Statistical convergence on probalistic normed spaces

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Abstract. *In this paper we define concepts of statistical convergence and statistical Cauchy on probabilistic normed spaces. Then we give a useful characterization for statistically convergent sequences. Furthermore, we display an example such that our method of convergence is stronger than the usual convergence on probabilistic normed spaces. We also introduce statistical limit points, statistical cluster points on probabilistic normed spaces and then we give the relations between these and limit points of sequence on probabilistic normed spaces.*

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1. Introduction

An interesting and important generalization of the notion of a metric space was introduced by Menger [10] under the name of statistical metric, which is now called a probabilistic metric space. The theory of a probabilistic metric space was developed by numerous authors, as it can be realized upon consulting the list of references in [5], as well as those in [13] and [14]. An important family of probabilistic metric spaces are probabilistic normed spaces. The theory of probabilistic normed spaces is important as a generalization of deterministic results of linear normed spaces. It seems therefore reasonable to think if the concept of statistical convergence can be extended to probabilistic normed spaces and in that case enquire how the basic properties are affected. But basic properties do not hold on probabilistic normed spaces. The problem is that the triangle function in such spaces. In this paper we extend the concept of statistical convergence to probabilistic normed spaces and observe that some basic properties are also preserved on probabilistic normed spaces. We also display an example such that our method of convergence is stronger than the usual convergence on probabilistic normed spaces.

Now we recall some notations and definitions used in the paper.

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Definition 1. *A function* $f:\mathbb{R}\to\mathbb{R}_0^+$ *is called a distribution function if it is non-decreasing and left-continuous with* $\inf_{t\in\mathbb{R}} f(t)=0$*, and* $\sup_{t\in\mathbb{R}} f(t)=1$*.*

We will denote the set of all distribution functions by D*.*

Definition 2. *A triangular norm, a briefly t-norm, is a binary operation on* $[0,1]$ *which is continuous, commutative, associative, non-decreasing and has 1 as a neutral element, i.e., it is the continuous mapping* $*:[0,1]\times[0,1]\to[0,1]$ *such that for all* $a,b,c\in[0,1]$*:*

(1) $a*1=a$,

(2) $a*b=b*a$,

(3) $c*d\geq a*b$ *if* $c\geq a$ *and* $d\geq b$,

(4) $(a*b)*c=a*(b*c)$.

Example 1. *The* $*$ *operations* $a*b=\max\{a+b-1,0\}$, $a*b=ab$, *and* $a*b=\min\{a,b\}$ *on* $[0,1]$ *are* $t-$*norms.*

Definition 3. *A triplet* $(X,N,*)$ *is called a probabilistic normed space (briefly, a* $PN-$*space) if* X *is a real vector space,* N *a mapping from* X *into* D *(for* $x\in X$*, the distribution function* $N(x)$ *is denoted by* N_x*, and* $N_x(t)$ *is the value* N_x *at* $t\in\mathbb{R}$ *) and* $*$ *a* $t-$*norm satisfying the following conditions:*

(PN-1) $N_x(0)=0$,

(PN-2) $N_x(t)=1$ for all $t>0$ iff $x=0$,

(PN-3) $N_{\alpha x}(t)=N_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha\in\mathbb{R}/\{0\}$,

(PN-4) $N_{x+y}(s+t)\geq N_x(s)*N_y(t)$ for all $x,y\in X$, and $s,t\in\mathbb{R}_0^+$.

Example 2. *Suppose that* $(X,||.||)$ *is a normed space* $\mu\in D$ *with* $\mu(0)=0$*, and* $\mu\neq h$*, where*

$$h(t)=\begin{cases}0, & t\leq 0,\\ 1, & t>0.\end{cases}$$

Define

$$N_x(t)=\begin{cases}h(t), & x=0,\\ \mu\left(\frac{t}{||x||}\right), & x\neq 0,\end{cases}$$

where $x\in X$, $t\in\mathbb{R}$. Then $(X,N,*)$ is a $PN-$space. For example, if we define functions μ and $\mu^{'}$ on $\mathbb{R}$ by

$$\mu(x)=\begin{cases}0, & x\leq 0,\\ \frac{x}{1+x}, & x>0,\end{cases}\qquad\text{and}\qquad \mu^{'}(x)=\begin{cases}0, & x\leq 0,\\ \exp\left(\frac{-1}{x}\right), & x>0,\end{cases}$$

then we obtain the following well-known $*-$norms

$$N_x(t)=\begin{cases}h(t), & x=0,\\ \frac{t}{t+||x||}, & x\neq 0,\end{cases}\qquad\text{and}\qquad N_x^{'}(t)=\begin{cases}h(t), & x=0,\\ \exp\left(\frac{-||x||}{t}\right), & x\neq 0.\end{cases}$$

We recall that the concept of convergence and Cauchy sequence in a probabilistic normed space are studied in [1].

