Non-metric continua and multi-valued mappings

IVAN LONČAR*

Abstract. *A continuum is an arboroid if it is hereditarily unicoherent and arcwise connected. A metric arboroid is a dendroid. A generalized dendrite is a locally connected arboroid. Among other things, we shall prove that a locally connected continuum X is a generalized dendrite if and only if X has the fixed point property for continuous, closed set-valued mappings.*

Key words: *arcwise connected, arboroid, dendrite, hyperspace, inverse system*

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1. Introduction

All spaces in this paper are Tychonoff and all mappings are continuous. We shall use the notion of an inverse system as in [6, pp. 135-142]. An inverse system is denoted by $\mathbf{X} = \{X_a, p_{ab}, A\}$.

Let X be a space. We define its hyperspaces as the following sets:

$$\begin{aligned} 2^X &= \{F \subseteq X : F \text{ is closed and nonempty}\}, \\ \mathcal{C}(X) &= \{F \in 2^X : F \text{ is connected}\}. \end{aligned} \tag{1}$$

The topology on 2^X is the Vietoris topology and $\mathcal{C}(X)$ is a subspaces of 2^X.

Let X and Y be the spaces and let $f : X \rightarrow Y$ be a mapping. Define $2^f : 2^X \rightarrow 2^Y$ by $2^f(F) = f(F)$ for $F \in 2^X$. By [13, 5.10] 2^f is continuous and $2^f(\mathcal{C}(X)) \subset \mathcal{C}(Y)$. The restriction $2^f|\mathcal{C}(X)$ is denoted by $\mathcal{C}(f)$.

Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be an inverse system of compact spaces with natural projections $p_a : \lim X \rightarrow X_a$, $a \in A$. Then $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$ and $\mathcal{C}(\mathbf{X}) = \{\mathcal{C}(X_a), \mathcal{C}(p_{ab}), A\}$ are inverse systems.

Lemma 1. *[9, Lemma 2] Let $X = \lim \mathbf{X}$. Then $2^X = \lim 2^{\mathbf{X}}$ and $\mathcal{C}(X) = \lim \mathcal{C}(\mathbf{X})$.*

A function $F : X \rightarrow 2^Y$ is *upper semi-continuous at a point* $p \in X$ provided that for every open set $V \subset Y$ such that $F(p) \subset V$ there is an open set $U \subset X$

*Faculty of Organization and Informatics Varaždin, University of Zagreb, Pavlinska 2, HR-42 000 Varaždin, e-mails: `ivan.loncar1@vz.htnet.hr`, `ivan.loncar@foi.hr`

such that $p \in U$ and satisfying $F(x) \subset V$ for all $x \in U$. The function F is said to be *upper semi-continuous* if it is upper semi-continuous at each of its points.

We say that a function $F : X \to 2^Y$ is *lower semi-continuous at a point* $x_0 \in X$ provided for every open $G \subset Y$ such that $F(x_0) \cap G \neq \emptyset$ there exists a neighbourhood $U(x_0)$ of x_0 such that $F(x) \cap G \neq \emptyset$ for every $x \in U(x_0)$. The function F is said to be *lower semi-continuous* if it is lower semi-continuous at each of its points.

If $F : X \to 2^Y$ is both upper and lower semi-continuous, then F is said to be *continuous.*

Let X be a space and $\mathcal{C}$ a class of set-valued mappings of X into itself. We say that X has the fixed point property for $\mathcal{C}$ if, for each $f \in \mathcal{C}$, there exists $x \in X$ such that $x \in f(x)$.

Let A be a partially ordered directed set. We say that a subset $A_1 \subset A$ *majorates* [4, p. 9] another subset $A_2 \subset A$ if for each element $a_2 \in A_2$ there exists an element $a_1 \in A_1$ such that $a_1 \geq a_2$. A subset which majorates A is called *cofinal* in A. A subset of A is said to be a *chain* if every two elements of it are comparable. The symbol $\sup B$, where $B \subset A$, denotes the lower upper bound of B (if such an element exists in A). Let $\tau \geq \aleph_0$ be a cardinal number. A subset B of A is said to be τ-*closed* in A if for each chain $C \subset B$, with $|B| \leq \tau$, we have $\sup C \in B$, whenever the element $\sup C$ exists in A. Finally, a directed set A is said to be τ-*complete* if for each chain C of A of elements of A with $|C| \leq \tau$, there exists an element $\sup C$ in A.

