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INTEGRABLE SOLUTIONS OF A NONLINEAR INTEGRAL EQUATION RELATED TO SOME EPIDEMIC MODELS

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Abstract. In this paper, we discuss the existence of integrable solutions for a nonlinear integral equation related to some epidemic models. The analysis uses the techniques of measures of noncompactness and relies on an improved version of the Krasnosel'skii fixed point theorem.

1. Introduction

In 1981, Gripenberg ([15]) studied the qualitative behavior of solutions of the equation

$$x(t) = k\left[p(t) + \int_0^t A(t-s)x(s)ds\right] \times \left[q(t) + \int_0^t B(t-s)x(s)ds\right]. \tag{1.1}$$

This equation arises in the study of the spread of an infectious disease that does not induce permanent immunity (see, for example [5, 9, 14, 25]). In [15], the author studied the existence of a unique bounded continuous and nonnegative solution of (1.1) under appropriate assumptions on A and B. He also obtained sufficient conditions for the convergence of the solution as $t \to \infty$. Pachpatte ([23]) provided a new integral inequality and studied the boundedness, the asymptotic behavior and the growth of the solutions of (1.1). Abdeldaim ([1]) and Li et al. ([19]) generalized Pachpatte's inequality and some integral inequalities to study the boundedness and the asymptotic behavior of the continuous solutions of (1.1). Olaru ([22]) generalized (1.1)

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and studied the existence and the uniqueness of the continuous solution of the following integral equation

$$x(t) = \prod_{i=1}^{m} A_i x(t),\ t \in [a,b].$$

Here $A_i(x)(t) = g_i(t) + \int_a^t K_i(t,s,x(s))ds, t \in [a,b]$; where K_i is Lipschitz for $i = \overline{1,m}$.

In this paper, we consider the following nonlinear integral equation

$$\begin{aligned} x(t) = u(t,x(t)) + &\left[p(t) + \int_0^t k_1(t,s) f_1(s,x(s))ds\right] \\ &\times \left[q(t) + \int_0^t k_2(t,s) f_2(s,x(s))ds\right], \end{aligned} \tag{1.2}$$

for $t \in I = [0,1]$.

This equation includes many important integral and functional equations that arise in nonlinear analysis and its applications, in particular the integral equation (1.1).

In our considerations, we look for solutions to (1.2) in the Banach space of real functions being integrable on I. The main tool used in our considerations is the conjunction of the techniques of measures of noncompactness with an improved version of the Krasnosel'skii fixed point theorem. An example is presented to show the importance and the applicability of our results.

2. Auxiliary facts and results

In this section, we provide some notations, definitions and auxiliary facts which will be needed for stating our results. Denote by $L^1(I)$ the set of all Lebesgue integrable functions on $I = [0,1]$, endowed with the standard norm

$$\|x\| = \int_I |x(t)|dt.$$

For later use, we recall the following definitions.

Definition 2.1 ([16]). *Let M be a subset of a Banach space X. A continuous map $A : M \longrightarrow X$ is said to be (ws)–compact if for any weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ in M the sequence $(Ax_n)_{n\in\mathbb{N}}$ has a strongly convergent subsequence in X.*

Notice that the concept of ws-compact mappings arises naturally in the study of integral and partial differential equations (see [2, 10, 12, 13, 17, 18]).

Definition 2.2 ([20]). *Let (X,d) be a metric space. We say that $B : X \longrightarrow X$ is a separate contraction if there exist two functions $\varphi, \psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ satisfying*

(1) $\psi(0) = 0$, *ψ is strictly increasing,*

(2) $d(Bx, By) \leq \varphi(d(x, y))$,
(3) $\psi(r) + \varphi(r) \leq r$ *for* $r > 0$.

Consider a function $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ (I is a bounded or unbounded interval). We say that f satisfies the Carathéodory conditions if it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then to every measurable function $x(t)$ on I we may assign the function $(Fx)(t) = f(t, x(t)), t \in I$. The operator F defined in such a way is called the superposition operator generated by the function f.