Definition 4. *Let $(X, N, *)$ be a PN-space. Then, a sequence $x = (x_n)$ is said to be convergent to $L \in X$ with respect to the probabilistic norm N if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists a positive integer k_0 such that $N_{x_n - L}(\varepsilon) > 1 - \lambda$ whenever $n \geq k_0$. It is denoted by $N - \lim x = L$ or $x_n \overset{N}{\to} L$ as $n \to \infty$.*

Remark 1 [[1]]. *Let $(X, ||.||)$ be a real normed space, and $N_x(t) = \frac{t}{t+||x||}$, where $x \in X$ and $t \geq 0$ (standard $*-$norm induced by $||.||$). Then it is not hard to see that $x_n \overset{||\cdot||}{\longrightarrow} x$ if and only if $x_n \overset{N}{\to} x$.*

Definition 5. *Let $(X, N, *)$ be a PN- space . Then, a sequence $x = (x_n)$ is called a Cauchy sequence with respect to the probabilistic norm N if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists a positive integer k_0 such that $N_{x_n - x_m}(\varepsilon) > 1 - \lambda$ for all $n, m \geq k_0$.*

2. Statistical convergence on PN-spaces

In this paper we deal with the statistical convergence on probabilistic normed spaces. Before proceeding further, we should recall some notation on the statistical convergence. If K is a subset of $\mathbb{N}$, the set of natural numbers, then the natural density of K denoted by $\delta(K)$, is given by

$$\delta(K) := \lim_n \frac{1}{n} |\{k \leq n : k \in K\}|$$

whenever the limit exists, where $|A|$ denotes the cardinality of the set A. The natural density may not exist for each set K. But the upper density $\overline{\delta}$ always exists for each set K identified as follows:

$$\overline{\delta}(K) := \limsup_n \frac{1}{n} |\{k \leq n : k \in K\}| .$$

Moreover, the natural density of K is different from zero which means $\overline{\delta}(K) > 0$. A sequence $x = (x_k)$ of numbers is statistically convergent to L if

$$\delta(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$. In this case we write $st - \lim x = L$.

Note that every convergent sequence is statistically convergent to the same value. If x is statistically convergent, then x need not be convergent. It is also not necessarily bounded. For example, $x = (x_k)$ be defined as

$$x_k := \begin{cases} k, & \text{if } k \text{ is a square,} \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to see that $st - \lim x = 1$, since the cardinality of the set

$$|\{k \leq n : |x_k - 1| \geq \varepsilon\}| \leq \sqrt{n} \qquad \text{for every } \varepsilon > 0.$$

But x is neither convergent nor bounded.

The idea of the statistical convergence was first introduced by Steinhaus (1951) [12] but rapid developments started after the papers of Connor [2] and Fridy [7]. Statistical convergence and its some generalizations have appeared in the study of locally convex spaces [9]. It is also connected with the subsets of the Stone-Čech compactification of the set of natural numbers [3]. Some results on characterizing Banach spaces with separable duals via statistical convergence may be found in [4]. This notion of convergence is also considered in the measure theory [11], in the trigonometric series [15] and in the approximation theory [6].

We are now ready to obtain our main results.

Definition 6. *Let $(X, N, *)$ be a PN- space. We say that a sequence $x = (x_k)$ is statistically convergent to $L \in X$ with respect to the probabilistic norm N provided that for every $\varepsilon > 0$ and $\lambda \in (0,1)$*

$$\delta\left(\{k \in \mathbb{N} : N_{x_k - L}(\varepsilon) \leq 1 - \lambda\}\right) = 0, \tag{1}$$

or equivalently,

$$\lim_n \frac{1}{n} \left|\{k \leq n : N_{x_k - L}(\varepsilon) \leq 1 - \lambda\}\right| = 0.$$

In this case we write $st_N - \lim x = L$, where L is said to be $st_N -$limit.

By using (1) and well-known density properties, we easily get the following lemma.

Lemma 1. *Let $(X, N, *)$ be a $PN-$ space. Then, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, the following statements are equivalent:*

(i) $st_N - \lim x = L$,

(ii) $\delta\left(\{k \in \mathbb{N} : N_{x_k - L}(\varepsilon) \leq 1 - \lambda\}\right) = 0$,

(iii) $\delta\left(\{k \in \mathbb{N} : N_{x_k - L}(\varepsilon) > 1 - \lambda\}\right) = 1$,

(iv) $st - \lim N_{x_k - L}(\varepsilon) = 1$.