Suppose that we have two inverse systems $\mathbf{X} = \{X_a, p_{ab}, A\}$ and $\mathbf{Y} = \{Y_b, q_{bc}, B\}$. A *morphism of the system X into the system* $\mathbf{Y}$ [4, p. 15] is a family $\{\varphi, \{f_b : b \in B\}\}$ consisting of a nondecreasing function $\varphi : B \to A$ such that $\varphi(B)$ is cofinal in A, and of continuous maps $f_b : X_{\varphi(b)} \to Y_b$ defined for all $b \in B$ such that the following

$$\begin{array}{ccc} X_{\varphi(b)} & \overset{p_{\varphi(b)\varphi(c)}}{\longleftarrow} & X_{\varphi(c)} \\ \downarrow f_b & & \downarrow f_c \\ Y_b & \overset{q_{bc}}{\longleftarrow} & Y_c \end{array} \tag{2}$$

diagram commutes. Any morphism $\{\varphi, \{f_b : b \in B\}\} : \mathbf{X} \to \mathbf{Y}$ induces a continuous map, called the *limit map of the morphism*

$$\lim\{\varphi, \{f_b : b \in B\}\} : \lim \mathbf{X} \to \lim \mathbf{Y} \tag{3}$$

In the present paper we deal with the inverse systems defined on the same indexing set A. In this case, the map $\varphi : A \to A$ is taken to be the identity and we use the following notation $\{f_a : X_a \to Y_a; a \in A\} : \mathbf{X} \to \mathbf{Y}$.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be σ-*directed* if for each sequence $a_1, a_2, ..., a_k, ...$ of members of A there is an $a \in A$ such that $a \geq a_k$ for each $k \in \mathbb{N}$.

We say that an inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is *factorizing* [4, p. 17] if for each real-valued function $f : \lim \mathbf{X} \to \mathbb{R}$ there exists an $a \in A$ and a function $f_a : X_a \to \mathbb{R}$ such that $f = f_a p_a$.

An inverse system $\mathbf{X} = \{X_a,\ p_{ab},\ A\}$ is said to be τ-*continuous* [4, p. 19] if for each chain B in A with $|B| < \tau$ and $\sup B = b$, the diagonal product $\Delta\,\{p_{ab} : a \in B\}$ maps the space X_b homeomorphically into the space $\lim\{X_a, p_{ab}, B\}$.

Let us recall that the weight of a space X is the least cardinal number which is the cardinal number of a basis of open sets for the topology of X; we denote the weight of X by $w(X)$. Let $\omega_{\tau(X)}$ be the initial ordinal number of cardinality of $w(X)$. Let $W(X)$ be the set of all ordinal numbers $\alpha < \omega_{\tau(X)}$.

An inverse system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is said to be a *τ-system* [4, p. 19] if:

a) $w(X_a) \leq \tau$ for every $a \in A$,

b) The system $\mathbf{X} = \{X_a, p_{ab}, A\}$ is τ-continuous,

c) The indexing set A is τ-complete.

If $\tau = \aleph_0$, then τ-system is called a σ-system. If $\mathbf{X} = \{X_a, p_{ab}, A\}$ is a σ-system of compact spaces, then each X_a is metrizable. The following theorem is called the *Spectral Theorem* [4, p. 21].

Theorem 1. *[4, Theorem 1.3.4, p. 19]. If a τ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ with surjective limit projections is factorizing, then each map of its limit space into the limit space of another τ-system $\mathbf{Y} = \{Y_a, q_{ab}, A\}$ is induced by a morphism of cofinal and τ-closed subsystems. If two factorizing τ-systems with surjective limit projections and the same indexing set have homeomorphic limit spaces, then they contain isomorphic cofinal and τ-closed subsystems.*

Let us remark that the requirement of surjectivity of limit projections of systems in *Theorem 1* is essential [4, p. 21].

