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UNIQUENESS OF HYPERSPACES OF INDECOMPOSABLE ARC CONTINUA

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Abstract. Given a metric continuum X, we consider the hyperspace $C_n(X)$ of all nonempty closed subsets of X with at most n components. In this paper we prove that if $n \neq 2$, X is an indecomposable continuum such that all its proper nondegenerate subcontinua are arcs and Y is a continuum such that $C_n(X)$ is homeomorphic to $C_n(Y)$, then X is homeomorphic to Y (that is, X has unique hyperspace $C_n(X)$).

1. Introduction

A *continuum* is a nondegenerate compact connected metric space. Given a continuum X, we consider the following *hyperspaces* of X.

$$2^X = \{A \subset X : A \text{ is nonempty and closed in } X\},$$
$$C_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\},$$
$$F_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ points}\},$$
$$C(X) = C_1(X).$$

All hyperspaces are considered with the *Hausdorff metric* H.

The hyperspace $F_n(X)$ is known as the *n-th symmetric product* of X. The hyperspace $F_1(X)$ is an isometric copy of X embedded in each one of the hyperspaces.

A hyperspace $K(X) \in \{2^X, C_n(X), F_n(X)\}$ is said to be *rigid* provided that for each homeomorphism $h : K(X) \to K(X)$, we have, $h(F_1(X)) = F_1(X)$. The continuum X is said to have *unique hyperspace* $K(X)$ provided

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that the following implication holds: if Y is a continuum such that $K(X)$ is homeomorphic to $K(Y)$, then X is homeomorphic to Y.

Uniqueness of hyperspaces has been widely studied (see, for example, [3, 5,7–9,12] for recent references). A detailed survey of what is known about this subject can be found in [13]. In the study of hyperspaces, a useful technique is to find a topological property that characterizes the elements of $F_1(X)$ in the hyperspace $K(X)$. When it is possible to find such a characterization, the hyperspace $K(X)$ is rigid. This technique has been used in studying uniqueness of hyperspaces, so both topics are closely related.

Rigidity of hyperspaces was introduced in [9]. Rigidity of symmetric products was studied in [8].

A continuum X is *indecomposable* if it cannot be put as the union of two of its proper subcontinua. The continuum X is said to be arc continuum if each one of its nondegenerate proper subcontinuum is an arc. Examples of indecomposable arc continua are the Buckethandle continuum and the solenoids ([18, 2.8 and 2.9]).

As a consequence of [8, Theorem 5] and [3, Theorem 9], it follows that if X is an indecomposable arc continuum and $n \neq 3$, then X has unique hyperspace $F_n(X)$, the case $n = 3$, remains unsolved.

In this paper we prove that if X is an indecomposable arc continuum, then X has unique hyperspaces $C_n(X)$ and $C_n(X)$ is rigid for every $n \neq 2$. The case $n = 2$ remains unsolved.

2. Definitions and conventions

A *map* is a continuous function. Suppose that d is a metric for X. Given $\varepsilon > 0$, $p \in X$ and $A \in 2^X$, let $B(\varepsilon, p)$ be the ε-open ball around p in X, $N(\varepsilon, A) = \{p \in X : \text{there exists } a \in A \text{ such that } d(p,a) < \varepsilon\}$ and $B^H(\varepsilon, A) = \{B \in 2^X : H(A,B) < \varepsilon\}$ (we write $B_X(\varepsilon, p)$ and $N_X(\varepsilon, A)$ when the space X needs to be mentioned). A *simple n-od* is a finite graph G that is the union of n arcs emanating from a single point, v, and otherwise disjoint from one another. The point v is called the *vertex* of G. Simple 3-ods are called *simple triods*. Given subsets $A_1, \ldots, A_m$ of X, let $\langle A_1, \ldots, A_m \rangle = \{B \in 2^X : B \cap A_i \neq \emptyset$ for each $i \in \{1, \ldots, m\}$ and $B \subset A_1 \cup \ldots \cup A_m\}$.

We denote by S^1 the unit circle in the Euclidean plane. A *free arc* in the continuum X is an arc α with end points a and b such that $\alpha - \{a, b\}$ is open in X.

Proceeding as in [6, Lemma 2.1] and using [17, Lemma 1.48], the following lemma can be proved.

Lemma 2.1. *Let X be a continuum, $n \in \mathbb{N}$ and let $\mathcal{A}$ be a connected subset of 2^X such that $\mathcal{A} \cap C_n(X) \neq \emptyset$. Let $A_0 = \cup\{A : A \in \mathcal{A}\}$. Then*

(a) *A_0 has at most n components,*
(b) *if $\mathcal{A}$ is closed in 2^X, then $A_0 \in C_n(X)$,*

(c) *for each $A \in \mathcal{A}$, each component of A_0 intersects A.*

PROOF. (a) follows immediately from [6, Lemma 2.1], (b) In [17, Lemma 1.48], it is proven that the map $\cup : 2^{2^X} \to 2^X$ is onto, then $A_0 \in 2^X$, by (a) it has at most n components, therefore $A_0 \in C_n(X)$.

(c) To prove (c) we proceed as in [6, Lemma 2.1], we write here the proof for better understanding of the reader.

Let $A_1, \ldots, A_m$ with $m \leq n$ be the components of A_0, suppose to the contrary that there exist $B \in \mathcal{A}$ and A_i, component of A_0, such that $B \cap A_i = \emptyset$. Assume that $A_1, \ldots, A_k$ are such that $A_i \cap B \neq \emptyset$ for each $i \in \{1, \ldots, k\}$ and $A_i \cap B = \emptyset$ for each $i \in \{k+1, \ldots, m\}$.

Let $\mathcal{K} = \{C \in \mathcal{A} : C \subset A_1 \cup \ldots \cup A_k\}$ and $\mathcal{L} = \{C \in \mathcal{A} : C \cap (A_{k+1} \cup \ldots \cup A_m) \neq \emptyset\}$. Proceeding exactly as in the rest of the proof of [6, Lemma 2.1], we prove that $\mathcal{K}$ and $\mathcal{L}$ is a separation of $\mathcal{A}$ which contradicts the fact that $\mathcal{A}$ is connected. Therefore (c) follows and the lemma is proved. □

A *wire* in a continuum X is a subset α of X such that α is homeomorphic to one of the spaces $(0,1)$, $[0,1)$, $[0,1]$ or S^1 and α is a component of an open subset of X. By [17, Theorem 20.3], if a wire α in X is compact, then $\alpha = X$. So, if a wire is homeomorphic to $[0,1]$ or S^1, then X is an arc or a simple closed curve. Given a continuum X, let

$$W(X) = \bigcup\{\alpha \subset X : \alpha \text{ is a wire in } X\}.$$

The continuum X is said to be *wired* provided that $W(X)$ is dense in X.