We recall the following result due to Appell and Zabrejko ([4]).

Theorem 2.3. *The superposition operator F generated by the function $f(t, x)$ maps continuously the space $L^1(I)$ into itself (I is an interval) if and only if $|f(t, x)| \leq a(t) + b|x|$ for all $t \in I$ and $x \in \mathbb{R}$, where $a(t)$ is a function from $L^1(I)$ and b is a nonnegative constant.*

In the sequel we will utilize the following theorem of Scorza Dragoni ([24]).

Theorem 2.4. *Let I be a bounded interval and let $f : I \times \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying Carathéodory conditions. Then, for each $\varepsilon > 0$ there exists a closed subset D_ϵ of the interval I such that $meas(I \setminus D_\epsilon) < \epsilon$ and $f|_{D_\epsilon \times \mathbb{R}}$ is continuous.*

Recall also the following well known result in L^1 spaces.

Theorem 2.5 ([7, Theorem IV.9, page 58]). *Let Ω be a measurable set of $\mathbb{R}^n$ and (f_n) a sequence in $L^1(\Omega)$. Suppose that $f_n \longrightarrow f$ in $L^1(\Omega)$. Then, there exist a subsequence (f_{n_k}) of (f_n) and $h \in L^1(\Omega)$ such that*
(i) *$f_{n_k} \longrightarrow f$ a.e. in Ω,*
(ii) *$|f_{n_k}(x)| \leq |h(x)|$ for all $k \geq 1$ and a.e in Ω.*

3. Measure of weak noncompactness

In this section, we assume that X is a Banach space. Let $\mathcal{B}(X)$ denote the family of all nonempty bounded subsets of X and $\mathcal{W}(X)$ the subset of $\mathcal{B}(X)$ consisting of all relatively weakly compact subsets of X. Finally, let B_r denote the closed ball centered at 0 with radius r.
Recall the following definition of the concept of the axiomatic measure of weak noncompactness.

Definition 3.1 ([6]). *A function $\mu : \mathcal{B}(X) \longrightarrow \mathbb{R}_+$ is said to be a measure of weak noncompactness if it satisfies the following conditions:*

1. *the family $ker(\mu) = \{M \in \mathcal{B}(X) : \mu(M) = 0\}$ is nonempty and $ker(\mu) \subset \mathcal{W}(X)$,*
2. *$M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$,*
3. *$\mu(co(M)) = \mu(M)$, where $co(M)$ is the convex hull of M,*
4. *$\mu(\lambda M_1 + (1 - \lambda)M_2) \leq \lambda\mu(M_1) + (1 - \lambda)\mu(M_2)$ for $\lambda \in [0, 1]$,*

5. *if* $(M_n)_{n\geq 1}$ *is a sequence of nonempty, weakly closed subsets of* X *with* M_1 *bounded and* $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_n \supseteq \cdots$ *such that* $\lim_{n\to\infty} \mu(M_n) = 0$*, then* $M_\infty := \bigcap_{n=1}^{\infty} M_n$ *is nonempty.*

The first important example of measure of weak noncompactness has been defined by De Blasi (see [8]). In the space $L^1(I)$, there is a convenient and workable formula for the function μ which was given by Appel and De Pascale [3] as follows: For a nonempty and bounded subset M of the space $L^1(I)$

$$\mu(M) = \lim_{\epsilon\to 0}\{\sup_{x\in M}\{\sup[\int_D |x(t)|dt : D \subset I, meas(D) \leq \epsilon]\}\}. \tag{3.1}$$

We will use the following criterion for relatively weakly compact sets in $L^1(I)$.

Theorem 3.2 ([11]). *A bounded set* S *is relatively weakly compact in* $L^1(I)$ *if and only if for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $meas(D) \leq \delta$ *then* $\int_D |x(t)| \leq \epsilon$ *for all* $x \in S$.

4. Main result

Equation (1.2) will be studied under the following assumptions.