The Spectral Theorem and the following theorem are the main tools of this paper.

Theorem 2. *[10, Theorem 4, p. 202]. Let X be compact Hausdorff space such that $w(X) \geq \aleph_1$. There exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that X is homeomorphic to* $\lim \mathbf{X}$ *and A is the set of all countable subsets of $W(X)$ ordered by inclusion.*

A space X is said to be *rim-metrizable* if it has a basis $\mathcal{B}$ such that $Bd(U))$ is metrizable for each $U \in \mathcal{B}$. Equivalently, a space X is rim-metrizable if and only if for each pair F, G of disjoint closed subsets of X there exists a metrizable closed subset of X which separates F and G.

In the sequel we shall use the following theorem.

Theorem 3. *[10, Theorem 10, p. 207]. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-system of compact spaces and surjective bonding mappings p_{ab}. If* $\lim \mathbf{X}$ *is a locally connected space* (*rim-metrizable continuum*)*, then there exists an $a \in A$ such that the projection p_b is monotone, for every $b \geq a$.*

2. Decomposable continua and multi-valued mappings

A continuum X is said to be *decomposable* provided that X can be written as the union of two proper subcontinua. A continuum X is said to be *hereditarily decomposable* provided that each subcontinuum of X is decomposable.

A connected topological space X is said to be *unicoherent* provided that whenever A and B are closed, connected subsets of X such that $X = A \cup B$, then $A \cap B$ is connected. A connected topological space is said to be *hereditarily unicoherent* provided that each of its closed, connected subsets is unicoherent.

Proposition 1. *Every rim-metrizable hereditarily decomposable continuum is the limit of a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric hereditarily decomposable continua X_a.*

Proof. By *Theorem 2* we infer that there is an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that X is homeomorphic to $\lim \mathbf{X}$. It follows that each X_a is metric since $\mathbf{X}$ is a σ-system. Moreover, there exists a subset B cofinal in A such that the projection p_b is monotone for every $b \in B$ (*Theorem 3*). From [3, Theorem XIV, p. 217] it follows that each X_b is hereditarily decomposable since each p_b is monotone. □

Theorem 4. *[2, (2.8'), p. 334]. Let X be a hereditarily decomposable metric continuum. If X is not hereditarily unicoherent, then there exists an upper semi-continuous mapping $f : X \to C(X)$ which is fixed point free.*

We shall prove the following generalization.

Theorem 5. *Let X be a hereditarily decomposable non-metric locally connected (or rim-metrizable) continuum. If X is not hereditarily unicoherent, then there exists an upper semi-continuous mapping $f : X \to C(X)$ which is fixed point free.*

Proof. By *Theorem 3* there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ such that X is homeomorphic to $\lim \mathbf{X}$ and each projection $p_a : X \to X_a$ is a monotone surjection.Moreover, each X_a is metric since $\mathbf{X}$ is a σ-system. Let us prove that each X_a is hereditarily decomposable. This follows from [3, Theorem XIV, p. 217] since each p_b is monotone. From [15] it follows that there exists a subset B of A such that for each $b \in B$ the continuum X_b is not hereditarily unicoherent. From *Theorem 4* it follows that there exists an upper semi-continuous mapping $f_b : X_b \to C(X_b)$ which is fixed point free. Define $f : X \to C(X)$ by $f(x) = p_b^{-1} f_b p_b(x)$ since it is obvious that f is a continuum-valued mapping. Let us prove that f is upper semi-continuous. Let x be any point in X and U open set in X such that $f(x) \subset U$. This means that $p_b^{-1}(f_b p_b(x)) \subset U$. From the fact that p_b is closed, it follows that there is an open set U_b such that $f_b p_b(x) \subset U_b$ and $p_b^{-1}(U_b) \subset U$. There exists an open set V_b containing $p_b(x)$ such that $y \in V_b$ implies $f(y) \subset U_b$. Now, the set $V = p_b^{-1}(V_b)$ has the property that $x \in V$ implies $p_b^{-1}(f_b p_b(x)) \subset U$ and, consequently, $x \in V$ implies $f(x) \subset U$. Hence, f is upper semi-continuous. Finally, let us prove that f is fixed point free. Suppose that there exists a point $x \in X$ such that $x \in f(x)$, i.e., $x \in p_b^{-1}(f_b p_b(x))$. It follows that $p_b(x) \in f_b p_b(x))$, i.e., $p_b(x)$ is the fixed point of f_b which is impossible since $f_b : X_b \to C(X_b)$ is fixed point free. □

