On some Gronwall type inequalities with iterated integrals

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Abstract. *The main objective of the present paper is to establish some new Gronwall type inequalities involving iterated integrals.*

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1. Introduction

Let $u : [\alpha, \alpha + h] \to R$ be a continuous real-valued function satisfying the inequality

$$0 \leq u(t) \leq \int_{\alpha}^{t} [a + bu(s)]\, ds, \quad \text{for all} \quad t \in [\alpha, \alpha + h],$$

where a, b are nonnegative constants. Then $u(t) \leq ahe^{bh}$ for $t \in [\alpha, \alpha + h]$. This result was proved by T. H. Gronwall [8] in the year 1919, and it is a prototype for the study of several integral inequalities of Volterra type, and also for obtaining explicit bounds of the unknown function. Among the several publications on this subject, the paper of Bellman [3] is very well known. It is clear that Bellman's result contains that of Gronwall. This is the reason why inequalities of this type were called "Gronwall-Bellman inequalities" or "Inequalities of Gronwall type". The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations, integral equations and inequalities of various types (see Gronwall [8] and Guiliano [9]). Some applications of this result to the study of stability of the solution of linear and nonlinear differential equations may be found in Bellman [3]. Some applications to existence and uniqueness theory of differential equations may be found in Nemyckii-Stepanov [13], Bihari [4], and

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Langenhop [10]. During the past few years several authors (see references below and some of the references cited therein) have established several Gronwall type integral inequalities in two or more independent real variables. Of course, such results have application in the theory of partial differential equations and Volterra integral equations.

Bainov and Simeonov [1, p. 101] proved the following interesting integral inequality involving iterated integrals:

Let $u(t), a(t)$, and $b(t)$ be nonnegative continuous functions in $J = [\alpha, \beta]$, and suppose that

$$u(t) \le a(t) + b(t)\Big[\int_\alpha^t k_1(t,t_1)u(t_1)\,dt_1 + \cdots + \int_\alpha^t \Big(\int_\alpha^{t_1} \cdots \Big(\int_\alpha^{t_{n-1}} k_n(t,t_1,\dots,t_n)u(t_n)\,dt_n\Big)\dots\Big)dt_1\Big] \tag{1.1}$$

for all $t \in J$, where $k_i(t,t_1,\dots,t_i)$ are nonnegative continuous functions in $J_{i+1}, i = 1,2,\dots,n$, which are nondecreasing in $t \in J$ for all fixed $(t_1,\dots,t_i) \in J_i, i = 1,2,\dots,n$. Then, for all $t \in J$

$$u(t) \le a(t) + b(t)\int_\alpha^t \widehat{R}[a](t,s)\exp\Big(\int_s^t \widehat{R}[b](t,\tau)\,d\tau\Big)\,ds,$$

where, for all $(t,s) \in J_2$,

$$\widehat{R}[w](t,s)) = k_1(t,s)w(s) + \int_\alpha^s k_2(t,s,t_2)w(t_2)dt_2 + \sum_{i=3}^n \int_\alpha^s \Big(\int_\alpha^{t_2} \cdots \Big(\int_\alpha^{t_{i-1}} k_i(t,s,t_2,\dots,t_i)w(t_i)\,dt_i\Big)\cdots\Big)\,dt_2,$$

for each continuous function $w(t)$ in J.

In this paper we consider simple inequalities involving iterated integrals in inequality (1.1) for functions when the function u on the right-hand side of inequality (1.1) is replaced by the function u^p for some p, We also provide some integral inequalities involving iterated integrals.

2. The main results

In this section we state and prove some new nonlinear integral inequalities involving iterated integrals. Throughout the paper all functions which appear in inequalities are assumed to be real-valued.

Before considering our first integral inequality, we need the following lemmas, which appeared in [1, p. 2, p. 38].