(i) The functions $p, q : I \longrightarrow \mathbb{R}$ are such that $p \in L^\infty(I)$ and $q \in L^1(I)$. Let $\|p\|$ be the supremum norm of p on I and $\|q\|$ be the norm of q in $L^1(I)$.

(ii) The function $u : I \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the Carathéodory conditions and is Lipschitzian with respect to the second variable with a Lipschitz constant α, that is, $|u(t,x) - u(t,y)| \leq \alpha|x-y|$ for all $t \in I$ and all $x, y \in \mathbb{R}$. Let $\beta(t) = |u(t,0)| \in L^1(I)$.

(iii) The functions $f_i : I \times \mathbb{R} \longrightarrow \mathbb{R}$ $(i = 1,2)$ satisfy Carathéodory conditions and there exist constants b_i and functions $a_i \in L^1(I)$ such that $|f_i(t,x)| \leq a_i(t) + b_i|x|$ for all $t \in I$ and all $x \in \mathbb{R}$.

(iv) The function $k_1 : I \times I \longrightarrow \mathbb{R}$ is measurable and the linear Volterra operator

$$K_1x(t) = \int_0^t k_1(t,s)x(s)ds, t \in I$$

maps continuously $L^1(I)$ into $L^\infty(I)$. Let $\|K_1\|$ be the norm of this bounded linear operator.

(v) The function $k_2 : I \times I \longrightarrow \mathbb{R}$ is measurable and the linear Volterra operator

$$K_2x(t) = \int_0^t k_2(t,s)x(s)ds, t \in I$$

maps continuously $L^1(I)$ into itself. Let $\|K_2\|$ be the norm of this bounded linear operator.

(vi)

$$\alpha+(\|p\|+\|K_1\|\|a_1\|)\|K_2\|b_2+(\|q\|+\|K_2\|\|a_2\|)\|K_1\|b_1$$
$$+2\sqrt{\|K_1\|\|K_2\|b_1b_2[\|\beta\|+(\|p\|+\|K_1\|\|a_1\|)(\|q\|+\|K_2\|\|a_2\|)]}<1.$$

REMARK 4.1. It is easy to check that, if the function k_1 is bounded on the set $\Delta=\{(t,s)\in I\times I\,;0\le s\le t\le 1\}$, then the linear operator K_1 transforms the space $L^1(I)$ into $L^\infty(I)$ and the norm $\|K_1\|$ of this operator is majorized by $\|k_1\|_{L^\infty(\Delta)}$.

The following result gives a sufficient condition that the operator K_2 transforms the space $L^1(I)$ into itself.

PROPOSITION 4.2. *Assume that*

$$ess\sup_{0\le s\le 1}\int_s^1|k_2(t,s)|dt<\infty.$$

Then the operator K_2 maps the space $L^1(I)$ into itself and the norm $\|K_2\|$ of this operator is majorized by the number ess $\sup\limits_{0\le s\le 1}\int_s^1|k_2(t,s)|dt$.

PROOF. Let k_2 be a measurable function on $I\times I$ such that

$$\operatorname{ess\,sup}_{0\le s\le 1}\int_s^1|k_2(t,s)|dt<\infty.$$

Then for all $x\in L^1(I)$ we have

$$\begin{aligned}\|K_2x\|&=\int_0^1\left|\int_0^t k_2(t,s)x(s)ds\right|dt\\&\le\int_0^1\int_s^1|k_2(t,s)||x(s)|dtds\\&\le\operatorname{ess\,sup}_{0\le s\le 1}\int_s^1|k_2(t,s)|dt\,\|x\|.\end{aligned}$$

This implies that K_2 transforms $L^1(I)$ into itself and $\|K_2\|$ is majorized by

$$\operatorname{ess\,sup}_{0\le s\le 1}\int_s^1|k_2(t,s)|dt.$$

□

The following fixed point theorem is crucial for our purposes. For a proof we refer the reader to [21].

