GLASNIK MATEMATIČKI
Vol. 49(69)(2014), 333 – 336

FINITE p-GROUPS IN WHICH THE NORMAL CLOSURE OF EACH NON-NORMAL CYCLIC SUBGROUP IS NONABELIAN

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Abstract. We determine up to isomorphism finite non-Dedekindian p-groups G (i.e., p-groups which possess non-normal subgroups) such that the normal closure of each non-normal cyclic subgroup in G is nonabelian. It turns out that we must have $p = 2$ and G has an abelian maximal subgroup A of exponent 2^e, $e \geq 3$, and an element $v \in G - A$ such that for all $h \in A$ we have either $h^v = h^{-1}$ or $h^v = h^{-1+2^{e-1}}$.

Let G be a finite p-group. Then it is well known (see [3, Theorem 224.1]) that the normal closure of each cyclic subgroup in G is abelian if and only if each two-generator subgroup of G is of class ≤ 2. If each two-generator subgroup of a p-group G is of class ≤ 2, then either G is of class ≤ 2 or $p = 3$ and G is of class 3.

It is natural to ask what happens if G is a non-Dedekindian finite p-group in which the normal closure of each non-normal cyclic subgroup is nonabelian. It turns out as a big surprise that in this case we must have $p = 2$ and G can be determined up to isomorphism (Theorem 2).

All groups considered here are finite p-groups and our notation is standard (see [1]).

Definition 1. *Let M be a 2-group possessing an abelian maximal subgroup H of exponent ≥ 4 such that there is an element $v \in M - H$ which inverts each element of H. Then $o(v) \leq 4$ since v inverts $\langle v^2 \rangle \leq H$.*

If $o(v) = 2$, then all elements in $M - H$ are involutions and M is called "quasidihedral" (or generalized dihedral).

2010 *Mathematics Subject Classification.* 20D15.
Key words and phrases. Finite p-groups, normal closure, quasidihedral 2-groups, quasi-generalized quaternion groups, exponent of a p-group.

If $o(v) = 4$, then all elements in $M - H$ are of order 4 with the same square v^2 and then M is called "quasi-generalized quaternion".

Theorem 2. *Let G be a non-Dedekindian p-group in which the normal closure of any non-normal cyclic subgroup is nonabelian. Then $p = 2$, G has an abelian maximal subgroup A of exponent 2^e, $e \geq 3$, and for an element $v \in G - A$ and for all $h \in A$ we have either $h^v = h^{-1}$ or $h^v = h^{-1+2^{e-1}}$.*

Conversely, let G be a 2-group just defined. Then each subgroup of A is G-invariant and for each $v \in G - A$, $o(v) \leq 4$, $\langle v\rangle$ is non-normal in G and $\langle v\rangle^G = [A, \langle v\rangle]\langle v\rangle$, where v inverts each element of $[A, \langle v\rangle] = G'$ (of exponent $2^{e-1} \geq 4$) so that $\langle v\rangle^G$ is either quasidihedral (in case $o(v) = 2$) or quasi-generalized quaternion (in case $o(v) = 4$) and so in any case $\langle v\rangle^G$ is nonabelian.

In the proof of Theorem 2, we shall use [2, Theorem 125.1] and therefore we state here that theorem for convenience:

Lemma 3. *Let G be a nonabelian p-group containing a maximal subgroup H such that all subgroups of H are G-invariant. Then there is an element $g \in G - H$ such that one of the following holds:*

(i) *$p = 2$, H is Hamiltonian, i.e., $H = Q \times V$, where $Q \cong \mathrm{Q}_8$, $\exp(V) \leq 2$, and $g \in \mathrm{Z}(G)$, $\mathrm{o}(g) \leq 4$.*
(ii) *$p = 2$, H is abelian of exponent 2^e, $e \geq 2$, and g either inverts each element in H, or $e \geq 3$ and $h^g = h^{-1+2^{e-1}}$ for all $h \in H$. In both cases $\mathrm{Z}(G) = \mathrm{C}_H(g) = \Omega_1(H)$ is elementary abelian and $\mathrm{o}(g) \leq 4$.*
(iii) *$p = 2$, H is abelian of exponent 2^e, $e \geq 3$, and $h^g = h^{1+2^{e-1}}$ for all $h \in H$, where $\mathrm{Z}(G) = \mathrm{C}_H(g) = \Omega_{e-1}(H)$.*
(iv) *$p > 2$, H is abelian of exponent p^e, $e \geq 2$, and $h^g = h^{1+p^{e-1}}$ for all $h \in H$, where $\mathrm{Z}(G) = \mathrm{C}_H(g) = \Omega_{e-1}(H)$.*

Proof of Theorem 2. Suppose that G is a title p-group. Let $A < G$ be a maximal normal abelian subgroup of G. Since each cyclic subgroup in A is normal in G, it follows that each subgroup in A is G-invariant.