Lemma 2.1. *Let $b(t)$ and $f(t)$ be continuous functions for $t \ge \alpha$, let $v(t)$ be a differentiable function for $t \ge \alpha$, and suppose*

$$v'(t) \le b(t)v(t) + f(t), \quad t \ge \alpha$$

and $v(\alpha) \le v_0$. *Then, for all* $t \ge \alpha$,

$$v(t) \le v_0 \exp\biggl(\int_\alpha^t b(s)\,ds\biggr) + \int_\alpha^t f(s)\exp\biggl(\int_s^t b(\tau)\,d\tau\biggr)\,ds.$$

Lemma 2.2. *Let* $v(t)$ *be a positive differential function satisfying the inequality*

$$v'(t) \le b(t)v(t) + k(t)v^p(t), \quad t \in J = [\alpha,\beta],$$

where functions b *and* k *are continuous in* J, *and* $p \ge 0$, $p \ne 1$, *is a constant. Then*

$$v(t) \le \exp\biggl(\int_\alpha^t b(s)\,ds\biggr)\biggl[v^q(\alpha) + q\int_\alpha^t k(s)\exp\biggl(-q\int_\alpha^s b(\tau)\,d\tau\biggr)\,ds\biggr]^{1/q}$$

for all $t \in [\alpha,\beta_1)$, *where* β_1 *is chosen so that the expression between* [...] *is positive in the subinterval* $[\alpha,\beta_1)$.

In the next theorems we consider some simple inequalities involving iterated integrals.

Let $\alpha < \beta$, and set

$$J_i = \{(t_1,t_2,\dots,t_i) \in R^i : \alpha \le t_i \le \cdots \le t_1 \le \beta\},\ i = 1,\cdots,n.$$

Theorem 2.1. *Let* $u(t), a(t)$ *and* $b(t)$ *be nonnegative continuous functions in* $J = [\alpha,\beta]$ *and let* $p > 1$ *be a constant. Suppose that* $\frac{a(t)}{b(t)}$ *is nondecreasing in* J *and*

$$\begin{aligned} u(t) \le a(t) + b(t)\biggl[&\int_\alpha^t k_1(t,t_1)u^p(t_1)\,dt_1 + \cdots \\ &+ \int_\alpha^t\biggl(\int_\alpha^{t_1}\cdots\biggl(\int_\alpha^{t_{n-1}} k_n(t,t_1,\cdots,t_n)u^p(t_n)\,dt_n\biggr)\cdots\biggr)dt_1\biggr] \end{aligned} \tag{2.1}$$

for any $t \in J$, *where* $k_i(t,t_1,\dots,t_i)$ *are nonnegative continuous functions in* J_{i+1} *for* $i = 1,2,\cdots,n$, *which are nondecreasing in* $t \in J$ *for all fixed* $(t_1,\cdots,t_i) \in J_i$, $i = 1,2,\cdots,n$. *Then, for any* $t \in [\alpha,\beta_p)$

$$u(t) \le a(t)\biggl[1 - (p-1)\int_\alpha^t \biggl(\frac{a(s)}{b(s)}\biggr)^{p-1} R[b^p](t,s)\,ds\biggr]^{\frac{1}{1-p}} \tag{2.2}$$

where, for any $(t,s) \in J_2$,

$$\beta_p = \sup\{t \in J : (p-1)\int_\alpha^t \biggl(\frac{a(s)}{b(s)}\biggr)^{p-1} R[b^p](t,s)\,ds < 1\},$$

and

$$\begin{aligned} R[w](t,s) &= k_1(t,s)w(s) + \int_\alpha^s k_2(t,s,t_2)w(t_2)dt_2 \\ &\quad + \sum_{i=3}^n \int_\alpha^s\biggl(\int_\alpha^{t_2}\cdots\biggl(\int_\alpha^{t_{i-1}} k_i(t,s,t_2,\cdots,t_i)w(t_i)\,dt_i\biggr)\cdots\biggr)\,dt_2, \end{aligned}$$

for each continuous function $w(t)$ in J.

Proof. For a fixed $T \in (\alpha, \beta]$ and $\alpha \le t \le T$ we have

$$u(t) \le a(t) + b(t)v(t), \tag{2.3}$$

where

$$\begin{aligned} v(t) = &\int_\alpha^t k_1(T, t_1)u^p(t_1)\,dt_1 + \cdots \\ &+ \int_\alpha^t \biggl(\int_\alpha^{t_1} \cdots \biggl(\int_\alpha^{t_{n-1}} k_n(T, t_1, \cdots, t_n)u^p(t_n)\,dt_n\biggr)\cdots\biggr)dt_1. \end{aligned}$$