THEOREM 4.3. *Let $\mathcal{M}$ be a nonempty bounded closed convex subset of a Banach space X. Suppose that $A:\mathcal{M}\longrightarrow X$ and $B:\mathcal{M}\longrightarrow X$ verify:*

(i) *A is (ws)–compact,*

(ii) *there exists* $\gamma \in [0,1)$ *such that* $\mu(AS+BS) \leq \gamma\mu(S)$ *for all* $S \subset \mathcal{M}$*; here* μ *is an arbitrary measure of weak noncompactness on* X*,*
(iii) B *is a separation contraction,*
(iv) $A\mathcal{M} + B\mathcal{M} \subseteq \mathcal{M}$*.*

Then there exists $x \in \mathcal{M}$ *such that* $Ax + Bx = x$*.*

Now we are in a position to state our main result.

THEOREM 4.4. *Under the assumptions above the nonlinear integral equation (1.2) has at least one solution* $x \in L^1(I)$*.*

PROOF. Solving (1.2) is equivalent to finding a fixed point of the operator $A+B$, where $Ax(t) = [p(t) + \int_0^t k_1(t,s)f_1(s,x(s))ds] \times [q(t) + \int_0^t k_2(t,s)f_2(s,x(s))ds]$ and $Bx(t) = u(t,x(t))$. We will show that A and B satisfy the conditions of Theorem 4.3. The proof is split into four steps.

STEP 1. We first show that there exists B_{r_0} from $L^1(I)$ such that $A(B_{r_0})+B(B_{r_0}) \subset B_{r_0}$. To see this, let $x, y \in B_r$. Then

$$\begin{aligned}
\|Ax+By\| \leq\ & \int_0^1 |u(t,y(t))|dt + \int_0^1 \left|p(t) + \int_0^t k_1(t,s)f_1(s,x(s))ds\right| \\
& \times \left|q(t) + \int_0^t k_2(t,s)f_2(s,x(s))ds\right| dt \\
\leq\ & \|\beta\| + \alpha\|y\| + (\|p\| + \|K_1\|(\|a_1\| + b_1\|x\|)) \\
& \times (\|q\| + \|K_2\|(\|a_2\| + b_2\|x\|)) \\
\leq\ & \|\beta\| + (\|p\| + \|K_1\|\|a_1\|)(\|q\| + \|K_2\|\|a_2\|) \\
& +[\alpha + (\|p\| + \|K_1\|\|a_1\|)\|K_2\|b_2 + (\|q\| + \|K_2\|\|a_2\|)\|K_1\|b_1]r \\
& +\|K_1\|\|K_2\|b_1b_2r^2.
\end{aligned}$$

We define the function

$$f(r) = \|\beta\| + (\|p\| + \|K_1\|\|a_1\|)(\|q\| + \|K_2\|\|a_2\|) - \xi r + \|K_1\|\|K_2\|b_1b_2r^2, r > 0$$

where $\xi = 1 - \alpha - (\|p\| + \|K_1\|\|a_1\|)\|K_2\|b_2 - (\|q\| + \|K_2\|\|a_2\|)\|K_1\|b_1$. Note that

$$\Delta = \xi^2 - 4\|K_1\|\|K_2\|b_1b_2(\|\beta\| + (\|p\| + \|K_1\|\|a_1\|)(\|q\| + \|K_2\|\|a_2\|))$$

is nonnegative from assumption (vi) and for $r_0 = \frac{\xi - \sqrt{\Delta}}{2\|K_1\|\|K_2\|b_1b_2}$ we have the desired result.

STEP 2. We illustrate that there exists $\gamma \in [0,1)$ such that $\mu(AS+BS) \leq \gamma\mu(S)$ for all subset S of B_{r_0}. To see this, take an arbitrary number $\epsilon > 0$ and a nonempty subset D of I such that D is measurable and $meas(D) \leq \epsilon$.