Notice that if α is a free arc of a continuum X and p, q are the end points of α, then $\alpha - \{p, q\}$ is a wire in X. Thus, a continuum for which the union of its free arcs is dense is a wired continuum. Therefore, the class of wired continua includes finite graphs, dendrites with closed set of end points, almost meshed continua ([7]), compactifications of the ray $[0, \infty)$, compactifications of the real line and indecomposable arc continua.

An *m-od* in a continuum X is a subcontinuum B of X for which there exists $A \in C(B)$ such that $B - A$ has at least m components. By [14, Theorem 70.1], a continuum X contains an m-od if and only if $C(X)$ contains an m-cell. Given $A, B \in 2^X$ such that $A \subsetneq B$, an *order arc* from A to B is a continuous function $\alpha : [0,1] \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$ and $\alpha(s) \subsetneq \alpha(t)$ if $0 \leq s < t \leq 1$. It is known ([17, Theorem 1.25]), that there exists an order arc from A to B if and only if $A \subsetneq B$ and each component of B intersects A.

Given a continuum X and $n \in \mathbb{N}$, let

$$\mathcal{W}_n(X) = \{A \in C_n(X) : \text{ each component of } A \text{ is contained in a wire of } X\};$$

and

$$\begin{aligned}\mathcal{Z}_n(X) = &\{A \in \mathcal{W}_n(X) : \text{ there is a neighborhood } \mathcal{M} \text{ of } A \text{ in } C_n(X) \text{ such} \\ &\quad \text{that the component } \mathcal{C} \text{ of } \mathcal{M} \text{ that contains } A \text{ is a } 2n \text{ cell}\}.\end{aligned}$$

We will use the following two results of [9].

Lemma 2.2 ([9, Lemma 2]). *Let X be an indecomposable arc continuum. Then X is a wired continuum.*

Theorem 2.3 ([9, Theorem 8]). *Let X be a continuum and let $n \geq 3$. Then*

$$\begin{aligned}\mathcal{W}_1(X) = \{&A \in \mathcal{W}_n(X) - \mathcal{Z}_n(X) : A \textit{ has a basis } \mathcal{B} \textit{ of neighborhoods}\\ &\textit{in } C_n(X) \textit{ such that for each } \mathcal{U} \in \mathcal{B}, \textit{ if } \mathcal{C} \textit{ is the component of } \mathcal{U}\\ &\textit{that contains } A, \textit{ then } \mathcal{C} \cap \mathcal{Z}_n(X) \textit{ is connected}\}.\end{aligned}$$

3. Indecomposable arc continua

Theorem 3.1. *If X is an indecomposable arc continuum, then X has unique hyperspace $C_n(X)$ and $C_n(X)$ is rigid for every $n \neq 2$.*

Proof. For $n = 1$, the uniqueness of $C(X)$ was shown in [1, Theorem 2.3]. In [15, Theorem 3], it was shown that if $h : C(X) \to C(X)$ is a homeomorphism, then $h(F_1(X)) = F_1(X)$. That is, $C(X)$ is rigid.

Suppose then that $n \geq 3$. Let Y be a continuum such that there exists a homeomorphism $h : C_n(X) \to C_n(Y)$. Let $Y_0 \in C_n(Y)$ be such that $h(X) = Y_0$.

Claim 1. The only element that arcwise disconnects $C_n(X)$ is X and $C_n(X) - \{X\}$ has uncountably many arc components.

We prove Claim 1. By [11, Corollary 2.2], and also by [16, 3.9] $C_n(X) - \{X\}$ has uncountably many arc components. Let $A \in C_n(X) - \{X\}$. Let $\mathcal{C}$ be the arc component of $C_n(X) - \{A\}$ such that $X \in \mathcal{C}$. We claim that $\mathcal{C} = C_n(X) - \{A\}$. Take $D \in C_n(X) - \{A\}$. If D is not contained in A, take an order arc α from D to X. Notice that for each $t \in [0, 1]$, $\alpha(t) \neq A$. Then $\operatorname{Im} \alpha \subset \mathcal{C}$ and $D \in \mathcal{C}$. Now consider the case that $D \subset A$. Then, we have that A is not a one-point set. Reasoning as in [17, Theorem 11.3], it follows that if A is not connected, then there is an arc joining D and X in $C(X) - \{A\}$. Thus, we assume that A is connected. Let $B \in C(X) - \{X\}$ be such that $A \subsetneq B$. Then A and B are arcs. Let F be a finite set containing exactly one point in each one of the components of D. Then $F \in F_n(X) \subset C_n(X)$. Let β be an order arc joining F and D. Notice that $\operatorname{Im} \beta \subset C_n(X) - \{A\}$. By [4, 2(a)] $F_n(B)$ is arcwise connected, then there exists an arc γ in $F_n(B)$ joining F and an element $E \subset B - A$. By the first case, $E \in \mathcal{C}$. Since $\operatorname{Im} \gamma \subset C_n(X) - \{A\}$, we conclude that $D \in \mathcal{C}$. We have shown that $\mathcal{C} = C_n(X) - \{A\}$. Hence, $C_n(X) - \{A\}$ is arcwise connected. This ends the proof of Claim 1.

Claim 2. $Y_0 \in C(Y)$ and Y_0 is indecomposable.

To prove Claim 2 observe that if Y_0 is disconnected, then by [17, Theorem 11.3] it can be proved that $C_n(Y) - \{Y_0\}$ is arcwise connected. Since h is a homeomorphism, this contradicts Claim 1. Hence, Y_0 is connected. Now, suppose that Y_0 is decomposable. By [11, Lemma 2.4], $C_n(Y) - \{Y_0\}$ has at most two arc components. Since h is a homeomorphism, Claim 1 implies that $C_n(Y)-\{Y_0\}$ has uncountably many arc components. This contradiction ends the proof of Claim 2.

CLAIM 3. Let $k=2n+1$. Then $C_n(Y)$ does not contain k-cells.