Since $\frac{\partial k_i}{\partial t}(T, t_1, \dots, t_i) = 0$ for $i = 1, \dots, n$ and $t \in [\alpha, T]$, we have

$$v'(t) = R[u^p](T, t) \le (R[b^p](T, t))\biggl(\frac{a(t)}{b(t)} + v(t)\biggr)^p,$$

that is,

$$v'(t) \le Q(T, t)[a(T)/b(T) + v(t)], \tag{2.4}$$

where $Q(T, t) = (R[b^p](T, t))[a(t)/b(t) + v(t)]^{p-1}$. *Lemma 2.1* and (2.4) imply

$$v(t) + \frac{a(T)}{b(T)} \le \frac{a(T)}{b(T)} \exp\biggl(\int_\alpha^t Q(T, s)\,ds\biggr), \quad \alpha \le t \le T.$$

Hence, for $t = T$,

$$v(t) + \frac{a(t)}{b(t)} \le \frac{a(t)}{b(t)} \exp\biggl(\int_\alpha^t Q(T, s)\,ds\biggr). \tag{2.5}$$

From (2.5), we successively obtain

$$\begin{aligned} &\biggl[v(t) + \frac{a(t)}{b(t)}\biggr]^{p-1} \le \biggl[\frac{a(t)}{b(t)}\biggr]^{p-1} \exp\biggl(\int_\alpha^t (p-1)Q(T, s)\,ds\biggr), \\ &Q(T, t) \le (R[b^p](T, t))\biggl[\frac{a(t)}{b(t)}\biggr]^{p-1} \exp\biggl(\int_\alpha^t (p-1)Q(T, s)\,ds\biggr), \\ &Z(T, t) \le (p-1)(R[b^p](T, t))\biggl[\frac{a(t)}{b(t)}\biggr]^{p-1} \exp\biggl(\int_\alpha^t (p-1)Q(T, s)\,ds\biggr), \end{aligned}$$

where $Z(T, t) = (p-1)Q(T, t)$. Consequently, we have

$$Z(T, s)\exp\biggl(-\int_\alpha^s Z(T, s)\,ds\biggr) \le (p-1)R[b^p](T, s)\biggl[\frac{a(s)}{b(s)}\biggr]^{p-1}$$

or

$$\frac{d}{ds}\biggl[-\exp\biggl(-\int_\alpha^s Z(T, \tau)\,d\tau\biggr)\biggr] \le (p-1)R[b^p](T, s)\biggl[\frac{a(s)}{b(s)}\biggr]^{p-1}.$$

Integrating this from α to t yields

$$1-\exp\Big(-\int_\alpha^t Z(T,s)\,ds\Big) \le (p-1)\int_\alpha^t \Big(\frac{a(s)}{b(s)}\Big)^{p-1} R[b^p](T,s)\,ds,$$

from which we conclude that

$$\exp\Big(\int_\alpha^t Q(T,s)\,ds\Big) \le \Big[1-(p-1)\int_\alpha^t \Big(\frac{a(s)}{b(s)}\Big)^{p-1} R[b^p](T,s)\,ds\Big]^{\frac{1}{1-p}}.$$

This, together with (2.3) and (2.5), implies

$$u(t) \le a(t)\Big[1-(p-1)\int_\alpha^t \Big(\frac{a(s)}{b(s)}\Big)^{p-1} R[b^p](T,s)\,ds\Big]^{\frac{1}{1-p}}.$$

In particular, for $T=t$ we find (2.2). This completes the proof. □

Theorem 2.2. *Let $u(t)$ and $b(t)$ be nonnegative continuous functions in $J=[\alpha,\beta]$, and suppose that*

$$\begin{aligned} u(t) \le b(t)\Big[a(t) &+ \int_\alpha^t k_1(t,t_1)u^p(t_1)\,dt_1 + \cdots \\ &+ \int_\alpha^t \Big(\int_\alpha^{t_1} \cdots \Big(\int_\alpha^{t_{n-1}} k_n(t,t_1,\ldots,t_n)u^p(t_n)\,dt_n\Big)\ldots\Big)dt_1\Big] \end{aligned}$$

for $t\in J$, where $p\ge 0$, $p\ne 1$ be a constant, $a(t)>0$ is a nondecreasing continuous function in $t\in J$, and $k_i(t,t_1,\ldots,t_i)$ are nonnegative continuous functions in J_{i+1}, $i=1,2,\ldots,n$, which are nondecreasing in $t\in J$ for all fixed $(t_1,\ldots,t_i)\in J_i$, $i=1,2,\ldots,n$. Then

$$u(t) \le b(t)\Big[a^q(t) + q\int_\alpha^t R[b^p](t,s)\,ds\Big]^{1/q} \tag{2.6}$$

for any $t\in[\alpha,\beta_1)$, where $q=1-p$ and β_1 is chosen so that the expression between [...] is positive in the subinterval $[\alpha,\beta_1)$.