Suppose $p > 2$. Let B/A be a normal subgroup of order p in G/A. Applying Lemma 3 on the subgroup B, we get $\exp(A) = p^e$, $e \geq 2$, and there is $g \in B - A$ such that for all $h \in A$, $h^g = h^{1+p^{e-1}}$. Since B is nonabelian, there is $b \in B - A$ such that $\langle b\rangle$ is not normal in B (and so $\langle b\rangle$ is not normal in G) so that (replacing b with a suitable power b^i, $i \not\equiv 0 \pmod p$, if necessary) we have for all $h \in H$, $h^b = h^{1+p^{e-1}}$ and $M = \langle b\rangle^G$ is nonabelian with $M \leq B$. Also, $B' = [A, \langle b\rangle]$ is elementary abelian and $B' \leq \mathrm{Z}(G)$ because each subgroup of A is G-invariant. There is $k \in G$ such that $[b, b^k] \neq 1$, where $b^k \in M - A$. Note that $\langle b^p\rangle = \langle b\rangle \cap A$ and so $\langle b^p\rangle \trianglelefteq G$ and therefore $\langle b^k\rangle \cap \langle b\rangle = \langle b^p\rangle$. There is $b' \in \langle b^k\rangle - A$ such that $(b')^p = b^{-p}$ and $[b, b'] \neq 1$.

We compute

$$(bb')^p = b^p(b')^p[b',b]^{\binom{p}{2}} = 1,$$

and so $o(bb') = p$. Set $bb' = s$ and assume $s \in \mathrm{Z}(G)$. But then

$$[b,b'] = [b, b^{-1}s] = 1,$$

a contradiction. Thus, s is an element of order p in $B - A$ and $\langle s\rangle$ is not normal in G. By our basic assumption, $\langle s\rangle^G \le B$ is nonabelian and so there is $l \in G$ such that setting $s' = s^l \in B - A$, we have $[s,s'] \ne 1$. But $[s,s'] = z$ is an element of order p in $\mathrm{Z}(G)$ (noting that $B' \le \mathrm{Z}(G)$ and B' is elementary abelian). It follows that $K = \langle s, s'\rangle \cong \mathrm{S}(p^3)$ (the nonabelian group of order p^3 and exponent p). Since $K \le B$, we have $K \cap A \cong \mathrm{E}_{p^2}$. On the other hand, $K \cap A \le \mathrm{Z}(G)$ and $|K : (K \cap A)| = p$ so that K is abelian, a contradiction. We have proved that we must have $p = 2$.

Since each subgroup of A is G-invariant and $\mathrm{C}_G(A) = A$, we have $\exp(A) \ge 4$. Now assume, by way of contradiction, that $\exp(A) = 4$. Let B/A be any subgroup of order 2 in G/A. By Lemma 3, each element $b \in B - A$ inverts each element in A. This implies that G/A has only one subgroup of order 2 and so G/A is either cyclic or generalized quaternion. Suppose $|G/A| > 2$. Then there is $g \in G - A$ such that $(A\langle g\rangle)/A \cong \mathrm{C}_4$, where g^2 inverts each element in A. Let $y \in A$ be an element of order 4. Since $\langle y\rangle \trianglelefteq G$, g normalizes $\langle y\rangle \cong \mathrm{C}_4$ and g^2 inverts $\langle y\rangle$, a contradiction. Hence $|G/A| = 2$ and so $\exp(G) = 4$. Indeed, since each element $v \in G-A$ inverts each element in A, we get $o(v) \le 4$. Because (by our assumption) G is not Dedekindian, there is $v \in G - A$ such that $\langle v\rangle$ is not normal in G, where $o(v) \le 4$. We have $[A, \langle v\rangle] = G'$ is elementary abelian and $G' \le \mathrm{Z}(G)$. Set $R = \langle v\rangle^G$ so that our basic assumption implies that R is nonabelian. On the other hand,

$$[A,\langle v\rangle]\langle v\rangle \le R \text{ and } [A,\langle v\rangle]\langle v\rangle = G'\langle v\rangle \trianglelefteq G$$

$$\text{so that } \; R \le [A,\langle v\rangle]\langle v\rangle \text{ and } R = \langle v\rangle^G = [A,\langle v\rangle]\langle v\rangle = G'\langle v\rangle.$$