3. Arcwise connected continua and multi-valued mappings

A continuum X with precisely two non-separating points is called a *generalized arc*. A continuum X is said to be *arcwise connected* provided for every pair x, y of points of X there is a generalized arc with the end points x, y.

The following result is known.

Theorem 6. *[22, Theorem 2]. Let X be an arcwise connected metric continuum. If X is not hereditarily unicoherent then there exists an upper semi-continuous mapping $f : X \to C(X)$ which is fixed point free.*

We shall generalize this result as follows.

Theorem 7. *Let X be an arcwise connected non-metric locally connected (or rim-metrizable) continuum. If X is not hereditarily unicoherent, then there exists* *an upper semi-continuous mapping $f : X \to C(X)$ which is fixed point free.*

Proof. By *Theorem 3* there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua X_a such that X is homeomorphic to $\lim \mathbf{X}$ and each projection $p_a : X \to X_a$ is a monotone surjection. Let us prove that each X_a is connected by arcs. Let x_a, y_a be a pair of points in X_a. There exists a pair of points in X such that $x_a = p_a(x)$ and $y_a = p_a(y)$. Moreover, there exists a generalized arc L in X such that $x, y \in L$. This means that $x_a, y_a \in p_a(L)$. Finally, from [19] it follows that $p_a(L)$ is arcwise connected. Hence, each X_a is connected by arcs. From [15] it follows that there exists a subset B of A such that for each $b \in B$ the continuum X_b is not hereditarily unicoherent. By virtue of *Theorem 6* there exists an upper semi-continuous mapping $f_b : X_b \to C(X_b)$ which is fixed point free. Define $f : X \to C(X)$ by $f(x) = p_b^{-1} f_b p_b(x)$ since it is obvious that f is a continuum-valued mapping. Let us prove that f is upper semi-continuous. Let x be any point in X and U open set in X such that $f(x) \subset U$. This means that $p_b^{-1}(f_b p_b(x)) \subset U$. From the fact that p_b is closed, it follows that there is an open set U_b such that $f_b p_b(x) \subset U_b$ and $p_b^{-1}(U_b) \subset U$. There exists an open set V_b containing $p_b(x)$ such that $y \in V_b$ implies $f(y) \subset U_b$. Now, the set $V = p_b^{-1}(V_b)$ has the property that $x \in V$ implies $p_b^{-1}(f_b p_b(x)) \subset U$ and, consequently, $x \in V$ implies $f(x) \subset U$. Hence, f is upper semi-continuous. Finally, let us prove that f is fixed point free. Suppose that there exists a point $x \in X$ such that $x \in f(x)$, i.e., $x \in p_b^{-1}(f_b p_b(x))$. It follows that $p_b(x) \in f_b p_b(x))$, i.e., $p_b(x)$ is the fixed point of f_b which is impossible since $f_b : X_b \to C(X_b)$ is fixed point free. □

From *Theorems 7* and Theorem 1 [22] we have the following corollary.

Corollary 1. *Let X be an arcwise connected non-metric locally connected* (*or rim-metrizable*) *continuum. A necessary and sufficient condition that X has the fixed point property for the class of upper semi-continuous, continuum valued mappings is that X is hereditarily unicoherent.*

4. Arboroids

A continuum is an *arboroid* if it is hereditarily unicoherent and arcwise connected. A metric arboroid is a dendroid.