Suppose, contrary to Claim 3, that $C_n(Y)$ contains a k-cell. Then there exists a k-cell $\mathcal{M}$ in $C_n(X)$. Let $m=\max\{i\in\{1,\ldots,n\}:\mathcal{M}\cap(C_i(X)-C_{i-1}(X))\neq\emptyset\}$. Since $\mathcal{M}\cap(C_m(X)-C_{m-1}(X))$ is a nonempty open subset of $\mathcal{M}$, there exists $A\in\mathcal{M}\cap(C_m(X)-C_{m-1}(X))-\{X\}$. Let $\mathcal{N}$ be a k-cell such that $A\in\mathcal{N}\subset\mathcal{M}\cap(C_m(X)-C_{m-1}(X))-\{X\}$ and let $B=\cup\{C:C\in\mathcal{N}\}$. Let $A_1,\ldots,A_m$ the components of A, taking

$$\varepsilon<\min\left\{\frac{d(A_i,A_j)}{2}:i,j\in\{1,\ldots,m\}\text{ and } i\neq j\right\}$$

and $\mathcal{N}\subset B^H(\varepsilon.A)$, then B has at least m components and $B\neq X$. By Lemma 2.1, if $C\in\mathcal{N}$, then C intersects each component of B. Since $A\in\mathcal{N}$, A intersects each component of B. Since $A\subset B$, we have that B has exactly m components. Let $B_1,\ldots,B_m$ be the components of B. Then each B_i is an arc or a one-point set. Given $C\in\mathcal{N}$, $C\in\langle B_1,\ldots,B_m\rangle\cap C_n(X)$ and, by the choice of m, C has exactly m components. Thus, the components of C are the sets $C\cap B_1,\ldots,C\cap B_m$. Let $\varphi:\mathcal{N}\to C(B_1)\times\ldots\times C(B_m)$ be given by $\varphi(C)=(C\cap B_1,\ldots,C\cap B_m)$. It is easy to check that φ is continuous and one-to-one. Hence, $\mathcal{N}$ can be embedded in $C(B_1)\times\ldots\times C(B_m)$. Since $C([0,1])$ is a 2-cell, we conclude that $\mathcal{N}$ can be embedded in a j-cell for some $j\leq 2m\leq 2n$. This implies that $k\leq 2n$. This contradiction proves Claim 3.

CLAIM 4. If $Z\in C(Y)-F_1(Y)$ and $Y_0\not\subseteq Z$, then Z is decomposable.

Suppose, contrary to Claim 4, that Z is indecomposable. Since $Y_0\subset Y$, $Z\neq Y$. Let $\mathcal{B}$ be the arc component of $C_n(Y)-\{Z\}$ such that $Y\in\mathcal{B}$. By [14, Theorem 70.1] and Claim 3, Y does not contain $(2n+1)$-ods. By [11, Lemma 2.3], the set $\mathcal{K}=\{K\subset Z:K\text{ is composant of } Z\text{ and }\langle K\rangle\cap C_n(Y)\cap\mathcal{B}\neq\emptyset\}$ has at most $2n$ elements. Since Z has infinitely many composants [18, Theorem 11.15], we can take a composant K_0 of Z such that $K_0\notin\mathcal{K}$. Fix a point $z_0\in K_0$. Then $\{z_0\}\notin\mathcal{B}$. This proves that $C_n(Y)-\{Z\}$ is arcwise disconnected. Since h is a homeomorphism, $C_n(X)-\{h^{-1}(Z)\}$ is arcwise disconnected. By Claim 1, $X=h^{-1}(Z)$ and $Z=h(X)=Y_0$, a contradiction. Therefore, Z is decomposable.

CLAIM 5. If $Z\in C(Y)-F_1(Y)$ and $Y_0\not\subseteq Z$, then Z is an arc.

In order to prove Claim 5, let $\mathcal{W}=h^{-1}(C_n(Z))$. Since $Y_0\notin C_n(Z)$, we have that $X=h^{-1}(Y_0)\notin\mathcal{W}$. Let $B=\cup\{D:D\in\mathcal{W}\}$. By Lemma 2.1, $B\in C_n(X)$. Let $B_1,\ldots,B_m$ be the components of B, where $m\leq n$. By [11, Corollary 2.2], the arc component of $C_n(X)-\{X\}$ that contains $Z_0=h^{-1}(Z)$ is a set of the form $\langle K_1,\ldots,K_r\rangle\cap C_n(X)$, where $r\leq n$ and $K_1,\ldots,K_r$ are composants of X. Since $C_n(Z)$ is arcwise connected, $\mathcal{W}$ is an arcwise connected set and $X\notin\mathcal{W}$. Since $Z_0\in\mathcal{W}$, $\mathcal{W}\subset\langle K_1,\ldots,K_r\rangle\cap C_n(X)$.

This implies that $B \subset K_1 \cup \ldots \cup K_r$ and then $B \neq X$. Hence, each B_i is an arc or a one-point set.

We claim that Z is locally connected.

Suppose to the contrary that Z is not connected im kleinen at some element $z_0 \in Z$. Then there exist an open subset U of Z and a sequence of points $\{z_j\}_{j=1}^{\infty}$ in U such that $z_0 \in U$, $\lim z_j = z_0$ and if E_j is the component of U containing z_j ($j \in \mathbb{N} \cup \{0\}$), then $E_0, E_1, E_2, \ldots$ are all different. Note that $U \neq Z$. Let V be an open subset of Z such that $z_0 \in V$ and $\mathrm{cl}_Z(V) \subset U$. For each $j \in \mathbb{N}$, we assume that $z_j \in V$ and we take the component D_j of $\mathrm{cl}_Z(V)$ such that $z_j \in D_j$. We may assume that $\lim D_j = D_0$ for some $D_0 \in C(Z)$. Then $z_0 \in D_0 \subset E_0$, $D_j \subset E_j$ and $D_j \cap \mathrm{bd}_Z(V) \neq \emptyset$ [17, Theorem 2.3] for each $j \in \mathbb{N}$. Thus, $D_0 \cap \mathrm{bd}_Z(V) \neq \emptyset$ and D_0 is nondegenerate. Fix a nondegenerate continuum D such that $z_0 \in D \subset D_0 \cap V$.