Then for any $x, y \in S$ we have

$$\begin{aligned}\int_D |(Ax+By)(t)|dt \;&\leq\; \|\beta\|_{L^1(D)} + \alpha\|y\|_{L^1(D)} \\ &\quad + (\|p\| + \|K_1\|(\|a_1\| + b_1\|x\|_{L^1(I)})) \\ &\quad \times (\|q\|_{L^1(D)} + \|K_2\|_{L^1(D)}(\|a_2\|_{L^1(D)} + b_2\|x\|_{L^1(D)})) \\ &\leq\; \|\beta\|_{L^1(D)} + \alpha\|y\|_{L^1(D)} + (\|p\| + \|K_1\|(\|a_1\| + b_1 r_0)) \\ &\quad \times (\|q\|_{L^1(D)} + \|K_2\|(\|a_2\|_{L^1(D)} + b_2\|x\|_{L^1(D)})).\end{aligned}$$

Now using (3.1) we get

$$\mu(AS + BS) \leq \gamma\mu(S),$$

where $\gamma = \alpha + [\|p\| + \|K_1\|(\|a_1\| + b_1 r_0)]\|K_2\|b_2$. Notice that by assumption (vi) we have $\gamma \in [0, 1)$.

STEP 3. We prove that the operator $H : B_{r_0} \longrightarrow L^1(I)$ defined by

$$Hx(t) = q(t) + \int_0^t k_2(t,s) f_2(s, x(s))ds$$

is (ws)–compact. The continuity of H follows from assumptions (iii) and (v) on the basis of Theorem 2.3. Now, let (y_n) be a weakly convergent sequence in B_{r_0}, then the set $S = \{y_n, n \in \mathbb{N}\}$ is relatively weakly compact. Take an arbitrary number $\epsilon > 0$. In view of Theorem 3.2 there exists $\delta(\epsilon) > 0$ such that whenever $J \subset I$ and $meas(J) < \delta(\epsilon)$, we have

$$\int_J a_2(s)ds < \frac{\epsilon}{4\|K_2\|(1+b_2)} \;\text{ and }\; \int_J |y_n(s)|ds < \frac{\epsilon}{4\|K_2\|(1+b_2)} \;\text{for all}\, n \in \mathbb{N}.$$

Now, in view of Theorem 2.4, we can find a closed subset $D_\epsilon \subset I$ such that $meas(I \setminus D_\epsilon) \leq \delta(\epsilon)$ and the function $q|_{D_\epsilon}$ is continuous and $k_2|_{D_\epsilon \times I}$ is uniformly continuous. Accordingly, for all $n \in \mathbb{N}$

$$\text{(4.1)} \quad \int_{I\setminus D_\epsilon} a_2(s)ds < \frac{\epsilon}{4\|K_2\|(1+b_2)} \;\text{ and }\; \int_{I\setminus D_\epsilon} |y_n(s)|ds < \frac{\epsilon}{4\|K_2\|(1+b_2)}.$$

Now, take $t_1, t_2 \in D_\epsilon$ such that $t_1 \le t_2$. Then, for an arbitrary $n \in \mathbb{N}$, we have

$$\begin{aligned}
|Hy_n(t_2) - Hy_n(t_1)| &= \Big| \int_0^{t_2} k_2(t_2,s) f_2(s, y_n(s)) ds \\
&\quad - \int_0^{t_1} k_2(t_1,s) f_2(s, y_n(s)) ds \Big| \\
&\le \Big| \int_{t_1}^{t_2} k_2(t_2,s) f_2(s, y_n(s)) ds \Big| \\
&\quad + \Big| \int_0^{t_1} k_2(t_2,s) f_2(s, y_n(s)) ds \\
&\quad - \int_0^{t_1} k_2(t_1,s) f_2(s, y_n(s)) ds \Big| \\
&\le \int_{t_1}^{t_2} |k_2(t_2,s)| [a_2(s) + b_2 |y_n(s)|] ds \\
&\quad + \int_0^{t_1} |k_2(t_1,s) - k_2(t_2,s)| [a_2(s) + b_2 |y_n(s)|] ds \\
&\le \|k_2\|_{L^\infty(D_\epsilon \times I)} \int_{t_1}^{t_2} [a_2(s) + b_2 |y_n(s)|] ds \\
&\quad + \omega(k_2, |t_1 - t_2|) \int_0^{t_1} [a_2(s) + b_2 |y_n(s)|] ds \\
&\le \|k_2\|_{L^\infty(D_\epsilon \times I)} \int_{t_1}^{t_2} a_2(s) ds \\
&\quad + \|k_2\|_{L^\infty(D_\epsilon \times I)} b_2 \int_{t_1}^{t_2} |y_n(s)| ds \\
&\quad + \omega(k_2, |t_1 - t_2|)(\|a_2\| + b_2 r_0),
\end{aligned}$$