Proof. For a fixed $T\in(\alpha,\beta]$ and $\alpha\le t\le T$ we have

$$\begin{aligned} u(t) &\le b(t)v(t) \\ &\equiv b(t)\Big[a(T) + \int_\alpha^t k_1(T,t_1)u^p(t_1)\,dt_1 + \cdots \\ &\quad + \int_\alpha^t \Big(\int_\alpha^{t_1} \cdots \Big(\int_\alpha^{t_{n-1}} k_n(T,t_1,\ldots,t_n)u^p(t_n)\,dt_n\Big)\ldots\Big)dt_1\Big]. \end{aligned}$$

Since $v(\alpha)=a(T)$, $v(t)$ is nondecreasing and continuous in J and $\frac{\partial k_i}{\partial t}(T,t_1,\ldots,t_i)\equiv 0$ for $i=1,\ldots,n$ and $t\in[\alpha,T]$, we have

$$\begin{aligned} v'(t) &= R[u^p](T,t) \le R[b^p v^p](T,t) \\ &\le (R[b^p](t))v^p(T,t). \end{aligned} \tag{2.7}$$

Lemma 2.2 and (2.7) imply

$$v(t) \le \left[a^q(T) + q\int_\alpha^t R[b^p](T,s)\,ds,\right]^{1/q}$$

from which we obtain

$$u(t) \le b(t)\left[a^q(T) + q\int_\alpha^t R[b^p](T,s)\,ds\right]^{1/q}$$

for $\alpha \le t \le T$. In particular, for $T = t$ we find (2.6). This completes the proof. □

Theorem 2.3. *Let $u(t)$, $a(t)$ and $b(t)$ be nonnegative continuous functions in $J = [\alpha,\beta]$. Suppose that*

$$\begin{aligned} u(t) \le a(t) + b(t)\bigg[&\int_\alpha^t k_1(t,t_1)u^p(t_1)\,dt_1 + \cdots \\ &+ \int_\alpha^t\bigg(\int_\alpha^{t_1}\cdots\bigg(\int_\alpha^{t_{n-1}} k_n(t,t_1,\dots,t_n)u^p(t_n)\,dt_n\bigg)\dots\bigg)dt_1\bigg]\end{aligned}$$

for all $t \in J$, where $0 < p \le 1$ be a constant, $\frac{a(t)}{b(t)} \ge 1$ is nondecreasing in J and $k_i(t,t_1,\dots,t_i)$ are nonnegative continuous functions in $J_{i+1}, i = 1,2,\dots,n$, which are nondecreasing in $t \in J$ for all fixed $(t_1,\dots,t_i) \in J_i$, $i = 1,2,\dots,n$. Then

$$u(t) \le a(t)\exp\bigg(\int_\alpha^t R[b^p](t,\tau)\,d\tau\bigg) \tag{2.8}$$

for all $t \in [\alpha,\beta]$.

Proof. For a fixed $T \in (\alpha,\beta]$ and $\alpha \le t \le T$, we have

$$\begin{aligned} u(t) &\le a(t) + b(t)w(t) \\ &\equiv a(t) + b(t)\bigg[\int_\alpha^t k_1(T,t_1)u^p(t_1)\,dt_1 + \cdots \\ &\quad + \int_\alpha^t\bigg(\int_\alpha^{t_1}\cdots\bigg(\int_\alpha^{t_{n-1}} k_n(T,t_1,\dots,t_n)u^p(t_n)\,dt_n\bigg)\dots\bigg)dt_1\bigg].\end{aligned}$$

Since $w(\alpha) = 0$, $w(t)$ is nondecreasing and continuous in J and $\frac{\partial k_i}{\partial t}(T,t_1,\dots,t_i) \equiv 0$ for $i = 1,\dots,n$ and $t \in [\alpha,T]$, we have

$$\begin{aligned} w'(t) &= R[u^p](T,t) \le R[b^p](T,t)\bigg(\frac{a(t)}{b(t)} + w(t)\bigg)^p \\ &\le R[b^p](T,t)\bigg(\frac{a(t)}{b(t)} + w(t)\bigg). \end{aligned} \tag{2.9}$$