Since $\mathrm{cl}_Z(V) \neq Z$, we can choose pairwise disjoint nondegenerate subcontinua $G_1, \ldots, G_{n-1}$ of Z contained in $Z - \mathrm{cl}_Z(V)$. By Claim 4, each G_i is decomposable. By [14, Exercise 14.19] G_i contains a 2-od. So, we may assume that each G_i is a 2-od. For each $i \in \{1, \ldots, n-1\}$, let $R_i \in C(G_i)$ be such that $G_i - R_i$ is disconnected. By the proof of [17, Theorem 1.100], there exists a 2-cell $\mathcal{G}_i$ in $C(G_i)$ such that $R_i, G_i \in \mathcal{G}_i$ and for each $L \in \mathcal{G}_i$, $R_i \subset L \subset G_i$. Let $\mathcal{G} = \{\{y\} \cup L_1 \cup \ldots \cup L_{n-1} \in C_n(Z) : y \in D$ and $L_i \in \mathcal{G}_i$ for each $i \in \{1, \ldots, n-1\}\}$. Notice that $\mathcal{G}$ is homeomorphic to $D \times \mathcal{G}_1 \times \ldots \times \mathcal{G}_{n-1}$, so $\dim(\mathcal{G}) \geq 2n-1$ ([10, Remark at the end of Section 4 of Chapter III]). Let

$$\mathcal{M} = h^{-1}(\mathcal{G}).$$

Then $\mathcal{M}$ is a subcontinuum of $C_n(X)$ such that $\mathcal{M} \subset \mathcal{W}$ and $\dim(\mathcal{M}) \geq 2n-1$. Notice that $X \notin \mathcal{M}$.

Let

$$m_0 = \max\{i \in \{1, \ldots, n\} : \mathcal{M} \cap (C_i(X) - C_{i-1}(X)) \neq \emptyset\}.$$

Now we show that $m_0 = n$. If $m_0 = 1$, then $\mathcal{M} \subset C(X) \cap \mathcal{W}$. This implies that each element of $\mathcal{M}$ is contained in $B_1 \cup \ldots \cup B_m$. Thus, $\mathcal{M} \subset C(B_1) \cup \ldots \cup C(B_m)$, in fact $\mathcal{M} \subset C(B_k)$ for some $k \in \{1, \ldots, m\}$ and so $\dim(\mathcal{M}) \leq 2$. Since each $C(B_i)$ is a one-point set or a 2-cell, we conclude that $2n-1 \leq \dim(\mathcal{M}) \leq 2$. Hence, $n = 1$, contrary to our assumption. Therefore, $m_0 \geq 2$.

Let $M_0 \in \mathcal{M} \cap (C_{m_0}(X) - C_{m_0-1}(X))$. Let $M_1, \ldots, M_{m_0}$ be the components of M_0. Suppose that $M_0 = h^{-1}(\{y_0\} \cup L_1^{(0)} \cup \ldots \cup L_{n-1}^{(0)})$, where $y_0 \in D$ and $L_i^{(0)} \in \mathcal{G}_i$ for each $i \in \{1, \ldots, n-1\}$. Let $\varepsilon > 0$ be such that the sets $N(\varepsilon, M_1), \ldots, N(\varepsilon, M_{m_0})$ are pairwise disjoint. Since $X \notin \mathcal{M}$, $M_0 \neq X$, so we can ask that $X \neq N(\varepsilon, M_1) \cup \ldots \cup N(\varepsilon, M_{m_0})$.

Since $C_{m_0-1}(X)$ is closed in $C_n(X)$ and h^{-1} is continuous, there exists a nondegenerate continuum D' of D and for each $i \in \{1, \ldots, n-1\}$ there exists a 2-cell $\mathcal{G}_i'$ such that $L_i^{(0)} \in \mathcal{G}_i' \subset \mathcal{G}_i$, $H(M_0, h^{-1}(L)) < \varepsilon$ and $h^{-1}(L) \notin$

$C_{m_0-1}(X)$ for each $L \in \mathcal{G}' = \{\{y\} \cup L_1 \cup \ldots \cup L_{n-1} \in C_n(Z) : y \in D'$ and $L_i \in \mathcal{G}'_i$ for each $i \in \{1, \ldots, n-1\}\}$.

Given $L \in \mathcal{G}'$, $h^{-1}(L) \in \mathcal{M}$, then $h^{-1}(L) \in \langle N(\varepsilon, M_1), \ldots, N(\varepsilon, M_{m_0})\rangle$, so $h^{-1}(L)$ has at least m_0 components and, by definition of m_0, $h^{-1}(L)$ has at most m_0 components. Thus, $h^{-1}(L)$ has exactly m_0 components. Since $h^{-1}(L) \in \langle N(\varepsilon, M_1), \ldots, N(\varepsilon, M_{m_0})\rangle \cap C_n(X)$, we have that the components of $h^{-1}(L)$ are the sets $h^{-1}(L) \cap N(\varepsilon, M_1), \ldots, h^{-1}(L) \cap N(\varepsilon, M_{m_0})$. Let $L_0 = \cup\{h^{-1}(L) : L \in \mathcal{G}'\}$. By Lemma 2.1, L_0 has at most m_0 components, but $L_0 \in \langle N(\varepsilon, M_1), \ldots, N(\varepsilon, M_{m_0})\rangle \cap C_n(X)$, so L_0 has exactly m_0 components and they are $L_0 \cap N(\varepsilon, M_1), \ldots, L_0 \cap N(\varepsilon, M_{m_0})$. This implies that each set $L_0 \cap N(\varepsilon, M_i)$ is an arc or a one-point set. Notice that $\mathcal{G}'$ is homeomorphic to $D' \times \mathcal{G}'_1 \times \ldots \times \mathcal{G}'_{n-1}$, so $\dim(\mathcal{G}') \geq 2n-1$ and $\dim(h^{-1}(\mathcal{G}')) \geq 2n-1$.

Notice that the map $\psi : \mathcal{G}' \to C(L_0 \cap N(\varepsilon, M_1)) \times \ldots \times C(L_0 \cap N(\varepsilon, M_{m_0}))$ given by $\psi(L) = (h^{-1}(L) \cap N(\varepsilon, M_1), \ldots, h^{-1}(L) \cap N(\varepsilon, M_{m_0}))$ is an embedding. This shows that $\dim(C(L_0 \cap N(\varepsilon, M_1)) \times \ldots \times C(L_0 \cap N(\varepsilon, M_{m_0}))) \geq 2n-1$. Since for each $i \in \{1, \ldots, m_0\}$, $C(L_0 \cap N(\varepsilon, M_i))$ is either a one-point set or a 2-cell [14, Theorem 5.1], we obtain that $2m_0 \geq \dim(C(L_0 \cap N(\varepsilon, M_1)) \times \ldots \times C(L_0 \cap N(\varepsilon, M_{m_0})))$. Thus, $m_0 \geq n$. Hence, $m_0 = n$.