Now we shall prove the expanding theorem of arboroids into inverse systems of dendroids.

A *chain* $\{U_1, ..., U_n\}$ is a finite collection of sets U_i such that $U_i \bigcap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. A continuum X is said to be *chainable* or *arc-like* if each open covering of X can be refined by an open covering $u = \{U_1, ..., U_n\}$ such that $\{U_1, ..., U_n\}$ is a chain.

If $\{A_1, ..., A_n\}$ is a chain and A_1 intersects A_n, then it is a circular chain. A collection $\mathcal{B}$ of sets is *coherent* if, for each nonempty proper subcollection $\mathcal{C}$ of $\mathcal{B}$, there is an element of $\mathcal{C}$ that intersects an element of $\mathcal{B} \setminus \mathcal{C}$.

A finite coherent collection $\mathcal{T}$ of open sets is a *tree chain* if no three elements of $\mathcal{T}$ have a point in common and no subcollection of $\mathcal{T}$ is a circular chain.

A metric continuum M is *tree-like* if for each positive number ε, there is a tree chain with mesh less than ε covering M. Every tree-like continuum is hereditarily unicoherent.

A Hausdorff continuum M is *tree-like* if for each open cover u of X, there is a tree chain covering M which refines u. It follows that a continuum X is tree-like if and only if for each open cover u of X there is a metric tree (i.e., a connected acyclic graph) X_u and an u-mapping $f_u : X \to X_u$ (the inverse image of each point is contained in a member of u).

Theorem 8. *Every non-metric arboroid X is the limit of an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of dendroids.*

Proof. By [5, Corollary, p.20] X is tree-like. Theorem 4 of [12, p. 19] implies that for a tree-like continuum X there is a σ-system $X = \{X_a, p_{ab}, A\}$ of metric tree-like continua X_a such that $X = \lim X$. Moreover, every X_a is hereditarily unicoherent since every tree-like continuum is hereditarily unicoherent. Let us prove that every X_a is arcwise connected. Let x_a, y_a be a pair of points in X_a. There exists a pair of points in X such that $x_a = p_a(x)$ and $y_a = p_a(y)$. Moreover, there exists a generalized arc L in X such that $x, y \in L$. This means that $x_a, y_a \in p_a(L)$. From [19] it follows that $p_a(L)$ is arcwise connected. Hence, each X_a is connected by generalized arcs. Finally, each X_a is a dendroid. □

A λ-*arboroid* is an hereditarily decomposable and hereditarily unicoherent continuum. For λ-arboroids we have the following result.

Theorem 9. *Every non-metric rim-metrizable λ-arboroid X is the limit of an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of λ-dendroids.*

Proof. By [5, Corollary, p.20] X is tree-like. Theorem 4 of [12, p. 19] implies that there is a σ-system $X = \{X_a, p_{ab}, A\}$ of metric tree-like continua X_a such that $X = \lim X$. Moreover, every X_a is hereditarily unicoherent since every every tree-like continuum is hereditarily unicoherent. Using *Theorem 3* we may assume that each projection $p_a : \lim \mathbf{X} \to X_a$ is monotone. In order to complete the proof it suffices to prove that X is hereditarily decomposable. This follows from [3, Theorem XIV, p. 217] since p_a is monotone. □

We close this section with the following result.

Theorem 10. *Let X be a non-metric rim-metrizable and arcwise connected continuum. The following conditions are equivalent:*

(a) *X has the fixed point property for the class of upper semi-continuous, continuum valued mappings,*

(b) *X is an arboroid.*

Proof. Apply *Theorem 1.* □

5. Dendrites

A *generalized dendrite* is a locally connected arboroid. In this section we shall use the following results.

