Strong convergence of an explicit iterative process with mean errors for a finite family of Ćirić quasi-contractive operators in normed spaces*

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Abstract. *The purpose of this paper is to establish a strong convergence of an explicit iteration scheme with mean errors to a common fixed point for a finite family of Ćirić quasi-contractive operators in normed spaces. The results presented in this paper generalize and improve the corresponding results of V. Berinde [1], A. Rafiq [9], B. E. Rhoades [10] and T. Zamfirescu [12].*

Key words: *Ćirić quasi-contractive operators, explicit iteration process with mean errors, common fixed points*

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1. Introduction and preliminaries

Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be *a-contraction*, if

$$d(Tx, Ty) \le ad(x, y) \ \ \forall\, x, y \in X, \tag{1.1}$$

where $a \in (0, 1)$.

A mapping $T : X \to X$ is said to be a *Kannan mapping* [7], if there exists $b \in (0, 1/2)$ such that

$$d(Tx, Ty) \le b[d(x, Tx) + d(y, Ty)] \ \ \forall\, x, y \in X. \tag{1.2}$$

A mapping $T : X \to X$ is said to be *Chatterjea mapping* [3], if there exists $c \in (0, 1/2)$ such that

$$d(Tx, Ty) \le c[d(x, Ty) + d(y, Tx)] \ \ \forall\, x, y \in X. \tag{1.3}$$

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Combining these three definitions, Zamfirescu [12] proved the following important result.

Theorem Z([12]). *Let (X,d) be a complete metric space and $T : X \to X$ a mapping for which there exist real numbers a, b and c satisfying $a \in (0,1)$, $b, c \in (0, 1/2)$ such that for each pair $x, y \in X$, at least one of the following conditions holds:*

(z_1) $d(Tx,Ty) \leq ad(x,y)$,

(z_2) $d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)]$,

(z_3) $d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)]$.

Then T has a unique fixed point p and the Picard iteration $\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}, \tag{1.4}$$

converges to p for any arbitrary but fixed $x_1 \in X$.

Remark 1.1. *An operator T satisfying the contractive conditions $(z_1) - (z_3)$ in the above theorem is called a Z-operator.*

Remark 1.2. *The conditions $(z_1) - (z_3)$ can be written in the following equivalent form*

$$d(Tx,Ty) \leq h \max\left\{d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}, \frac{d(x,Ty) + d(y,Tx)}{2}\right\},$$

$\forall\ x, y \in X, 0 < h < 1$. Thus, a class of mappings satisfying the contractive conditions $(z_1) - (z_3)$ is a subclass of mappings satisfying the following condition

$$d(Tx,Ty) \leq h \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty) + d(y,Tx)}{2}\right\}, \tag{CG}$$

$0 < h < 1$. The class of mappings satisfying (CG) was introduced and investigated by Ćirić [5] in 1971.

Remark 1.3. *A mapping satisfying (CG) is commonly called a Ćirić generalized contraction.*

In 2000, Berinde [1] introduced a new class of operators on a normed space E satisfying

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|Tx - x\|, \tag{1.5}$$

for any $x, y \in E$, $0 \leq \delta < 1$ and $L \geq 0$.

It may be noted that (1.5) is equivalent to

$$\|Tx - Ty\| \leq \delta\|x - y\| + L \min\{\|Tx - x\|, \|Ty - y\|\}, \tag{1.6}$$

for any $x, y \in E$, $0 \leq \delta < 1$ and $L \geq 0$.

Berinde [1] proved that this class is wider than the class of Zamfiresu operators and used the Mann [8] iteration process to approximate fixed points of this class of operators in a normed space given in the form of following theorem:

Theorem B([1]). *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (1.5) and $F(T) \neq \emptyset$. For given $x_0 \in C$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n,\ n \geq 0, \tag{1.7}$$

where $\{\alpha_n\}$ is a real sequence in [0, 1]. If $\sum_{n=1}^{\infty} \alpha_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

Recently, Rafiq [9] considered a class of mappings satisfying the following condition

$$\|Tx - Ty\| \leq h \max\left\{\|x - y\|, \frac{\|x - Tx\| + \|y - Ty\|}{2}, \|x - Ty\|, \|y - Tx\|\right\}, \quad (CR)$$

$0 < h < 1$. This class of mappings is a subclass of mappings satisfying the following condition

$$\|Tx - Ty\| \leq h \max\left\{\|x - y\|, \|x - Tx\|, \|y - Ty\|, \|x - Ty\|, \|y - Tx\|\right\}, \quad (CQ)$$

$0 < h < 1$. The class of mappings satisfying (CQ) was introduced and investigated by Ćirić [6] in 1974 and a mapping satisfying is commonly called Ćirić quasi contraction.

Rafiq [9] proved the following result:

Theorem R([9]). *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying the condition (CR). For given $x_0 \in C$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n,\ n \geq 0, \tag{1.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in [0, 1] satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, $\{u_n\}$ is a bounded sequences in C. If $\sum_{n=1}^{\infty} \beta_n = \infty$ and $\gamma_n = o(\alpha_n)$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.