Lemma 2.1 and (2.9) imply

$$w(t) \le \int_\alpha^t R[b^p](T,s)\bigg(\frac{a(s)}{b(s)}\bigg)\exp\bigg(\int_s^t R[b^p](T,\tau)\,d\tau\bigg)\,ds,$$

from which we obtain

$$u(t) \le a(t) + b(t) \int_\alpha^t R[b^p](T,s)\left(\frac{a(s)}{b(s)}\right) \exp\left(\int_s^t R[b^p](T,\tau)\,d\tau\right) ds. \tag{2.10}$$

Indeed, (2.10) implies that

$$\begin{aligned} u(t) &\le a(t)\left[1 + \int_\alpha^t R[b^p](T,s) \exp\left(\int_s^t R[b^p](T,\tau)\,d\tau\right) ds\right] \\ &= a(t) \exp\left(\int_\alpha^t R[b^p](T,\tau)\,d\tau\right) \end{aligned}$$

for $\alpha \le t \le T$. In particular, for $T = t$ we find (2.8). This completes the proof. □

Theorem 2.4. *Let $u, f_1, \dots, f_n$ be nonnegative continuous functions in $J = [\alpha, \beta]$, and suppose that*

$$\begin{aligned} u(t) \le a &+ \int_\alpha^t f_1(t_1) u^p(t_1)\,dt_1 + \cdots \\ &+ \int_\alpha^t f_1(t_1)\left(\int_\alpha^{t_1} f_2(t_2) \cdots \left(\int_\alpha^{t_{n-1}} f_n(t_n) u^p(t_n)\,dt_n\right) \cdots\right) dt_1 \end{aligned} \tag{2.11}$$

for all $t \in J$, where $a \ge 1$ and $0 < p \le 1$ are constants. Then

$$u(t) \le a R_1(t), \quad t \in J, \tag{2.12}$$

where

$$R_n(t) = \exp\left(\int_\alpha^t f_n(s)\,ds\right), \quad t \in J,$$

and

$$R_n(t) = 1 + \int_\alpha^t f_i(t) R_{i+1}(s) \exp\left(\int_\alpha^s f_i(\tau)\,d\tau\right) ds$$

for all $t \in J$ and $i = n-1, \cdots, 1$.

Proof. We set

$$u_1(t) = a + L_1[u^p](t), \quad u_{j+1}(t) = u_j + L_{j+1}[u^p](t)$$

for all $t \in J$ and $j = 1, \cdots, n-1$, where

$$\begin{aligned} L_k[u^p](t) = &\int_\alpha^t f_k(t_k) u^p(t_k)\,dt_k + \cdots \\ &+ \int_\alpha^t f_k(t_k)\left(\int_\alpha^t f_{k+1}(t_{k+1}) \cdots \left(\int_\alpha^t f_n(t_n)\,dt_n\right) \cdots\right) dt_k \end{aligned}$$

for all $t \in J$ and $k = 1, \cdots, n$. Now, (2.11) implies

$$u(t) \le u_1(t). \tag{2.13}$$

Taking into account that

$$
\begin{aligned}
u_k(t) &\le u_{k+1}(t),\\
(L_k[u^p])' &= f_k(u^p(t) + L_{k+1}[u^p]), \quad k = 1, \cdots, n-1,
\end{aligned}
$$

and

$$
(L_n[u^p])' = f_n(t)u^p(t).
$$

We successively find

$$
\begin{aligned}
u_1'(t) &= (L_1[u^p](t))' = f_1[u^p(t) + L_2[u^p]]\\
&\le f_1[u_1(t) + L_2[u^p]] = f_1 u_2,\\
u_k'(t) &\le (f_1 + \cdots + f_{k-1})u_k(t) + f_k u_k(t), \quad k = 2, \ldots, n-1, \qquad (2.14)\\
u_n'(t) &\le (f_1 + \cdots + f_n)u_n(t).
\end{aligned}
$$

Since $u_k(\alpha) = a$, $k = 1, \ldots, n$, (2.14) gives, by successive application of *Lemma 2.1*,

$$
u_k(t) \le aR_k(t)\exp\biggl(\int_\alpha^t \sum_{j=1}^{k-1} f_j(s)\,ds\biggr), \quad k = n, n-1, \ldots, 1.
$$