Theorem 11. *[8]. Let $\mathbf{X} = \{X_a, p_{ab}, A\}$ be a σ-directed inverse system of (hereditarily) locally connected continua and surjective bonding mappings. Then $X = \lim \mathbf{X}$ is (hereditarily) locally connected. Moreover, if each X_a is a generalized arc, then $\lim \mathbf{X}$ is a generalized arc.*

Lemma 2. *Let $f : X \to Y$ be a monotone surjection. If X is a generalized arc, then Y is a generalized arc.*

Proof. See [21, (1.1), p. 165]. □

Theorem 12. *[11, Corollary 2.7, p. 233]. A continuum X is a generalized dendrite if and only if there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of dendrites X_a and monotone bonding mappings p_{ab} such that X is homeomorphic to* $\lim \mathbf{X}$.

Corollary 2. *[11, Theorem 2.8, p. 233]. Each generalized dendrite is hereditarily locally connected.*

Theorem 13. *A continuum X is a generalized dendrite if and only if it is hereditarily locally connected and hereditarily unicoherent.*

In this section we generalize the following theorem.

Theorem 14. *[18]. A Peano continuum X has the fixed point property for continuous closed set-valued mappings if and only if X is a dendrite.*

We start with the following lemma.

Lemma 3. *A generalized dendrite X has the fixed point property for continuous mappings $f : X \to 2^X$, i.e., for continuous closed set-valued mappings.*

Proof. By *Theorem 12* there exists an inverse σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric dendrites X_a and monotone bonding mappings p_{ab} such that X is homeomorphic to $\lim \mathbf{X}$. By *Lemma 1* we have the inverse system $2^{\mathbf{X}} = \{2^{X_a}, 2^{p_{ab}}, A\}$ whose limit is 2^X. Let $f : X \to 2^X$ be a continuous mapping. From *Theorem 1* it follows that there exists a subset B cofinal in A such that for every $b \in B$ there exists a continuous mapping $f_b : X_b \to 2^{X_b}$ with the property that $\{f_b : b \in B\}$ is a morphism which induce f. From *Theorem 14* it follows that the set $F_b \subset X_b, b \in B$, of fixed points of f_b is non-empty. Let us prove that F_b is a closed subset of X_b. We shall prove that $U_b = X_b \diagdown F_b$ is open. Let $x_b \in U_b$. This means that x_b and $f_b(x_b)$ are disjoint closed subset of X_b. By the normality of X_b there exists a pair of open sets U, V such that $x \in U$ and $F_b \subset V$. ¿From the upper semi-continuity of f_b it follows that there exists an open set $W \subset U$ such that for every $x \in W$ we have $f(x) \subset V$. Hence, U_b is open and, consequently, F_b is closed. Now, we shall prove that the collection $\{F_b, p_{bc}|F_c, B\}$ is an inverse system. To do this we have to prove that if $c > b$, then $p_{bc}(F_c) \subset F_b$. Let x_c be a point of F_c. This means that $x_c \in f_c(x_c)$. Hence, $p_{bc}(x_c) \in p_{bc}(f_c(x_c)) = f_b p_{bc}(x_c)$. We conclude that the point $x_b = p_{bc}(x_c)$ has the property $x_b \in f_b(x_b)$, i.e., $x_b = p_{bc}(x_c) \in F_b$. Finally, $p_{bc}(F_c) \subset F_b$. and $\{F_b, p_{bc}|F_c, B\}$ is an inverse system with non-empty limit. Let $F = \lim \{F_b, p_{bc}|F_c, B\}$. In order to complete the proof we shall prove that for every $x \in F$ we have $x \in f(x)$. Now we have $p_b(x) \in F_b$, i.e., $p_b(x) \in f_b(p_b(x)) = p_b f(x)$, for every $b \in B$. It follows that $x \in f(x)$ since $x \notin f(x)$ implies that there is a $b \in B$ such that $p_b(x) \notin p_b f(x)$. We conclude that f has the fixed point property. □

The obtained results can be summarized as follows.