Theorem 1. *Let $(X, N, *)$ be a $PN-$ space. If a sequence $x = (x_k)$ is statistically convergent with respect to the probabilistic norm N, then $st_N -$limit is unique.*

Proof. Assume that $st_N - \lim x = L_1$ and $st_N - \lim x = L_2$. For a given $\lambda > 0$ choose $\gamma \in (0,1)$ such that $(1-\gamma) * (1-\gamma) > 1 - \lambda$. Then, for any $\varepsilon > 0$, define the following sets:

$$K_{N,\,1}(\gamma, \varepsilon) := \left\{k \in \mathbb{N} : N_{x_k - L_1}(\varepsilon) \leq 1 - \gamma\right\},$$
$$K_{N,\,2}(\gamma, \varepsilon) := \left\{k \in \mathbb{N} : N_{x_k - L_2}(\varepsilon) \leq 1 - \gamma\right\}.$$

Since $st_N - \lim x = L_1$, $\delta\left\{K_{N,\,1}(\gamma, \varepsilon)\right\} = 0$ for all $\varepsilon > 0$. Furthermore, using $st_N - \lim x = L_2$, we get $\delta\left\{K_{N,\,2}(\gamma, \varepsilon)\right\} = 0$ for all $\varepsilon > 0$. Now let $K_N(\gamma, \varepsilon) = K_{N,\,1}(\gamma, \varepsilon) \cap K_{N,\,2}(\gamma, \varepsilon)$. Then we observe that $\delta\left\{K_N(\gamma, \varepsilon)\right\} = 0$ which implies $\delta\left\{\mathbb{N}/K_N(\gamma, \varepsilon)\right\} = 1$. If $k \in \mathbb{N}/K_N(\gamma, \varepsilon)$, then we have

$$N_{L_1 - L_2}(\varepsilon) \geq N_{x_k - L_1}\left(\frac{\varepsilon}{2}\right) * N_{x_k - L_2}\left(\frac{\varepsilon}{2}\right) > (1-\gamma) * (1-\gamma).$$

Since $(1-\gamma)*(1-\gamma)>1-\lambda$, it follows that

$$N_{L_1-L_2}(\varepsilon)>1-\lambda. \tag{2}$$

Since $\lambda>0$ was arbitrary, by (2) we get $N_{L_1-L_2}(\varepsilon)=1$ for all $\varepsilon>0$, which yields $L_1=L_2$. Therefore, we conclude that st_N-limit is unique. □

Theorem 2. *Let* $(X,N,*)$ *be a PN− space. If* $N-\lim x=L$, *then* $st_N-\lim x=L$.

Proof. By hypothesis, for every $\lambda\in(0,1)$ and $\varepsilon>0$, there is a number $k_0\in\mathbb{N}$ such that $N_{x_n-L}(\varepsilon)>1-\lambda$ for all $n\geq k_0$. This guaranties that the set $\{n\in\mathbb{N}:N_{x_n-L}(\varepsilon)\leq 1-\lambda\}$ has at most finitely many terms. Since every finite subset of the natural numbers has density zero, we immediately see that $\delta\left(\{n\in\mathbb{N}:N_{x_n-L}(\varepsilon)\leq 1-\lambda\}\right)=0$,whence the result. □

The following example shows that the converse of *Theorem 2* is not valid.

Example 3. *Let* $(\mathbb{R},|.|)$ *denote the space of real numbers with the usual norm. Let* $a*b=ab$ *and* $N_x(t)=\frac{t}{t+|x|}$, *where* $x\in X$ *and* $t\geq 0$. *In this case observe that* $(\mathbb{R},N,*)$ *is a PN− space. Now we define a sequence* $x=(x_k)$ *whose terms are given by*

$$x_k:=\begin{cases}1, \text{ if } k=m^2 \quad (m\in\mathbb{N}),\\ 0, \text{ otherwise.}\end{cases} \tag{3}$$

Then, for every $\lambda\in(0,1)$ *and for any* $\varepsilon>0$, *let* $K_n(\lambda,\varepsilon):=\{k\leq n:N_{x_k}(\varepsilon)\leq 1-\lambda\}$. *Since*

$$\begin{aligned}K_n(\lambda,\varepsilon)&=\left\{k\leq n:\frac{t}{t+|x_k|}\leq 1-\lambda\right\}\\&=\left\{k\leq n\ :|x_k|\geq\frac{\lambda t}{1-\lambda}>0\right\}\\&=\{k\leq n\ :x_k=1\}\\&=\left\{k\leq n\ :\ k=m^2 \text{ and } m\in\mathbb{N}\right\},\end{aligned}$$

we get

$$\frac{1}{n}\left|K_n(\lambda,\varepsilon)\right|\leq\frac{1}{n}\left|\left\{k\leq n\ :\ k=m^2 \text{ and } m\in\mathbb{N}\right\}\right|\leq\frac{\sqrt{n}}{n}$$

which implies that $\lim\limits_n\frac{1}{n}\left|K_n(\lambda,\varepsilon)\right|=0$. *Hence, by* Definition 6, *we get* $st_N-\lim x=0$. *However, since the sequence* $x=(x_k)$ *given by (3) is not convergence in the space* $(\mathbb{R},|.|)$, *by* Remark 1, *we also see that* x *is not convergent with respect to the probabilistic norm* N.