For $k = 1$ this and (2.13) imply (2.12). □

Remark 2.1. *In the case when $a \ge 0, p = 1$, the inequality given in (2.11) reduces to the inequality established earlier by Ráb in [16] (see also [1, Theorem 11.6, p. 102]).*

Corollary 2.1. *Let u, f, g be nonnegative continuous functions in $J = [\alpha, \beta]$, $u_0 \ge 1$ and suppose that*

$$
u(t) \le u_0 + \int_\alpha^t f(s)\biggl[u^p(s) + \int_\alpha^s g(\tau)u^p(\tau)\,d\tau\biggr]ds
$$

for all $t \in J$, where $0 < p \le 1$ is a constant. Then

$$
u(t) \le u_0\biggl[1 + \int_\alpha^t f(s)\exp\biggl(\int_\alpha^s (f(\tau) + g(\tau))\,d\tau\biggr)ds\biggr]
$$

for all $t \in J$.

Theorem 2.5. *Let u, f_i, $i = 1, \cdots, n$ be nonnegative continuous functions in $J = [\alpha, \beta]$, and suppose that*

$$
\begin{aligned}
u(t) \le\ & a(t) + \int_\alpha^t f_1(t_1)u^p(t_1)\,dt_1 + \cdots\\
&+ \int_\alpha^t f_1(t_1)\biggl(\int_\alpha^{t_1} f_2(t_2)\cdots\biggl(\int_\alpha^{t_{n-1}} f_n(t_n)u^p(t_n)\,dt_n\biggr)\cdots\biggr)dt_1 \qquad (2.15)
\end{aligned}
$$

for all $t \in J$, where $a(t) \ge 1$ is a continuous function in J and $0 < p \le 1$ is a constant. Then

$$
u(t) \le a(t) + \int_\alpha^t f_1(s)[a(s) + v_2(s)]\,ds, \qquad (2.16)
$$

where

$$u_n(t) = \int_\alpha^t (f_1(s) + \cdots + f_n(s))a(s) \exp\left(\int_s^t (f_1(\tau) + \cdots + f_n(\tau))\, d\tau\right) ds$$

and

$$u_k(t) = \int_\alpha^t \left[\left(\sum_{j=1}^{k} f_j(s)\right) a(s) + f_k(s) v_{k+1}(s)\right] \exp\left(\int_s^t \sum_{j=1}^{k-1} f_j(\tau)\, d\tau\right) ds$$

for all $t \in J$ *and* $k = n-1, \dots, 2$.

Proof. Let $L_k[u^p](t)$ be defined as in *Theorem 2.4* and put

$$v_1(t) = L_1[u^p](t), \quad v_{k+1}(t) = v_k + L_{k+1}[u^p](t)$$

for all $t \in J$ and $k = 1, \cdots, n-1$. Then (2.15) implies

$$u(t) \le a(t) + v_1(t) \tag{2.17}$$

and we successively find

$$\begin{aligned} v_1'(t) &= (L_1[u^p](t))' = f_1[u^p(t) + L_2[u^p]] \\ &\le f_1[a(t) + v_1(t) + L_2[u^p]] = f_1[a(t) + v_2(t)], \\ v_k'(t) &\le (f_1 + \cdots + f_{k-1}) v_k(t) + (f_1 + \cdots + f_k) a + f_k v_{k+1}(t), \\ &\quad k = 2, \cdots, n-1, \\ v_n'(t) &\le (f_1 + \cdots + f_n) v_n(t) + (f_1 + \cdots + f_n) a(t). \end{aligned} \tag{2.18}$$

Since $v_k(\alpha) = 0$, $k = 1, \dots, n$, solving the system (2.18) 'backward' and applying *Lemma 2.1*, we arrive at

$$v_1(t) \le \int_\alpha^t f_1(s)[a(s) + v_2(s)]\, ds, \tag{2.19}$$

where

$$v_k(t) = \int_\alpha^t \left[\left(\sum_{j=1}^{k} f_j(s)\right) a(s) + f_k(s) v_{k+1}(s)\right] \exp\left(\int_s^t \sum_{j=1}^{k-1} f_j(\tau)\, d\tau\right) ds,$$

for all $t \in J$, $k = n-1, \dots, 2$, and

$$u_n(t) = \int_\alpha^t (f_1(s) + \cdots + f_n(s))a(s) \exp\left(\int_s^t (f_1(\tau) + \cdots + f_n(\tau))\, d\tau\right) ds.$$