Theorem 15. *If X is a locally connected arcwise connected continuum, then the following statements are equivalent.*

(1) *X is a generalized dendrite,*

(2) *X has the fixed point property for the class of continuous, closed set-valued mappings,*

(3) *X has the fixed point property for the class of upper semi-continuous, continuum-valued mappings.*

Proof. (1) $\Rightarrow$ (2). Apply *Lemma 3.* (2) $\Longrightarrow$ (3). Obviously. (3) $\Longrightarrow$ (1). Apply *Lemma 1.* □

Corollary 3. *If X is a hereditarily locally connected continuum, then the following statements are equivalent.*

(1) *X is a generalized dendrite,*

(2) *X has the fixed point property for the class of continuous, closed set-valued mappings,*

(3) *X has the fixed point property for the class of upper semi-continuous, continuum-valued mappings,*

Proof. Every hereditarily locally connected continuum X is a continuous image of a generalized arc [17]. This means that X is arcwise connected [19]. Apply *Theorem 15.* □

Theorem 16. *For a locally connected continuum X the following conditions are equivalent:*

a) *X is a dendrite,*

b) *for every two upper semi-continuous functions $F_1 : X \to C(X)$ and $F_2 : X \to C(X)$ there are two points x_1 and x_2 in X such that $x_1 \in F_2(x_2)$ and $x_2 \in F_1(x_1)$.*

Proof. *a)* $\Longrightarrow$ *b)*. Now X has property (3) from *Theorem 15.* Let $F_1 : X \to C(X)$ and $F_2 : X \to C(X)$ be upper semi-continuous. Define the mapping $F_2F_1 : X \to C(X)$ by $F_2F_1(x) = \cup\{F_2(y) : y \in F_1(x)\}$ [2, (4.1), p. 337]. It is obvious that the definition of F_2F_1 is correct. By (3) of *Theorem 15* there is a point $x_1 \in F_2F_1(x_1)$. This means that there is a point $x_2 \in F_1(x_1)$ such that $x_1 \in F_2(x_2)$.

b) $\Longrightarrow$ *a)*. Let $F : X \to C(X)$ be an upper semi-continuous function. Set $F_1 = F$ and $F_2(x) = x$. By b) there exists x_1 such that $x_1 \in F_2(x_2) = x_2$ (i.e., $x_1 = x_2$) and $x_2 \in F_1(x_1) = F(x_1)$. It follows that $x_1 \in F(x_1)$. By (3) of *Theorem 15* we conclude that X is a generalized dendrite. □

A continuum X is hereditarily unicoherent if and only if for each closed subset A of there exists a unique continuum M_A such that M_a is irreducible about A. Obviously, $M_A = \cap\{M \in C(X) : A \subset M\}$. This characterization of hereditarily unicoherent continua induces a natural function $f : 2^X \to C(X)$ defined by $f(A) = M_A$.

Theorem 17. *[7, Theorem 1, p. 3]. The function $f : 2^X \to C(X)$ is continuous if and only if X is a dendrite.*

In another formulation [1, Theorem 1.2 (1)(13), pp. 230-231] we have the following result.

Theorem 18. *A hereditarily unicoherent continuum is a dendrite if and only if the function $f : 2^X \to C(X)$ is continuous.*

Now we shall prove the following generalization of this result.

Theorem 19. *A hereditarily unicoherent continuum is a generalized dendrite if and only if the function $f : 2^X \to C(X)$ is continuous.*

Proof. *The only if part.* If X is a generalized dendrite, then it is locally connected and $f : 2^X \to C(X)$ is continuous. See the proof of Theorem 1 in [7, p. 3].