But $G' \le \mathrm{Z}(G)$ implies that v centralizes G' and so R is abelian, a contradiction. We have proved that $\exp(A) = 2^e$ with $e \ge 3$.

Suppose, by way of contradiction, that G/A possesses a subgroup B/A of order 2 such that for all $b \in B - A$ and $h \in A$, we have $h^b = h^{1+2^{e-1}}$. In this case $B' = [A, \langle b\rangle]$ is elementary abelian and $B' \le \mathrm{Z}(G)$. Also, we have $\mathrm{Z}(B) = \mathrm{C}_A(b) = \Omega_{e-1}(A)$. If B is not normal in G, then there is $x \in G$ such $b^x \in G - B$ for some $b \in G - A$. But then $bb^x \in G - A$ and bb^x centralizes A (since b^x acts on A the same way as b does), a contradiction. Hence $B \trianglelefteq G$. Because B is nonabelian and $\exp(B) \ge 8$, it follows that B is not Dedekindian. Hence there is $b \in B - A$ such that $\langle b\rangle$ is not normal in B. By our basic assumption, $\langle b\rangle^G \le B$ is nonabelian. There is $g \in G$ such that $b^g \in B - A$ and $[b, b^g] \ne 1$. On the other hand, $\langle b^2\rangle \trianglelefteq G$ and so $\langle b\rangle \cap \langle b^g\rangle = \langle b^2\rangle$. There is $c \in \langle b^g\rangle - A$ such that $c^2 = b^{-2}$ and $[b,c] \ne 1$. We compute $(bc)^2 = b^2c^2[c,b] = [c,b] \neq 1$. Since $[c,b]$ is of order 2, it follows that $d = bc$ is an element of order 4 and $d \in A$. Hence $d \in \mathrm{Z}(B)$ and so we get

$$[b,c] = [b, b^{-1}d] = [b, b^{-1}][b,d] = 1,$$

a contradiction. We have proved that such a group B/A of order 2 in G/A does not exist. Then using again Lemma 3, we see that G/A has exactly one subgroup C/A of order 2 such that for all $c \in C - A$ and $h \in A$, we have either $h^c = h^{-1}$ or $h^c = h^{-1+2^{e-1}}$. In any case $o(c) \leq 4$.

Assume $|G/A| > 2$. Then there is $g \in G - A$ such that $g^2 \in C - A$ so that if h is an element of order 4 in A, then g normalizes $\langle h \rangle \trianglelefteq G$ and g^2 inverts $\langle h \rangle$, a contradiction. We have proved that $|G/A| = 2$, all elements in $G - A$ are of order ≤ 4, $\exp(A) = 2^e$, $e \geq 3$, and for each $v \in G - A$ and $h \in A$, either $h^v = h^{-1}$ or $h^v = h^{-1+2^{e-1}}$. The structure of our group G is determined.

Conversely, let G be a 2-group defined above and let v be any element in $G - A$. Then $o(v) \leq 4$ and the way in which v acts on A insures that each subgroup of A is G-invariant. Set $R = \langle v \rangle^G \trianglelefteq G$ and then $[A, \langle v \rangle]\langle v \rangle \leq R$. On the other hand, $[A, \langle v \rangle] = G'$ and therefore $[A, \langle v \rangle]\langle v \rangle \trianglelefteq G$ so that $R \leq [A, \langle v \rangle]\langle v \rangle$. We have proved that $[A, \langle v \rangle]\langle v \rangle = R$. But $\exp([A, \langle v \rangle]) = 2^{e-1} \geq 4$ and so v inverts each element of $[A, \langle v \rangle] = G'$ so that $\langle v \rangle^G$ is either quasidihedral (if $o(v) = 2$) or quasi-generalized quaternion (if $o(v) = 4$). Hence in any case $\langle v \rangle^G$ is nonabelian. Our theorem is proved. □

References

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Received: 20.1.2014.