Theorem 3. *Let* $(X,N,*)$ *be a PN− space. Then,* $st_N-\lim x=L$ *if and only if there exists an increasing index sequence* $K=\{k_n\}_{n\in\mathbb{N}}$ *of the natural numbers such that* $\delta\{K\}=1$ *and* $N-\lim\limits_{n\in K}x_n=L$, *i.e.,* $N-\lim\limits_n x_{k_n}=L$.

Proof. *Necessity:* We first assume that $st_N-\lim x=L$. Now, for any $\varepsilon>0$ and $j\in\mathbb{N}$, let

$$K(j,\varepsilon):=\left\{n\in\mathbb{N}:N_{x_n-L}(\varepsilon)>1-\frac{1}{j}\right\}$$

Then observe that, for $\varepsilon > 0$ and $j \in \mathbb{N}$,

$$K(j+1,\varepsilon) \subset K(j,\varepsilon). \tag{4}$$

Since $st_N - \lim x = L$, it is clear that

$$\delta\{K(j,\varepsilon)\} = 1, \quad (j \in \mathbb{N} \text{ and } \varepsilon > 0). \tag{5}$$

Now let p_1 be an arbitrary number of $K(1,\varepsilon)$. Then, by (5), there is a number $p_2 \in K(2,\varepsilon)$, $(p_2 > p_1)$, such that, for all $n \geq p_2$,

$$\frac{1}{n}\left|\left\{k \leq n : N_{x_k - L}(\varepsilon) > 1 - \frac{1}{2}\right\}\right| > \frac{1}{2}$$

Further, again by (5), there is a number $p_3 \in K(3,\varepsilon)$, $(p_3 > p_2)$, such that, for all $n \geq p_3$,

$$\frac{1}{n}\left|\left\{k \leq n : N_{x_k - L}(\varepsilon) > 1 - \frac{1}{3}\right\}\right| > \frac{2}{3},$$

and so on. So, by induction we can construct an increasing index sequence $\{p_j\}_{j\in\mathbb{N}}$ of natural numbers such that $p_j \in K(j,\varepsilon)$ and that the following statement holds for all $n \geq p_j$ $(j \in \mathbb{N})$:

$$\frac{1}{n}\left|\left\{k \leq n : N_{x_k - L}(\varepsilon) > 1 - \frac{1}{j}\right\}\right| > \frac{j-1}{j} \tag{6}$$

Now we construct the increasing index sequence K as follows:

$$K := \{n \in \mathbb{N} : 1 < n < p_1\} \cup \left[\bigcup_{j\in\mathbb{N}} \{n \in K(j,\varepsilon) : p_j \leq n < p_{j+1}\}\right]. \tag{7}$$

Then by (4), (6) and (7) we conclude, for all n, $(p_j \leq n < p_{j+1})$, that

$$\frac{1}{n}\left|\{k \leq n : k \in K\}\right| \geq \frac{1}{n}\left|\left\{k \leq n : N_{x_k - L}(\varepsilon) > 1 - \frac{1}{j}\right\}\right| > \frac{j-1}{j}.$$

Hence it follows that $\delta(K) = 1$. Now let $\varepsilon > 0$ and choose a number $j \in \mathbb{N}$ such that $\frac{1}{j} < \varepsilon$. Assume that $n \geq v_j$ and $n \in K$. Then, by the definition of K, there exists a number $m \geq j$ such that $v_m \leq n < v_{m+1}$ and $n \in K(j,\varepsilon)$. Hence, we have, for every $\varepsilon > 0$,

$$N_{x_n - L}(\varepsilon) > 1 - \frac{1}{j} > 1 - \varepsilon$$

for all $n \geq v_j$ and $n \in K$. This indicates that

$$N - \lim_{n\in K} x_n = L.$$

So the proof of necessity is completed.

Sufficiency: Suppose that there exists an increasing index sequence $K=\{k_n\}_{n\in\mathbb{N}}$ of natural numbers such that $\delta\{K\}=1$ and $N-\lim_{n\in K} x_n=L$. Then, for every $\varepsilon>0$, there is a number n_0 such that for each $n\geq n_0$ the inequalities $N_{x_n-L}(\varepsilon)>1-\varepsilon$ hold. Now define $M(\lambda,\varepsilon):=\{n\in\mathbb{N}:N_{x_n-L}(\varepsilon)\leq 1-\lambda\}$. Then we have

$$M(\lambda,\varepsilon)\subset\mathbb{N}-\{k_{n_0},k_{n_0+1},k_{n_0+2},...\}.$$

Since $\delta\{K\}=1$, we get $\delta\{\mathbb{N}-\{k_{n_0},k_{n_0+1},k_{n_0+2},...\}\}=0$, which yields that

$$\delta\{M(\lambda,\varepsilon)\}=0.$$

Therefore, we conclude that $st_N-\lim x=L$. □

Remark 2. *If $st_N-\lim_n x_n=L$, then there exits a sequence $y=(y_n)$ such that $N-\lim_n y_n=L$ and $\delta\{n\in\mathbb{N}:x_n=y_n\}=1$.*

We now introduce the notion of a statistical Cauchy sequence on a probabilistic norm space and give a characterization.