Since $M_0 \in \mathcal{M} \subset \mathcal{W}$, we have $M_0 \subset B$ and by Lemma 2.1, each B_i intersects M_0. Since B is a finite union of arcs or one-point sets, there exist pairwise disjoint subarcs (or one-point sets), $Q_1, \ldots, Q_n$ of B such that for each $i \in \{1, \ldots, n\}$, $M_i \subset \operatorname{int}_B(Q_i)$. Notice that if Q is contained in a degenerate component of B, then Q_i is a one-point set open in B. Then $M_0 \in C_n(X) \cap \mathcal{W} \cap \langle \operatorname{int}_B(Q_1), \ldots, \operatorname{int}_B(Q_n)\rangle$, which is an open subset of $\mathcal{W}$. We are going to see that each Q_i is an arc.

Since $C_n(X) - C_{n-1}(X)$ is open in $C_n(X)$ and $M_0 \in \mathcal{M} \cap (C_n(X) - C_{n-1}(X))$, there exists $\varepsilon_0 > 0$ and for each $i \in \{1, \ldots, n-1\}$ there exists a 2-cell $\mathcal{L}_i$ such that $B_Z(\varepsilon_0, y_0) \subset V$, $L_i^{(0)} \in \mathcal{L}_i \subset \mathcal{G}_i$ and $h^{-1}(L) \in \langle \operatorname{int}_B(Q_1), \ldots, \operatorname{int}_B(Q_n)\rangle \cap C_n(X) \cap \mathcal{W}$ for each $L \in \mathcal{L}$, where

$$\mathcal{L} = \{A \cup L_1 \cup \ldots \cup L_{n-1} \in C_n(Z) : H(A, \{y_0\}) < \varepsilon_0 \text{ and } L_i \in \mathcal{L}_i \text{ for each } i \in \{1, \ldots, n-1\}\}.$$

Fix a sequence $\{y_m\}_{m=1}^{\infty}$ in Z such that $\lim y_m = y_0$ and $y_m \in D_m$ for each $m \in \mathbb{N}$. Let $N_0 \in \mathbb{N}$ be such that $y_m \in B_Y(\frac{\varepsilon_0}{2}, y_0)$ for each $m \geq N_0$. For each $m \geq N_0$, choose a subcontinuum P_m of Z such that diameter$(P_m) = \frac{\varepsilon_0}{2}$ and $y_m \in P_m$. Then $P_m \subset V$, so $P_m \subset D_m$. Taking a subsequence if necessary, we may assume that $\lim P_m = P_0$ for some $P_0 \in C(Z)$ and $\lim C(P_m) = \mathcal{P}$ and some $\mathcal{P} \in C(C(Z))$. Then $y_0 \in P_0$, diameter$(P_0) = \frac{\varepsilon_0}{2}$ and $\mathcal{P} \subset C(P_0)$. Then $P_0 \subset D_0$. Fix points $p_0, q_0 \in P_0$ such that $p_0 \neq q_0$ and choose sequences $\{p_m\}_{m=N_0}^{\infty}$, $\{q_m\}_{m=N_0}^{\infty}$ is Z such that $\lim p_m = p_0$, $\lim q_m = q_0$ and for each $m \geq N_0$, $p_m \in P_m$. Given $m \geq N_0$, choose order arcs α_m, β_m from $\{p_m\}$ to P_m and $\{q_m\}$ to P_m, respectively. Let $\mathcal{T}_m = \operatorname{Im} \alpha_m$ and $\mathcal{S}_m = \operatorname{Im} \beta_m$. We may assume also that $\lim \mathcal{T}_m = \mathcal{T}_0$ and $\lim \mathcal{S}_m = \mathcal{S}_0$, for some $\mathcal{T}_0, \mathcal{S}_0 \in C(C(P_0))$. By [17, Remark 1.34], each of the sets $\mathcal{T}_0$ and $\mathcal{S}_0$ are images of respective order arcs from $\{p_0\}$ to P_0 and $\{q_0\}$ to P_0. Notice that $F_1(P_0) \cup \mathcal{T}_0 \cup \mathcal{S}_0 \subset \mathcal{P}$.

Given $m \in \{0, N_0, N_0 + 1, \ldots\}$ and a subcontinuum A of P_m, since $A \subset P_m \subset B_Y(\varepsilon_0, y_0)$, $H(A, \{y_0\}) < \varepsilon_0$. Thus, for each choice of elements $L_i \in \mathcal{L}_i$ ($i \in \{1, \ldots, n-1\}$), $A \cup L_1 \cup \ldots \cup L_{n-1} \in \mathcal{L}$.

Given $L \in \mathcal{L}$, $h^{-1}(L) \in \langle \mathrm{int}_B(Q_1), \ldots, \mathrm{int}_B(Q_n) \rangle \cap C_n(X) \cap \mathcal{W}$. Since $Q_1, \ldots, Q_n$ are pairwise disjoint, we have that $h^{-1}(L)$ has exactly n components and they are $h^{-1}(L) \cap Q_1, \ldots, h^{-1}(L) \cap Q_n$. Let $\mathcal{A} = C(Q_1) \times \ldots \times C(Q_n)$. Define $\sigma : \mathcal{L} \to \mathcal{A}$ by $\sigma(L) = (h^{-1}(L) \cap Q_1, \ldots, h^{-1}(L) \cap Q_n)$. Clearly, σ is an embedding. By [17, Theorem 2.1], $\dim[C(P_0)] \geq 2$. Since $\mathcal{L}$ contains a topological copy of $C(P_0) \times \mathcal{L}_1 \times \ldots \times \mathcal{L}_{n-1}$ and the dimension of this set is $\dim[C(P_0)] + 2(n-1) \geq 2n$ [10, Remark at the end of Section 4 of Chapter III], we have that $\dim[\mathcal{A}] \geq 2n$. Since each $C(Q_i)$ is a one-point set or a 2-cell, $\dim[\mathcal{A}] \leq 2n$, so $\dim[\mathcal{A}] = 2n$. This implies that each Q_i is an arc and $\mathcal{A}$ is a $2n$-cell.