The if part. If $f : 2^X \to C(X)$ is continuous, then X is locally connected [7, Theorem 1, p.3]. It remains to prove that X is arcwise connected. By *Theorems 2* and *3* there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua and monotone surjection p_a such that X is homeomorphic to $\lim \mathbf{X}$. Each X_a is locally connected [23, Lemma 1.5, p. 70] and, consequently, arcwise connected. Let us prove that each X_a is hereditarily unicoherent. If K and L are subcontinua of X_a, then $p_a^{-1}(K)$ and $p_a^{-1}(L)$ are subcontinua of X since p_a is monotone. This means that $p_a^{-1}(K) \cap p_a^{-1}(L)$ is a subcontinuum of X since X is hereditarily unicoherent. It follows that $p_a(p_a^{-1}(K) \cap p_a^{-1}(L)) = K \cap L$ is a subcontinuum of X_a. Hence, X_a is hereditarily unicoherent and, consequently, it is a metric dendrite. Now we are ready to prove that X is arcwise connected. Let $x, y, x \neq y$, be a pair of points of X. There exists an $a \in A$ such that for every $b \geq a$ we have $p_b(x) \neq p_b(y)$. There exists a unique arc L_b in X_b with end points $p_b(x), p_b(y)$ since X_b is a dendrite. If $c \geq b$, then $p_{bc}(L_c) = L_b$ since $p_{bc}(L_c)$ is an arc by 2 and X_b is hereditarily unicoherent. Now we have a σ-directed inverse system $\mathbf{L} = \{L_b, p_{bc}|L_c, c \geq b\}$ of arcs. By *Theorem 11* $L = \lim \mathbf{Y}$ is a generalized arc. Hence, X is arcwise connected. □

A continuum is said to be *selectible* provided that there exists a mapping $s : C(X) \to X$ such that $s(A) \in A$ for each continuum $A \subset X$ [14, p. 253].

For metric continua we have the following result [1, Theorem 1.2 (1)(18), pp. 230-231]. See also [16, Exercise 10.53 (c), p. 190].

Theorem 20. *A locally connected continuum is a dendrite if and only if it is selectible.*

We shall prove the following generalization of this result.

Theorem 21. *A locally connected continuum X is a generalized dendrite if and only if it is selectible.*

Proof. *The only if part.* If X is a generalized dendrite, then we may define a continuous selection $s : C(X) \to X$ as in metric settings. See [16, Exercise 10.53 (b), p.190].

The if part. Suppose now that X is a locally connected continuum and there is a continuous selection $s : C(X) \to X$: The proof requires the following steps.

Step 1. *A selection $s : C(X) \to X$ is a surjection.*

Step 2. *X is arcwise connected.* By [20] $C(X)$ is arcwise connected. Hence, X is arcwise connected [19].

Step 3. *X is hereditarily unicoherent.* We shall use the inverse system method since the proof given in [14, pp. 256-257.] is not valid in non-metric settings. By *Theorem 2* there exists a σ-system $\mathbf{X} = \{X_a, p_{ab}, A\}$ of metric continua X_a such that X is homeomorphic to $\lim \mathbf{X}$. We may assume that the projection p_a are monotone surjections (*Theorem 3*). From [23, Lemma 1.5, p. 70] it follows that each X_a is locally connected since X is locally connected. Using *Theorem 1* for $\tau = \aleph_0, \mathbf{X}$, $C(\mathbf{X}) = \{C(X_a), C(p_{ab}), A\}$ and $s : C(X) \to X$ we obtain a collection of mappings $\{s_b : C(X_b) \to X_b;\ b \in B\} : C(\mathbf{X}) \to \mathbf{X}$, where B is cofinal in A. Let us prove that each s_b is a selection. For every subcontinuum K_b of X_b (i.e., $K_b \in C(X_b)$) there is a subcontinuum K of X (i.e., $K \in C(X)$) such that $C(p_b)(K) = K_b$ or $p_b(K) = K_b$. From the commutativity of the diagram

$$\begin{array}{ccc} C(X_b) & \overset{C(p_b)}{\longleftarrow} & C(X) \\ \downarrow s_b & & \downarrow s \\ X_b & \overset{p_b}{\longleftarrow} & X \end{array} \tag{4}$$

it follows that $p_b s(K) = s_b(C(p_b)(K)) = s_b(K_b)$. Since $s(K) \in K$ we conclude that $p_b s(K) \in p_b(K) = K_b$ and $s_b(K_b) \in K_b$. Hence, each s_b is a selection.

Final Step. From *Theorem 20* it follows that each X_b is a metric dendrite. Applying *Theorem 12* we conclude that X is a generalized dendrite. □

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