where $\omega(k_2, .)$ denotes the modulus of continuity of the function k_2 on the set $D_\epsilon \times I$. Now, in virtue of Theorem 3.2 we have the terms $\int_{t_1}^{t_2} |y_n(s)| ds$ and $\int_{t_1}^{t_2} |a_1(s)| ds$ are arbitrarily small provided that the number $t_2 - t_1$ is small enough. This means that (Hy_n) is a sequence of equicontinuous functions on D_ϵ. Moreover, for an arbitrary $t \in D_\epsilon$ and for $n \in \mathbb{N}$, we have

$$\begin{aligned}
|Hy_n(t)| &= \left| q(t) + \int_0^t k_2(t,s) f_2(s, y_n(s)) ds \right| \\
&\le |q(t)| + \int_0^t |k_2(t,s)| [a_2(s) + b_2 |y_n(s)|)] ds \\
&\le \|q\|_{L^\infty(D_\epsilon)} + \|k_2\|_{L^\infty(D_\epsilon \times I)} (\|a_2\| + b_2 r_0).
\end{aligned}$$

This means that the sequence (Hy_n) is uniformly bounded in $C(D_\epsilon)$. Hence the Arzela-Ascoli theorem guarantees that $\{Hy_n, n \in \mathbb{N}\}$ is a relatively compact subset of $C(D_\epsilon)$. This implies the existence of a convergent subsequence (Hy_{n_k}) of (Hy_n) in $C(D_\epsilon)$. This subsequence is a Cauchy sequence in $C(D_\epsilon)$. Thus, for a given $\epsilon > 0$, there exists k_0 such that for all $m, k \geq k_0$ we have

$$|Hy_{n_m}(t) - Hy_{n_k}(t)| \leq \frac{\epsilon}{2meas(D_\epsilon)} \tag{4.2}$$

for any $t \in D_\epsilon$.

Now, we prove that the subsequence (Hy_{n_k}) is convergent in $L^1(I)$. Since $L^1(I)$ is a complete metric space, it suffices to prove that the subsequence (Hy_{n_k}) is a Cauchy sequence. From (4.1) and (4.2), it follows that for all $m, k \geq k_0$ we have

$$\begin{aligned}\int_0^1 |Hy_{n_m}(t) - Hy_{n_k}(t)|dt &= \int_{D_\epsilon} |Hy_{n_m}(t) - Hy_{n_k}(t)|dt \\ &\quad + \int_{I\setminus D_\epsilon} |Hy_{n_m}(t) - Hy_{n_k}(t)|dt \\ &\leq \frac{\epsilon}{2} + \int_{I\setminus D_\epsilon} |(K_2(f_2(.,y_{n_m}(.)) - f_2(.,y_{n_k}(.)))(t)| \\ &\leq \frac{\epsilon}{2} + \|K_2\|_{L^1(I\setminus D_\epsilon)} \|f_2(t,y_{n_m}(t)) - f_2(t,y_{n_k}(t))\|_{L^1(I\setminus D_\epsilon)} \\ &\leq \frac{\epsilon}{2} + \|K_2\| \|2a_2(s) + b_2|y_{n_m}(s)| + b_2|y_{n_k}(s)|\|_{L^1(I\setminus D_\epsilon)} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\end{aligned}$$

which implies that (Hy_{n_k}) is a Cauchy sequence in $L^1(I)$. Finally, the operator H is (ws)–compact.