Definition 7. *Let $(X,N,*)$ be a $PN-$ space. We say that a sequence $x=(x_n)$ is statistically Cauchy with respect to the probabilistic norm N provided that, for every $\varepsilon>0$ and $\lambda\in(0,1)$, there exists a positive integer $m\in\mathbb{N}$ satisfying $\delta\{n\in\mathbb{N}:N_{x_n-x_m}(\varepsilon)\leq 1-\lambda\}=0$.*

Now using a similar technique in the proof of *Theorem 3* one can get the following result at once.

Theorem 4. *Let $(X,N,*)$ be a PN- space, and let $x=(x_k)$ be a sequence whose terms are in the vector space X. Then, the following conditions are equivalent:*

- (a) *x is a statistically Cauchy sequence with respect to the probabilistic norm N.*
- (b) *There exists an increasing index sequence $K=\{k_n\}$ of natural numbers such that $\delta\{K\}=1$ and the subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the probabilistic norm N.*

We show that statistically convergence on $PN-$spaces has some arithmetical properties similar to properties of the usual convergence on $\mathbb{R}$.

Lemma 2. *Let $(X,N,*)$ be a $PN-$ space.*

(1) *If* $st_N-\lim x_n=\xi$ and $st_N-\lim y_n=\eta$, then $st_N-\lim(x_n+y_n)=\xi+\eta$,

(2) *If* $st_N-\lim x_n=\xi$ and $\alpha\in\mathbb{R}$, then $st_N-\lim\alpha x_n=\alpha\xi$,

(3) *If* $st_N-\lim x_n=\xi$ and $st_N-\lim y_n=\eta$, then $st_N-\lim(x_n-y_n)=\xi-\eta$.

Proof. (1) Let $st_N-\lim x_n=\xi$, $st_N-\lim y_n=\eta$, $\varepsilon>0$ and $\lambda\in(0,1)$. Choose $\gamma\in(0,1)$ such that $(1-\gamma)*(1-\gamma)>1-\lambda$. Then we define the following sets:

$$\begin{aligned}K_{N,1}(\gamma,\varepsilon) &:= \{n\in\mathbb{N}:N_{x_n-\xi}(\varepsilon)\leq 1-\gamma\}\\ K_{N,2}(\gamma,\varepsilon) &:= \{n\in\mathbb{N}:N_{y_n-\eta}(\varepsilon)\leq 1-\gamma\}.\end{aligned}$$

Since $st_N-\lim x_n=\xi$, $\delta\{K_{N,1}(\gamma,\varepsilon)\}=0$ for all $\varepsilon>0$. Furthermore, using $st_N-\lim y_n=\eta$ we get $\delta\{K_{N,2}(\gamma,\varepsilon)\}=0$ for all $\varepsilon>0$. Now let $K_N(\gamma,\varepsilon)=$

$K_{N,1}(\gamma,\varepsilon) \cap K_{N,2}(\gamma,\varepsilon)$. Then we observe that $\delta\{K_N(\gamma,\varepsilon)\} = 0$ which implies $\delta\{\mathbb{N}/K_N(\gamma,\varepsilon)\} = 1$. If $n \in \mathbb{N}/K_N(\gamma,\varepsilon)$, then we have

$$\begin{aligned} N_{(x_n-\xi)+(y_n-\eta)}(\varepsilon) &\geq N_{x_n-\xi}\left(\frac{\varepsilon}{2}\right) * N_{y_n-\eta}\left(\frac{\varepsilon}{2}\right) \\ &> (1-\gamma)*(1-\gamma) > 1-\lambda. \end{aligned}$$

This shows that

$$\delta\left(\left\{n \in \mathbb{N} : N_{(x_n-\xi)+(y_n-\eta)}(\varepsilon) \leq 1-\lambda\right\}\right) = 0$$

so $st_N - \lim(x_n + y_n) = \xi + \eta$.

(2) Let $st_N - \lim x_n = \xi$, $\lambda \in (0,1)$ and $\varepsilon > 0$. First of all, we consider the case of $\alpha = 0$. In this case

$$N_{0x_n-0\xi}(\varepsilon) = N_0(\varepsilon) = 1 > 1-\lambda.$$

So we obtain $N - \lim 0x_n = 0$. Then from *Theorem 2* we have $st_N - \lim 0x_n = 0$.

