D(x) - G(x) + \zeta(x) - \theta(x)$$

for all $x \in R$, which completes the proof of the theorem. □

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