Step 4. We show that $A : B_{r_0} \longrightarrow L^1(I)$ is (ws)–compact. To see this, notice that $Ax = (Gx)(Hx)$, where G is defined by

$$Gx(t) = p(t) + \int_0^t k_1(t,s) f_1(s, x(s))ds.$$

The reasoning in *Step 3* shows that H and (similarly) G are (ws)–compact. Now, let (y_n) be a weakly convergent sequence in B_{r_0}. Then, up to a subsequence, we may assume that there exists a subsequence (y_{n_k}) such that (Hy_{n_k}) and (Gy_{n_k}) converge strongly to $h \in L^1(I)$ and $g \in L^1(I)$ respectively. Thanks to Theorem 2.5, we deduce that there exists a subsequence (w_k) of (y_{n_k}) such that (Gw_k) converges to g a.e in I. Keeping in mind that (Gw_k) is bounded by $\|p\| + \|K_1\|(\|a_1\| + b_1 r_0)$ we infer that

$$g \in L^\infty(I) \quad \text{and} \quad \|g\|_{L^\infty(I)} \leq \|p\| + \|K_1\|(\|a_1\| + b_1 r_0).$$

Now, we prove that $A(w_k)$ converges in $L^1(I)$ to gh. Notice for all $k \in \mathbb{N}$ we have

(4.3)
$$\begin{aligned}\int_0^1 |Aw_k(t) - gh(t)|dt \leq & \int_0^1 |Gw_k(t)||Hw_k(t) - h(t)|dt \\ & + \int_0^1 |h(t)||Gw_k(t) - g(t)|dt \\ \leq & (\|p\| + \|K_1\|(\|a_1\| + b_1 r_0)) \int_0^1 |Hw_k(t) - h(t)|dt \\ & + \int_0^1 |h(t)||Gw_k(t) - g(t)|dt.\end{aligned}$$

By Applying the Lebesgue dominated convergence theorem, we get

$$\lim_{k\to\infty} \int_0^1 |h(t)||Gw_k(t) - g(t)|dt = 0.$$

Hence by (4.3), we deduce that

$$\lim_{k\to\infty} \int_0^1 |Aw_k(t) - gh(t)|dt = 0.$$

Consequently, A is (ws)–compact.

Now, Theorem 4.3 guarantees the existence of a fixed point in B_{r_0} to $A + B$, where $Bx = u(., x)$ and hence an integrable solution to (1.2). □

5. Example

Consider the following integral equation

(5.1)
$$x(t) = t^2 + \frac{1}{5}x(t) + \left(\frac{1}{1+t} + \int_0^t \sin(ts)(s^2 + x(s))ds\right) \int_0^t \frac{1}{ts+\lambda} \ln(1+x^2(s))ds,$$

where $t \in [0, 1]$ and λ is a positive number. Set

$$u(t, x) = t^2 + \frac{1}{5}x, p(t) = \frac{1}{1+t}, k_1(t, s) = \sin(ts), f_1(t, x) = t^2 + x$$

and

$$q(t) = 0, k_2(t, s) = \frac{1}{ts+\lambda}, f_2(t, x) = \ln(1 + x^2).$$

Using the notations of Theorem 4.4, we can easily show that

$$\beta(t) = t^2, \alpha = \frac{1}{5}, \|K_1\| \leq 1, \|K_2\| \leq \frac{1}{\lambda}, a_1(t) = t^2, b_1 = 1, a_2(t) = 0, b_2 = 1.$$

Therefore, the inequality (vi) takes the form

$$\frac{1}{5}+\frac{4}{3\lambda}+2\sqrt{\frac{1}{3\lambda}}<1 \Longleftrightarrow \frac{4}{3}+\frac{2}{\sqrt{3}}\sqrt{\lambda}<\frac{4}{5}\lambda \Longleftrightarrow \lambda>\frac{65+5\sqrt{105}}{24}.$$

Then, by Theorem 4.4 we conclude that the integral equation (5.1) has a solution $x \in L^1(I)$ whenever $\lambda > \frac{65+5\sqrt{105}}{24} \cong 4.81$.

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