Since $F_1(P_0) \subset \mathcal{P}$, we have $\dim(\mathcal{P}) \geq 1$. To finish the proof that Z is locally connected, we analyze two cases.

Case 1. $\dim(\mathcal{P}) \geq 2$.

In this case, let $\mathcal{L}_0 = \{A \cup L_1 \cup \ldots \cup L_{n-1} \in C_n(Z) : A \in \mathcal{P}$ and $L_i \in \mathcal{L}_i$ for each $i \in \{1, \ldots, n-1\}\}$. Since $\mathcal{L}_0$ is homeomorphic to $\mathcal{P} \times [0,1]^{2(n-1)}$, $\dim[\mathcal{L}_0] \geq 2n$. Since $\sigma|_{\mathcal{L}_0} : \mathcal{L}_0 \to \mathcal{A}$ is an embedding, $\dim[\mathcal{L}_0] = 2n$. By [10, Theorem IV 3], $\mathrm{int}_{\mathcal{A}}[\sigma(\mathcal{L}_0)]$ is nonempty. Let $L = A \cup L_1 \cup \ldots \cup L_{n-1} \in C_n(Z)$ be such that $\sigma(L) \in \mathrm{int}_{\mathcal{A}}[\sigma(\mathcal{L}_0)]$, where $A \in \mathcal{P}$ and $L_i \in \mathcal{L}_i$ for each $i \in \{1, \ldots, n-1\}$. Since $A \in \mathcal{P} = \lim C(P_m)$, there exists a sequence $\{A_m\}_{m=1}^{\infty}$ in $C(Z)$ such that $\lim A_m = A$ and $A_m \in C(P_m)$ for each $m \in \mathbb{N}$. Then $\lim \sigma(A_m \cup L_1 \cup \ldots \cup L_{n-1}) = \sigma(A \cup L_1 \cup \ldots \cup L_{n-1}) = \sigma(L) \in \mathrm{int}_{\mathcal{A}}[\sigma(\mathcal{L}_0)]$. Thus, there exists $m \in \mathbb{N}$ such that $\sigma(A_m \cup L_1 \cup \ldots \cup L_{n-1}) \in \sigma(\mathcal{L}_0)$. Since σ is one-to-one, $A_m \cup L_1 \cup \ldots \cup L_{n-1} \in \mathcal{L}_0$. This implies that $A_m \cup L_1 \cup \ldots \cup L_{n-1} = A' \cup L_1' \cup \ldots \cup L_{n-1}'$, where $A' \in \mathcal{P}$ and $L_i' \in \mathcal{L}_i$ for each $i \in \{1, \ldots, n-1\}$. Intersecting these sets with $B_{C_n(Z)}(\varepsilon_0, \{y_0\})$, we obtain that $A_m = A'$. This is a contradiction since $A_m \in C(P_m)$, $A' \in \mathcal{P} \subset C(P_0)$ and $P_0 \cap P_m = \emptyset$. Therefore, this case is impossible.

Case 2. $\dim(\mathcal{P}) = 1$.

Let S^+ (respectively, S^-) be the upper (lower) half of S^1. Since $F_1(P_0) \cap (\mathcal{T}_0 \cup \mathcal{S}_0) = \{\{p_0\}, \{q_0\}\}$, by Urysohn's lemma for metric spaces, there exists a map $f : F_1(P_0) \cup \mathcal{T}_0 \cup \mathcal{S}_0 \to S^1$ such that $f(F_1(P_0)) = S^-$, $f(\{p_0\}) = \{(-1, 0)\}$, $f(\{q_0\}) = \{(1, 0)\}$ and $f(\mathcal{T}_0 \cup \mathcal{S}_0) = S^+$. Since $\dim(\mathcal{P}) = 1$, by [10, Theorem VI 4] the map f can be extended to a map (we also call f to the extension) $f : \mathcal{P} \to S^1$. Since S^1 is an ANR, f can be extended to a map (we also call f to the extension) $f : \mathcal{U} \to S^1$, where $\mathcal{U}$ is an open subset of $C(Z)$ such that $\mathcal{P} \subset \mathcal{U}$. Since $\lim \mathcal{T}_m \cup \mathcal{S}_m = \mathcal{T}_0 \cup \mathcal{S}_0$ and $\lim F_1(P_m) = F_1(P_0)$, there exists $m \geq N_0$ such that $C(P_m) \subset \mathcal{U}$, $f(\mathcal{T}_m \cup \mathcal{S}_m) \subset N_{S^1}(\frac{1}{10}, S^+)$, $f(F_1(P_m)) \subset$

$N_{S^1}(\frac{1}{10}, S^-)$, $f(\{p_m\}) \in N_{S^1}(\frac{1}{10}, \{(-1,0)\})$ and $f(\{q_m\}) \in N_{S^1}(\frac{1}{10}, \{(1,0)\})$. [19, Lemma 5.12] and the fact that $F_1(P_m) \cap (\mathcal{T}_m \cup \mathcal{S}_m) = \{p_m, q_m\}$ imply that $f|F_1(P_m) \cup \mathcal{T}_m \cup \mathcal{S}_m$ cannot be lifted (that is, there is not a map $f_1 : F_1(P_m) \cup \mathcal{T}_m \cup \mathcal{S}_m \to \mathbb{R}$ such that $f|F_1(P_m) \cup \mathcal{T}_m \cup \mathcal{S}_m = (\cos \circ f_1, \sin \circ f_1)$). But, by [2, Lemma 13], $f|C(P_m)$ can be lifted. Since $F_1(P_m) \cup \mathcal{T}_m \cup \mathcal{S}_m \subset C(P_m)$, we conclude that $f|F_1(P_m) \cup \mathcal{T}_m \cup \mathcal{S}_m$ can be lifted. This contradiction proves that this case is also impossible. Therefore, we have shown that Z is locally connected.