Now we consider the case of $\alpha \in \mathbb{R}\ (\alpha \neq 0)$. Since $st_N - \lim x_n = \xi$, if we define the set

$$K_N(\gamma,\varepsilon) := \{n \in \mathbb{N} : N_{x_n-\xi}(\varepsilon) \leq 1-\lambda\}$$

then we can say $\delta\{K_N(\gamma,\varepsilon)\} = 0$ for all $\varepsilon > 0$. In this case $\delta\{\mathbb{N}/K_N(\gamma,\varepsilon)\} = 1$. If $n \in \mathbb{N}/K_N(\gamma,\varepsilon)$, then

$$\begin{aligned} N_{\alpha x_n-\alpha\xi}(\varepsilon) &= N_{x_n-\xi}\left(\frac{\varepsilon}{|\alpha|}\right) \\ &\geq N_{x_n-\xi}(\varepsilon) * N_0\left(\frac{\varepsilon}{|\alpha|} - \varepsilon\right) \\ &= N_{x_n-\xi}(\varepsilon) * 1 = N_{x_n-\xi}(\varepsilon) > 1-\lambda \end{aligned}$$

for $\alpha \in \mathbb{R}\ (\alpha \neq 0)$. This shows that

$$\delta\left(\{n \in \mathbb{N} : N_{\alpha x_n-\alpha\xi}(\varepsilon) \leq 1-\lambda\}\right) = 0.$$

So $st_N - \lim \alpha x_n = \alpha\xi$.

(3) The proof is clear from (1) and (2). □

Definition 8. *Let $(X, N, *)$ be a $PN-$ space. For $x \in X$, $t > 0$ and $0 < r < 1$, the ball centered at x with radius r is defined by*

$$B(x,r,t) = \{y \in X : N_{x-y}(t) > 1-r\}.$$

Definition 9. *Let $(X, N, *)$ be a $PN-$space. A subset Y of X is said to be bounded on $PN-$spaces if for every $r \in (0,1)$ there exists $t_0 > 0$ such that $N_x(t_0) > 1-r$ for all $x \in Y$.*

It follows from *Lemma 2* that the set of all bounded statistically convergent sequences on $PN-$space is a linear subspace of the space $\ell_\infty^N(X)$ of all bounded sequences on $PN-$spaces.

Theorem 5. *Let $(X, N, *)$ be a $PN-$ space and $S_b^N(X)$ the space of bounded statistically convergent sequences on $PN-$spaces. Then the set $S_b^N(X)$ is a closed linear subspace of the set $\ell_\infty^N(X)$.*

Proof. It is clear that $S_b^N(X) \subset \overline{S_b^N(X)}$. Now we show that $\overline{S_b^N(X)} \subset S_b^N(X)$. Let $y \in \overline{S_b^N(X)}$. Since $B(y, r, t) \cap S_b^N(X) \neq \varnothing$, there is an $x \in B(y, r, t) \cap S_b^N(X)$.

Let $t > 0$ and $\varepsilon \in (0, 1)$. Choose $r \in (0, 1)$ such that $(1 - r) * (1 - r) > 1 - \varepsilon$. Since $x \in B(y, r, t) \cap S_b^N(X)$, there is a set $K \subseteq \mathbb{N}$ with $\delta(K) = 1$ such that

$$N_{y_n - x_n}\left(\frac{t}{2}\right) > 1 - r \quad \text{and} \quad N_{x_n}\left(\frac{t}{2}\right) > 1 - r$$

for all $n \in K$. Then we have

$$\begin{aligned} N_{y_n}(t) &= N_{y_n - x_n + x_n}(t) \\ &\geq N_{y_n - x_n}\left(\frac{t}{2}\right) * N_{x_n}\left(\frac{t}{2}\right) \\ &> (1 - r) * (1 - r) > 1 - \varepsilon \end{aligned}$$

for all $n \in K$. Hence $\delta\{n \in K : N_{y_n}(t) > 1 - \varepsilon\} = 1$ and thus $y \in S_b^N(X)$. □

3. Statistical limit points and statistical cluster points on IFNS

Fridy introduced the concepts of statistical limit points and statistical cluster points of real number sequences in 1993 [8]. Also he gives relations between them and the set of ordinary limit points. Now we study the analogues of these on probabilistic normed spaces.

Definition 10. *Let $(X, N, *)$ be a $PN-$space. $\ell \in X$ is called a limit point of the sequence $x = (x_k)$ with respect to the probabilistic norm N provided that there is a subsequence of x that converges to ℓ with respect to the probabilistic norm N. Let $L_N(x)$ denote the set of all limit points of the sequence x.*

Definition 11. *Let $(X, N, *)$ be a $PN-$space. If $\{x_{k(j)}\}$ is a subsequence of $x = (x_k)$ and $K := \{k(j) \in \mathbb{N} : j \in \mathbb{N}\}$ then we abbreviate $\{x_{k(j)}\}$ by $\{x\}_K$ which in case $\delta(K) = 0$. $\{x\}_K$ is called a subsequence of density zero or thin subsequence. On the other hand, $\{x\}_K$ is a nonthin subsequence of x if K does not have density zero.*