The inequalities (2.17) and (2.19) imply (2.16). This completes the proof. □

Remark 2.2. *In the case when* $a(t) \ge 0, p = 1$, *the inequality given in (2.15) reduces to the inequality established earlier by Young in [18] (see also [1, Theorem 11.7, p. 103]).*

Corollary 2.2. *Let u, f, g, h be nonnegative continuous functions in $J = [\alpha, \beta]$. Suppose that*

$$u(t) \leq u_0 + \int_\alpha^t (f(s)u^p(s) + h(s))\, ds + \int_\alpha^t f(s)\left(\int_\alpha^s g(\tau)u^p(\tau)\, d\tau\right) ds$$

for all $t \in J$, where $u_0 + \int_\alpha^t h(s)\, ds \geq 1$ is a continuous function in J and $0 < p \leq 1$ is a constant. Then

$$\begin{aligned} u(t) \leq u_0 &+ \int_\alpha^t h(s)\, ds \\ &+ \int_\alpha^t f(s)\left[u_0 + \int_\alpha^s h(\tau)\, d\tau\right] \exp\left(\int_\alpha^s (f(\tau) + g(\tau)\, \tau\right) ds. \end{aligned}$$

Proof. Indeed, the proof follows from *Theorem 2.5* with $f_1 = f, f_2 = g$ and $a(t) = u_0 + \int_\alpha^t h(s)\, ds$. □

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References

[1] D. BAINOV, P. SIMEONOV, *Integral Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, 1992.

[2] E. F. BECKENBACH, R. BELLMAN, *Inequalities*, Springer-Verlag, New York, 1961.

[3] R. BELLMAN, The stability of solutions of linear differential equations, Duke Math. J. **10**(1943), 643–647.

[4] I. BIHARI, *A generalization of a lemma of Bellman and its application to uniqueness problems of differential equations*, Acta Math. Acad. Sci. Hungar. **7**(1956), 71–94.

[5] YA. V. BYKOV, KH. M. SALPAGAROV, *On the theory of integro-differential equations* (in Russian), in: *Investigations in Integro-Differential Equations in Khirghizia*, 2 Izd., Akad. Nauk Khirghizia SSR, 1962.

[6] S. S. DRAGOMIR, N. M. IONESCU, *On nonlinear integral inequalities in two independent variables*, Studia Univ. Babeş-Bolyai, Math. **34**(1989), 11–17.

[7] S. S. DRAGOMIR, Y. H. KIM, *On certain new integral inequalities and their applications*, J. Inequal. Pure and Appl. Math. **3**(2002), 1–8.

[8] S. S. DRAGOMIR, Y. H. KIM, *Some integral inequalities for function of two variables*, Electron. J. Differ. Equat. **10**(2003), 1–13.

[9] T. H. Gronwall, *Note on the derivatives with respect to a parameter of solutions of a system of differential equations*, Ann. Math. **20**(1919), 292–296.

[10] L. Guiliano, *Generalazzioni di un lemma di Gronwall*, Rend. Accad., Lincei, 1946.

[11] C. E. Langenhop, *Bounds on the norm of a solution of a general differential equation*, Proc. Amer. Math. Soc. **11**(1960), 795–799.

[12] A. Mate, P. Neval, *Sublinear perturbations of the differential equation* $y^{(n)} = 0$ *and of the analogous difference equation*, J. Differential Equations **52**(1984), 234–257.

[13] D. S. Mitrinović, J. E. Pečarić, A. M. Fink, *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1991

[14] V. V. Nemyckii V. V. Stepanov, *Qualitative Theory of Differential Equations* (in Russian) OGIZ, Moscow, 1947.

[15] B. G. Pachpatte, *On some fundamental integral inequalities and their discrete analogues*, J. Inequal. Pure Appl. Math. **2**(2001), 1–13.

[16] M. Ráb, *Linear integral inequalities*, Arch. Math. 1. Scripta Fac. Sci. Nat. Ujep Brunensis **XV**(1979), 37–46.

[17] Ju. A. Vedķrime, *On perturbations of linear homogeneous differential equations with variable coefficients* (in Russian), in: Issled. Integro-Differents. Uravn. Kirghizia **3** Ilim. Frunze, 1965.

[18] E. C. Young, *On integral inequalities of Gronwall-Bellman type*, Proc. Amer. Math. Soc. **94**(1985), 636–640.