Now, suppose that Z contains a simple triod T, we may assume that $T \neq Z$, so we can construct arcs $J_1, \ldots, J_{n-1}$ is Z such that $T, J_1, \ldots, J_{n-1}$ are pairwise disjoint. Since $C(T) \times C(J_1) \times \ldots \times C(J_{n-1})$ is naturally embedded in $C_n(Z)$. By [14, Examples 5.1 and 5.4], $C(T) \times C(J_1) \times \ldots \times C(J_{n-1})$ contains a $(2n+1)$-cell. This contradicts Claim 3 and ends the proof that Z does not contain simple triods. Hence, Z is an arc or a simple closed curve. Using an order arc from Z to Y, it is possible to construct a subcontinuum Z_1 of Y such that $Z \subsetneq Z_1$ and $Y_0 \not\subseteq Z_1$. Thus, we can apply what we have proved to Z_1 and conclude that Z_1 is an arc or a simple closed curve. Therefore, Z is an arc. This completes the proof of Claim 5.

Claim 6. If $D \in C_n(Y)$ and $Y_0 \not\subseteq D$, then $D \in \mathcal{W}_n(Y)$. Moreover, $\mathcal{W}_n(X) = C_n(X) - \{X\}$.

We prove the first part of Claim 6, the second one can be made with similar arguments. Let V be an open subset of Y such that $D \subset V$ and $Y_0 \not\subseteq \mathrm{cl}_Y(V)$. Let Z be a component of D. Let W be the component of V containing Z. By Claim 5, Z is an arc or a one-point set. Let B be the component of $\mathrm{cl}_Y(V)$ such that $Z \subset B$. Then B is nondegenerate. By Claim 5, B is an arc. By [18, Theorem 12.10], $\mathrm{cl}_Y(W) \cap (Y - V) \neq \emptyset$. Thus, W is not compact. Then W is a non compact connected subset of B. Hence, W is homeomorphic either to $[0,1)$ or $(0,1)$. That is, W is a wire. This ends the proof of Claim 6.

Claim 7. If $Z \in C(Y) - F_1(Y)$ and $Y_0 \not\subseteq Z$, then $h^{-1}(Z)$ is connected.

We prove Claim 7. Let $A = h^{-1}(Z)$. By Claim 6, $Z \in W_1(Y)$, and by Theorem 2.3, $Z \notin \mathcal{Z}_n(Y)$. Since $A \neq X$, by Claim 6, $A \in \mathcal{W}_n(X)$. Since h is a homeomorphism, $Z \notin \mathcal{Z}_n(Y)$ and the definition of $\mathcal{Z}_n(X)$ is given in terms of topological properties that are preserved under homeomorphisms, we obtain that $A \notin \mathcal{Z}_n(X)$. By Theorem 2.3, Z has a basis $\mathcal{B}$ of neighborhoods in $C_n(Y)$ such that for each $\mathcal{V} \in \mathcal{B}$, if $\mathcal{C}$ is the component of $\mathcal{V}$ that contains Z, then $\mathcal{C} \cap \mathcal{Z}_n(Y)$ is connected. Since $Y_0 \not\subseteq Z$, we can ask that for each $\mathcal{V} \in \mathcal{B}$ and each $B \in \mathcal{V}$, $Y_0 \not\subseteq B$, then by Claim 6, $B \in \mathcal{W}_n(Y)$ and $h(X) \notin \mathcal{V}$. Using the fact that h is a homeomorphism and the second part of Claim 6, it is easy to show that if $\mathcal{V} \in \mathcal{B}$ and $\mathcal{C}$ is the component of $\mathcal{V}$ that contains Z, then $h^{-1}(\mathcal{C}) \cap \mathcal{Z}_n(X) = h^{-1}(\mathcal{C} \cap \mathcal{Z}_n(Y))$. Define $h^{-1}(\mathcal{B}) = \{h^{-1}(\mathcal{V}) \subset C_n(X) : \mathcal{V} \in \mathcal{B}\}$. Then $h^{-1}(\mathcal{B})$ is a basis of neighborhoods of A in $C_n(X)$. Given $\mathcal{V} \in \mathcal{B}$ and $\mathcal{C}$ the component of $\mathcal{V}$ that contains Z, the equality $h^{-1}(\mathcal{C}) \cap \mathcal{Z}_n(X) = h^{-1}(\mathcal{C} \cap \mathcal{Z}_n(Y))$ implies that $h^{-1}(\mathcal{C}) \cap \mathcal{Z}_n(X)$ is connected. Hence, we can apply Theorem 2.3 to conclude that $A \in \mathcal{W}_1(X)$. In particular, A is connected. Hence, $h^{-1}(Z)$ is connected.

Claim 8. Let $K_1, \ldots, K_r$ be composants of X, where $r \leq n$. Then $C(X) \subset \mathrm{cl}_{C_n(X)}(\langle K_1, \ldots, K_r \rangle \cap C_n(X))$.

We prove Claim 8. Since $C(X) - (\{X\} \cup F_1(X))$ is dense in $C(X)$, it is enough to show that $C(X) - (\{X\} \cup F_1(X)) \subset \mathrm{cl}_{C_n(X)}(\langle K_1, \ldots, K_r \rangle \cap C_n(X))$. Let $E \in C(X) - (\{X\} \cup F_1(X))$. Then E is an arc. Let a_1, a_2 be the end points of E. Let K be a composant of X. Given $i \in \{1, 2\}$, let $A_i(K) = \{p \in E :$ there exists a sequence $\{B_m\}_{m=1}^{\infty}$ in $\langle K \rangle \cap C(X)$ converging to a subcontinuum B of E and $p, a_i \in B\}$. Since K is dense in X, $\{a_i\} \in A_i(K)$. It is easy to show that $A_i(K)$ is closed in E and that if $p \in A_i(K)$, then the subarc of E joining a_i and p is contained in $A_i(K)$. Thus $A_i(K)$ is a subcontinuum of E.

We claim that $E = A_1(K) \cup A_2(K)$. Take $p \in E - \{a_1, a_2\}$. Let $\{p_m\}_{m=1}^{\infty}$ be a sequence in K such hat $\lim p_m = p$. Let $\mu : C(X) \to [0, 1]$ be a Whitney map, where $\mu(X) = 1$ ([14, Theorem 13.4]). Using order arcs, it is possible to find a subcontinuum B_m of X such that $p_m \in B_m$ and $\mu(B_m) = \mu(E)$, for each $m \in \mathbb{N}$. We may assume that $\lim B_m = B$ for some $B \in C(X)$. For each $m \in \mathbb{N}$, since $E \neq X$, we have that $B_m \neq X$. This implies that $B_m \in \langle K \rangle \cap C(X)$. Notice that $p \in B$. Since E and B are proper subcontinua of X, $E \cup B$ is a subcontinuum of X, so $E \cup B$ is an arc. Since $\mu(E) = \mu(B)$, it is not possible that $B \subsetneq E$. This implies that $a_1 \in B$ or $a_2 \in B$.