Definition 12. *Let $(X, N, *)$ be a PN-space. Then $\xi \in X$ is called a statistical limit point of sequence $x = (x_k)$ with respect to the probabilistic norm N provided that there is a nonthin subsequence of x that converges to ξ with respect to the probabilistic norm N. In this case we say ξ is an st_N-limit point of seqence $x = (x_k)$. Let $\Lambda_N(x)$ denote the set of all st_N-limit points of the sequence x.*

Definition 13. *Let $(X, N, *)$ be a $PN-$space.Then $\gamma \in X$ is called a statistical cluster point of sequence $x = (x_k)$ with respect to the probabilistic norm N provided that for every $\varepsilon > 0$ and $\lambda \in (0, 1)$.*

$$\bar{\delta}(\{k \in \mathbb{N} : N_{x_k - \gamma}(\varepsilon) > 1 - \lambda\}) > 0.$$

In this case we say γ is an st_N-cluster point of sequence $x=(x_k)$. Let $\Gamma_N(x)$ denote the set of all st_N-cluster points of the sequence x.

Theorem 6. *Let $(X,N,*)$ be a $PN-$space. For any sequence $x\in X$, $\Lambda_N(x)\subset\Gamma_N(x)$.*

Proof. Suppose $\xi\in\Lambda_N(x)$, then there is a nonthin subsequence $\left(x_{k(j)}\right)$ of $x=(x_k)$ that converges to ξ with respect to the probabilistic norm N, i.e.

$$\delta\left(\left\{k(j)\in\mathbb{N}:N_{x_{k(j)}-\xi}(\varepsilon)>1-\lambda\right\}\right)=d>0.$$

Since

$$\{k\in\mathbb{N}:N_{x_k-\xi}(\varepsilon)>1-\lambda\}\supset\left\{k(j)\in\mathbb{N}:N_{x_{k(j)}-\xi}(\varepsilon)>1-\lambda\right\}$$

for every $\varepsilon>0$, we have

$$\begin{aligned}&\{k\in\mathbb{N}:N_{x_k-\xi}(\varepsilon)>1-\lambda\}\\&\qquad\supseteq\{k(j)\in\mathbb{N}:j\in\mathbb{N}\}\setminus\left\{k(j)\in\mathbb{N}:N_{x_{k(j)}-\xi}(\varepsilon)\le 1-\lambda\right\}.\end{aligned}$$

Since $\left(x_{k(j)}\right)$ converges to ξ with respect to the probabilistic norm N, the set

$$\left\{k(j)\in\mathbb{N}:N_{x_{k(j)}-\xi}(\varepsilon)\le 1-\lambda\right\}$$

is finite for any $\varepsilon>0$. Therefore,

$$\begin{aligned}\bar{\delta}(\{k\in\mathbb{N}:N_{x_k-\xi}(\varepsilon)>1-\lambda\})\ge\ &\bar{\delta}\{k(j)\in\mathbb{N}:j\in\mathbb{N}\}\\&-\bar{\delta}\left\{k(j)\in\mathbb{N}:N_{x_{k(j)}-\xi}(\varepsilon)\le 1-\lambda\right\}.\end{aligned}$$

Hence

$$\bar{\delta}(\{k\in\mathbb{N}:N_{x_k-\xi}(\varepsilon)>1-\lambda\})>0$$

which means that $\xi\in\Gamma_N(x)$. □

Theorem 7. *Let $(X,N,*)$ be a $PN-$space. For any sequence $x\in X$, $\Gamma_N(x)\subseteq L_N(x)$.*

Proof. Let $\gamma\in\Gamma_N(x)$, then

$$\delta(\{k\in\mathbb{N}:N_{x_k-\gamma}(\varepsilon)>1-\lambda\})>0$$

for every $\varepsilon>0$ and $\lambda\in(0,1)$. We set $\{x\}_K$ a nonthin subsequence of x such that

$$K:=\left\{k(j)\in\mathbb{N}:N_{x_{k(j)}-\gamma}(\varepsilon)>1-\lambda\right\}$$

for every $\varepsilon>0$ and $\delta(K)\ne 0$. Since there are infinitely many elements in K, $\gamma\in L_N(x)$. □

Theorem 8. *Let $(X,N,*)$ be a $PN-$space. For a sequence $x=(x_k)$, $st_N-\lim x=x_0$ then $\Lambda_N(x)=\Gamma_N(x)=\{x_0\}$.*