Let C be a nonempty closed convex subset of a normed space E.

In [11], Xu and Ori introduced the following implicit iteration process for a finite family of nonexpansive mappings $\{T_i\}_{i \in I}$ (here $I = \{1, 2, \cdots, N\}$), with $\{\alpha_n\}$ a real sequence in (0, 1), and an initial point $x_0 \in C$:

$$x_n = (1 - \alpha_n)x_{n-1} + \alpha_n T_n x_n, \quad \forall n \geq 1, \tag{1.9}$$

where $T_n = T_{n(modN)}$ (here the mod$N$ function takes values in I). Xu and Ori proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that it is yet unclear what assumptions on the mappings and/or the parameters $\{\alpha_n\}$ are sufficient to guarantee the strong convergence of the sequence $\{x_n\}$.

Zhou-Chang [13] and Chidume-Shahzad [4] studied the weak and strong convergences of this implicit process to a common fixed point for a finite family of nonexpansive mappings, respectively.

Inspired and motivated by the above said facts, we introduced an explicit iteration process with mean errors as follows:

$$x_{n+1} = \alpha_n x_n + \beta_n T_n x_n + \gamma_n u_n,\ n \geq 1, \tag{1.10}$$

where $T_n = T_{n(modN)}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in [0, 1] satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, $\{u_n\}$ is a bounded sequences in C and x_0 is a given point.

The purpose of this paper is to study the convergence of an explicit iterative sequence $\{x_n\}$ defined by (1.10) to a common fixed point for a finite family of Ćirić quasi-contractive operators in normed spaces. The results presented in this paper generalized and extend the corresponding results of Berinde [1], Rafiq [9], Rhoades [10] and Zamfirescu [12].

In order to prove the main results of this paper, we need the following Lemma:

Lemma 1.1([2]). *Suppose that $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are three nonnegative real sequences satisfying the following condition:*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n,\ \ \forall n \geq n_0,$$

where n_0 is some nonnegative integer, $t_n \in [0, 1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main results

We are now in a position to prove our main results in this paper.

Theorem 2.1. *Let C be a nonempty closed convex subset of a normed space E. Let $\{T_i\}_{i=1}^{N} : C \to C$ be N operators satisfying the condition (CR) with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_i\}_{i=1}^{N}$). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in [0, 1] satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, $\{u_n\}$ is a bounded sequences in C satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ or $\gamma_n = o(\beta_n)$.

Suppose further that $x_0 \in C$ is any given point and $\{x_n\}$ is an explicit iteration sequence defined by (1.10), then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{N}$.

Proof. Since $\{T_i\}_{i=1}^{N} : C \to C$ is an N Ćirić operator satisfying the condition (CR), hence there exists $0 < h_i < 1$ ($i \in I = \{1, 2, \cdots, N\}$) such that

$$\|T_i x - T_i y\| \leq h_i \max\left\{\|x - y\|, \frac{\|x - T_i x\| + \|y - T_i y\|}{2}, \|x - T_i y\|, \|y - T_i x\|\right\}. \tag{2.1}$$

For each fixed $i \in I = \{1, 2, \cdots, N\}$. Denote $h = \max\{h_1, h_2, \cdots, h_N\}$, then $0 < h < 1$ and

$$\|T_i x - T_i y\| \leq h \max\left\{\|x - y\|, \frac{\|x - T_i x\| + \|y - T_i y\|}{2}, \|x - T_i y\|, \|y - T_i x\|\right\} \tag{2.2}$$

hold for each fixed $i \in I = \{1, 2, \cdots, N\}$. If from (2.2) we have

$$\|T_i x - T_i y\| \le \frac{h}{2}[\|x - T_i x\| + \|y - T_i y\|],$$

then

$$\begin{aligned}\|T_i x - T_i y\| &\le \frac{h}{2}[\|x - T_i x\| + \|y - T_i y\|] \\ &\le \frac{h}{2}[\|x - T_i x\| + \|y - x\| + \|x - T_i x\| + \|T_i x - T_i y\|].\end{aligned}$$

Hence

$$(1 - \frac{h}{2})\|T_i x - T_i y\| \le \frac{h}{2}\|x - y\| + h\|x - T_i x\|,$$

which yields (using the fact that $0 < h < 1$)

$$\|T_i x - T_i y\| \le \frac{\frac{h}{2}}{1 - \frac{h}{2}}\|x - y\| + \frac{h}{1 - \frac{h}{2}}\|x - T_i x\|. \tag{2.3}$$