For each $m \in \mathbb{N}$, let $\alpha_m : [0, 1] \to C(B_m)$ be an order arc from $\{p_m\}$ to B_m. We may assume that $\lim \mathrm{Im}\, \alpha_m = \gamma$ for some $\gamma \in C(C(X))$. By [17, Remark 1.34], γ is the image of an order arc $\alpha : [0, 1] \to C(X)$ that joins $\{p\}$ to B. Let $s_0 = \min\{s \in [0, 1] : \gamma(s) \cap \{a_1, a_2\} \neq \emptyset\}$. Given $s < s_0$, $\gamma(s) \cap \{a_1, a_2\} = \emptyset$, $\gamma(s)$ intersects the arc E and $\gamma(s)$ is contained in the arc $E \cup B$. This implies that $\gamma(s) \subset E$. Hence, $\gamma(s_0) \subset E$. Since $\gamma(s_0)$ belongs to $\lim \mathrm{Im}\, \alpha_m$, $\gamma(s_0)$ satisfies the conditions in the definition of $A_i(K)$, this allows us to conclude that $p \in A_1(K) \cup A_2(K)$. We have shown that $E = A_1(K) \cup A_2(K)$.

In the case that $r = 1$, by the connectedness of E, we conclude that there exists a point $p \in A_1(K_1) \cap A_2(K_1)$. Let $\{B_m\}_{m=1}^{\infty}$ and $\{C_m\}_{m=1}^{\infty}$ be sequences in $\langle K_1 \rangle \cap C(X)$ converging to respective subcontinua B and C of E satisfying $p, a_1 \in B$ and $p, a_2 \in C$. Then $B \cup C$ is a subcontinuum of E containing a_1 and a_2. Thus, $E = B \cup C$. Hence, $E = \lim B_m \cup C_m$. Since $B_m \cup C_m \in \langle K_1 \rangle \cap C_n(X)$ for each $m \in \mathbb{N}$, we conclude that $E \in \mathrm{cl}_{C_n(X)}(\langle K_1 \rangle \cap C_n(X))$.

In the case $r \geq 2$, take the natural order in E such that $a_1 < a_2$. By the connectedness of E, we can choose points $p_1 \in A_1(K_1) \cap A_2(K_1)$ and $p_2 \in A_1(K_2) \cap A_2(K_2)$. We can assume that $p_1 \leq p_2$. Let $\{B_m\}_{m=1}^{\infty}$ and $\{C_m\}_{m=1}^{\infty}$ be sequences in $\langle K_1 \rangle \cap C(X)$ and $\langle K_2 \rangle \cap C(X)$, respectively, converging to subcontinua B and C, respectively, of E satisfying $p_1, a_2 \in B$ and $p_2, a_1 \in C$. Thus, $E = B \cup C$ and $E = \lim B_m \cup C_m$. For each $i \in \{3, \ldots, r\}$, choose a sequence $\{x_m^{(i)}\}_{m=1}^{\infty}$ in K_i such that $\lim x_m^{(i)} = a_1$. For each $m \in \mathbb{N}$, let $E_m = B_m \cup C_m \cup \{x_m^{(3)}, \ldots, x_m^{(r)}\}$. Then $E_m \in \langle K_1, \ldots, K_r \rangle \cap C_n(X)$ and $\lim E_m = E$. This ends the proof of Claim 8.

CLAIM 9. $Y_0 = Y$.

Since $C_n(X) - \{X\}$ has uncountably many arc components (Claim 1), $C_n(Y) - \{Y_0\}$ has uncountably many arc components. Let $\mathcal{G}$ be an arc component of $C(Y) - \{Y_0\}$ such that $Y \notin \mathcal{G}$. Given $G \in \mathcal{G}$, if $G \not\subseteq Y_0$, then an order arc from G to Y is a path connecting G to Y without passing through Y_0, a contradiction. Thus, $G \subset Y_0$ and $\mathcal{G} \subset C_n(Y_0)$. By [11, Corollary 2.2], $h^{-1}(\mathcal{G})$ is of the form $h^{-1}(\mathcal{G}) = \langle K_1, \ldots, K_r \rangle \cap C_n(X)$ for some $r \leq n$ and composants $K_1, \ldots, K_r$ of X. Suppose that $Y_0 \neq Y$. Take a point $y \in Y - Y_0$. Then there exists a nondegenerate subcontinuum Z of Y such that $y \in Z \subset Y - Y_0$. Let $E = h^{-1}(Z)$. By Claim 7, E is a subcontinuum of X. By Claim 8, $E \in \mathrm{cl}_{C_n(X)}(\langle K_1, \ldots, K_r \rangle \cap C_n(X))$. Then $Z \in \mathrm{cl}_{C_n(Y)}(h(\langle K_1, \ldots, K_r \rangle \cap C_n(X))) = \mathrm{cl}_{C_n(Y)}(\mathcal{G}) \subset C_n(Y_0)$ and $Z \subset Y_0$. This contradicts the choice of Z and completes the proof of Claim 9.

We have shown that Y is an indecomposable continuum (Claim 2) such that each one of its nondegenerate proper subcontinua are arcs (Claim 5). Moreover, $h^{-1}(Z) \in C(X)$ for each $Z \in C(Y)$ (this follows from Claim 7). Thus, Y satisfies the initial conditions we had for X. By symmetry, we can conclude that $h(W) \in C(Y)$ for each $W \in C(X)$. Hence, $h|_{C(X)} : C(X) \to C(Y)$ is a homeomorphism. By [15, Theorem 3], $h(F_1(X)) = F_1(Y)$. This proves that X has unique hyperspace $C_n(X)$ and $C_n(X)$ is rigid. ∎

QUESTION 3.1. *Suppose that X is a wired continuum. Is it true that $C_2(X)$ is not rigid? It would be interesting to determine if $C_2(X)$ is rigid for the Buckethandle continuum (see [18, 2.9] for a description), the solenoids (see [18, 2.8] for a description) or the cone over the Cantor set.*

QUESTION 3.2 ([13, Problem 23]). *Suppose that X is an indecomposable arc continuum. Does X have unique hyperspace $C_2(X)$? It would be interesting to solve this question for the case that X is the buckethandle or a solenoid.*

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