Proof. First we show that $\Lambda_N(x)=\{x_0\}$. We suppose that $\Lambda_N(x)=\{x_0,y_0\}$ such that $x_0\ne y_0$. In this case, there exist $\left\{x_{k(j)}\right\}$ and $\left\{x_{l(i)}\right\}$ nonthin subsequences of $x=(x_k)$ that converge to x_0, y_0 with respect to the probabilistic norm N, respectively. Since $\{x_{l(i)}\}$ converge to y_0 with respect to the probabilistic norm N for every $\varepsilon > 0$ and $\lambda \in (0,1)$

$$K := \{l(i) \in \mathbb{N} : N_{x_{l(i)}-y_0}(\varepsilon) \leq 1-\lambda\}$$

is a finite set so $\delta(K) = 0$. Then we observe that

$$\{l(i) \in \mathbb{N} : i \in \mathbb{N}\} = \{l(i) \in \mathbb{N} : N_{x_{l(i)}-y_0}(\varepsilon) > 1-\lambda\} \\ \cup \{l(i) \in \mathbb{N} : N_{x_{l(i)}-y_0}(\varepsilon) \leq 1-\lambda\}$$

which implies that

$$\delta\left(\{l(i) \in \mathbb{N} : N_{x_{l(i)}-y_0}(\varepsilon) > 1-\lambda\}\right) \neq 0. \tag{8}$$

Since $st_N - \lim x = x_0$,

$$\delta(\{k \in \mathbb{N} : N_{x_k-x_0}(\varepsilon) \leq 1-\lambda\}) = 0 \tag{9}$$

for every $\varepsilon > 0$. Therefore, we can write

$$\delta(\{k \in \mathbb{N} : N_{x_k-x_0}(\varepsilon) > 1-\lambda\}) \neq 0.$$

For every $x_0 \neq y_0$

$$\{l(i) \in \mathbb{N} : N_{x_{l(i)}-y_0}(\varepsilon) > 1-\lambda\} \cap \{k \in \mathbb{N} : N_{x_k-x_0}(\varepsilon) > 1-\lambda\} = \varnothing.$$

Hence

$$\{l(i) \in \mathbb{N} : N_{x_{l(i)}-y_0}(\varepsilon) > 1-\lambda\} \subseteq \{k \in \mathbb{N} : N_{x_k-x_0}(\varepsilon) \leq 1-\lambda\}.$$

Therefore

$$\bar{\delta}\left(\{l(i) \in \mathbb{N} : N_{x_{l(i)}-y_0}(\varepsilon) > 1-\lambda\}\right) \leq \bar{\delta}(\{k \in \mathbb{N} : N_{x_k-x_0}(\varepsilon) \leq 1-\lambda\}) = 0.$$

This contradicts (8). Hence $\Lambda_N(x) = \{x_0\}$.

Now we assume that $\Gamma_N(x) = \{x_0, z_0\}$ such that $x_0 \neq z_0$. Then

$$\bar{\delta}(\{k \in \mathbb{N} : N_{x_k-z_0}(\varepsilon) > 1-\lambda\}) \neq 0. \tag{10}$$

Since

$$\{k \in \mathbb{N} : N_{x_k-x_0}(\varepsilon) > 1-\lambda\} \cap \{k \in \mathbb{N} : N_{x_k-z_0}(\varepsilon) > 1-\lambda\} = \varnothing$$

for every $x_0 \neq z_0$, so

$$\{k \in \mathbb{N} : N_{x_k-x_0}(\varepsilon) \leq 1-\lambda\} \supseteq \{k \in \mathbb{N} : N_{x_k-z_0}(\varepsilon) > 1-\lambda\}.$$

Therefore

$$\bar{\delta}(\{k \in \mathbb{N} : N_{x_k-x_0}(\varepsilon) \leq 1-\lambda\}) \geq \bar{\delta}(\{k \in \mathbb{N} : N_{x_k-z_0}(\varepsilon) > 1-\lambda\}). \tag{11}$$

From (10), the right hand-side of (11) is greater than zero and from (9), the left hand-side of (11) equals zero. This is a contradiction. Hence $\Gamma_N(x) = \{x_0\}$. □

Theorem 9. *Let $(X, N, *)$ be a $PN-$space. Then the set Γ_N is closed in X for each $x = (x_k)$ of elements of X.*

Proof. Let $y \in \overline{\Gamma_N(x)}$. Take $0 < r < 1$ and $t > 0$. There exists $\gamma \in \Gamma_N(x) \cap B(y, r, t)$ such that

$$B(y, r, t) = \{x \in X : N_{y-x}(t) > 1 - r\}.$$

Choose $\eta > 0$ such that $B(\gamma, \eta, t) \subset B(y, r, t)$. We have

$$\{k \in \mathbb{N} : N_{y-x_k}(t) > 1 - r\} \supset \{k \in \mathbb{N} : N_{\gamma-x_k}(t) > 1 - \eta\}$$

hence

$$\delta(\{k \in \mathbb{N} : N_{y-x_k}(t) > 1 - r\}) \neq 0$$

and $y \in \Gamma_N$. □

Conclusion 1. *In this paper we obtained results on statistical convergence in probabilistic normed spaces. As every ordinary norm induces a probabilistic norm, the results obtained here are more general than the corresponding of normed spaces.*

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