Also, from (2.2), if

$$\|T_i x - T_i y\| \le h \max\{\|x - T_i y\|, \|y - T_i x\|\} \tag{2.4}$$

holds, then

(a) $\|T_i x - T_i y\| \le h \|x - T_i y\|$, which implies $\|T_i x - T_i y\| \le h \|x - T_i x\| + h \|T_i x - T_i y\|$ and hence, as $h < 1$,

$$\|T_i x - T_i y\| \le \frac{h}{1 - h} \|x - T_i x\|, \tag{2.5}$$

or

(b) $\|T_i x - T_i y\| \le h \|y - T_i x\|$, which implies

$$\|T_i x - T_i y\| \le h \|y - x\| + h \|x - T_i x\|. \tag{2.6}$$

Thus, if (2.4) holds, then from (2.5) and (2.6) we have

$$\|T_i x - T_i y\| \le h \|y - x\| + \frac{h}{1 - h} \|x - T_i x\|, \tag{2.7}$$

Denote

$$\begin{aligned}\delta &= \max\left\{h, \frac{\frac{h}{2}}{1 - \frac{h}{2}}\right\} = h, \\ L &= \max\left\{h, \frac{h}{1 - \frac{h}{2}}, \frac{h}{1 - h}\right\} = \frac{h}{1 - h}\end{aligned}$$

Then we have $0 < \delta < 1$ and $L \ge 0$. In view of (2.2), (2.3) and (2.7) it results that the inequality

$$\|T_i x - T_i y\| \le \delta\|x - y\| + L\|x - T_i x\|. \tag{2.8}$$

holds for all $x, y \in C$ and $i \in I$.

Let $p \in F = \cap_{i=1}^{N} F(T_i)$, using (1.10) we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + \beta_n \|T_n x_n - p\| + \gamma_n \|u_n - p\| \\ &\leq \alpha_n \|x_n - p\| + \beta_n \|T_n x_n - p\| + \gamma_n M, \qquad (2.9)\end{aligned}$$

where $M = \sup_{n \geq 1}\{||u_n - p||\}$. Now for $y = x_n$ and $x = p$, (2.8) gives

$$\|T_n x_n - p\| = \|T_n x_n - T_n p\| \leq \delta \|x_n - p\|. \qquad (2.10)$$

Substituting (2.10) into (2.9), we obtain that

$$\begin{aligned}||x_{n+1} - p|| &\leq (\alpha_n + \beta_n \delta)||x_n - p|| + \gamma_n M \\ &= (1 - \beta_n - \gamma_n + \beta_n \delta)||x_n - p|| + \gamma_n M \\ &\leq [1 - \beta_n(1 - \delta)]||x_n - p|| + \gamma_n M \qquad (2.11)\end{aligned}$$

From the conditions (i)-(ii), using (2.11) and *Lemma 1.1* we have $\lim_{n\to\infty} \|x_n - p\| = 0$, and so $\lim_{n\to\infty} x_n = p$. This completes the proof of *Theorem 2.1.* □

Corollary 2.1([9]). *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying the condition (CR). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in [0, 1] satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, $\{u_n\}$ is a bounded sequence in C satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ or $\gamma_n = o(\beta_n)$.

Suppose further that $x_0 \in C$ is any given point and $\{x_n\}$ is an explicit iteration sequence as follows:

$$x_{n+1} = \alpha_n x_n + \beta_n T\ x_n + \gamma_n u_n,\ n \geq 1, \qquad (2.12)$$

then $\{x_n\}$ converges strongly to the unique fixed point of T.

Proof. By Ćirić [6], we know that T has a unique fixed point in C. Taking $N = 1$ in *Theorem 2.1*, the conclusion of *Corollary 2.1* can be obtained from *Theorem 2.1* immediately. This completes the proof of *Corollary 2.1.* □

Theorem 2.2. *Let C be a nonempty closed convex subset of a normed space E. Let $\{T_i\}_{i=1}^{N} : C \to C$ be N operators satisfying the condition (CR) with $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_i\}_{i=1}^{N}$). Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two real sequences in [0, 1] with $\alpha_n + \beta_n = 1$ for all $n \geq 1$ satisfying the following condition:*

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$.

Suppose further that $x_0 \in C$ is any given point and $\{x_n\}$ is an explicit iteration sequence as follows:

$$x_{n+1} = \alpha_n x_n + \beta_n T_n x_n,\ n \geq 1, \qquad (2.13)$$

then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^{N}$.

Proof. Taking $\gamma_n = 0$, $\forall n \geq 1$ in *Theorem 2.1*, the conclusion of *Theorem 2.2* can be obtained from *Theorem 2.1* immediately. This completes the proof of *Theorem 2.2.* □

Corollary 2.2. *Let C be a nonempty closed convex subset of a normed space E. Let $\{T_i\}_{i=1}^N : C \to C$ be N operators satisfying the condition (2.8) with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ (the set of common fixed points of $\{T_i\}_{i=1}^N$). Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in [0, 1] satisfying $\alpha_n + \beta_n + \gamma_n = 1$ for all $n \geq 1$, $\{u_n\}$ is a bounded sequences in C satisfying the following conditions:*

(i) $\sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$ or $\gamma_n = o(\beta_n)$.

Suppose further that $x_0 \in C$ is any given point and $\{x_n\}$ is explicit iteration sequence defined by (1.10), then $\{x_n\}$ converges strongly to a common fixed point of $\{T_i\}_{i=1}^N$.

Remark 2.1. Theorem 2.2 *and* Corollary 2.2 *improve and extend the corresponding results of Berinde [1], Rafiq [9], Rhoades [10] and Zamfirescu [12].*

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