Existence, uniqueness and non-regularity of the solution to the Neumann problem for the mono-dimensional compressible Reynolds equation

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Abstract. *In this paper we construct the solution of the Dirichlet problem for a mono-dimensional compressible Reynolds equation and we prove its uniqueness. The solution is smooth, if the total mass of the fluid is sufficiently large, but it is not regular if the total mass is insufficient. The reason for such deterioration of the solution is vacuum that appears in part of the domain.*

1. Introduction

Reynolds equation is a differential equation describing a motion of a thin fluid film that lubricates a bearing. Fluid film bearings are machine elements that can be simplified as two rigid surfaces in relative motion and a thin gap between them filled by a fluid (lubricant). In this paper we study the case when the fluid is not a liquid but a gas, typically, clean dry air. The most common examples where gas lubrication appears are computer hard discs, magnetic tapes and some high precision measuring devices. In case of incompressible fluids, the engineering model for describing the process of lubrication is linear elliptic equation and its theory is simple. For a compressible fluid due to nonlinearity of the continuity equation, the model becomes a nonlinear, degenerated elliptic equation called the compressible Reynolds equation. The compressible Reynolds equation has been first derived in the engineering literature as for e.g. [1], [2], [6], [10].

In the case of Dirichlet boundary condition, i.e. when the pressure is prescribed on the boundary of the domain, the model has been well-studied in [5] and particularly in [4] (see also [7] for the evolutional case and [3], [9] for the mono-dimensional model). The existence and uniqueness of the solution was proved. For the proof of

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uniqueness, a comparison principle was used, relying heavily on the fact that the boundary condition is of the Dirichlet type.

In [8] a rigorous derivation of the compressible Reynolds model was derived via asymptotic analysis, starting from the compressible, isothermal Navier-Stokes system. However, the obtained model has a Neumann boundary condition for the pressure or, equivalently, Dirichlet condition for the velocity. Besides the boundary condition, the non-negativity of the solution is imposed as well as the total mass of the fluid in the domain.

The existence of the H^1 solution for such problem was proved in [8] by the limiting procedure. The uniqueness proof was left out. The situation is significantly different from the case of Dirichlet problem and the results from [5], [3],[9] or [4] do not apply.

The goal of the present paper is to prove the uniqueness in case of mono-dimensional problem, i.e.in case when the Reynolds equation becomes an ODE. In such situation, the solution can be constructed and one can see that it can be ir-regular. Indeed, in case when the total mass of the fluid is insufficient, in part of the domain vacuum appears causing the degeneration of the equation and irregularity of the solution. Such solution is only weak (i.e. H^1) but not classical. However, the non-smoothness does not cause the non-uniqueness of the solution.

2.   Statement of the problem and the main result

Let $O \subset \mathbf{R}^n$ be a smooth bounded domain and let $h \in C^1(\overline{O})$ be a strictly positive function describing the shape of the upper (rough) surface of the thin fluid film Ω_ε with thickness ε between two surfaces of the bearing

$$\Omega_\varepsilon = \{(x, x_{n+1}) \in \mathbf{R}^{n+1} \; ; \; x = (x_1, \dots, x_n) \in O \; , \; 0 < x_{n+1} < \varepsilon\, h(x) \; \}.$$

We assume that the velocity of relative motion of two surfaces is $\mathbf{V} \in C^1(\overline{O})^n$. In [8] the following problem for the compressible Navier-Stokes system, related to the gas-lubrication was studied:

$$-\mu\Delta u^\varepsilon - (\lambda + \mu)\nabla(\operatorname{div} u^\varepsilon) + \nabla p^\varepsilon = 0 \; , \; \operatorname{div}(\rho^\varepsilon\, u^\varepsilon) = 0 \; \text{in} \; \Omega_\varepsilon \qquad (1)$$

$$u^\varepsilon = 0 \; \text{for} \; x_{n+1} = \varepsilon\, h(x) \; , \; u^\varepsilon = \mathbf{V} \; \text{for} \; x_{n+1} = 0 \; , \; u^\varepsilon = w_\varepsilon \; \text{on} \; \Gamma_\varepsilon$$

$$p^\varepsilon \geq 0 \; , \; \frac{1}{|\Omega_\varepsilon|}\int_\Omega p^\varepsilon = M_\varepsilon. \qquad (2)$$

The model is isothermal so that the density is proportional to the pressure $p^\varepsilon = a_\varepsilon\, \rho^\varepsilon$. Thus we must impose $p^\varepsilon \geq 0$. The other equation in (2) means that the total mass of the fluid in the domain Ω_ε is prescribed. That condition is necessarily imposed to the compressible stationary Navier-Stokes system in order to avoid trivial, meaningless solutions with zero density.

The main result from [8] can be stated as follows:

Theorem 1. *Assume that the limit* $M = \lim\limits_{\varepsilon \to 0} \varepsilon^2\, M_\varepsilon$ *exists. Let* $U^\varepsilon(x, y) = u^\varepsilon(x, \varepsilon y)$, $P^\varepsilon(x, y) = p^\varepsilon(x, \varepsilon\, y)$. *Then there is a subsequence, denoted again by the same symbol, such that*

$$(U^\varepsilon, \varepsilon^2 P^\varepsilon) \to (U, P) \;\; \text{weakly in} \;\; L^2(\Omega) \qquad (3)$$

where (U, P) is a solution of the compressible Reynolds equations:

$$U = -\frac{1}{2\mu} y (h - y) \nabla_x P + (1 - \frac{y}{h(x)}) \mathbf{V} \tag{4}$$

$$div_x (P \overline{U}) = 0 \quad in \ O \ , \quad P\overline{U} \cdot \mathbf{n} = 0 \ on \ \partial O \tag{5}$$

$$\frac{1}{|\Omega|} \int_\Omega P = M \ , \quad P \geq 0 \tag{6}$$

and $\overline{U}(x) = \int_0^{h(x)} U(x, \xi) \, d\xi$ *and*

$$\Omega = \{(x, y) \in \mathbf{R}^{n+1} \ ; \ x \in O \ , \ 0 < y < \ h(x) \ \}.$$

Here U is the Reynolds velocity and P is the pressure.
The problem (4) can be written in the usual form

$$6\mu\mathrm{div} (\mathbf{V} \ h \ P) = \mathrm{div} (h^3 \ P \nabla P) = \frac{1}{2}\mathrm{div} (h^3 \nabla P^2)$$

$$\mathbf{n} \cdot (h^2 \ P \ \nabla P - 6\mu \ P \ \mathbf{V}) = 0 \ \ on \ \partial O$$

$$\int_O h(x) \ P(x) \, dx = M \int_O h(x) \, dx \ , \quad P \geq 0.$$

In [8] the uniqueness of solution for that problem was not proved. Thus the convergence stated in the theorem remains up to a subsequence. By proving the uniqueness one would actually prove that the whole sequence $(U^\varepsilon, \varepsilon^2 P^\varepsilon)$ converges towards (U, P). Uniqueness results from [5], [3],[9] and [4] are all based on the assumption that the equation does not degerenate, i.e. that the pressure remains strictly positive. Such assumption is reasonable in case of Dirichlet problem and can be deduced from the assumption that the boundary pressure is positive using the monotonicity. However, in our case, as we shall see later, the equation can degenerate if the total mass of the fluid is insufficient. We concentrate here on mono-dimensional case and we use the methods adapted to such situation. We prove that the degeneration of the equation does not cause the non-uniqueness of the solution.

In case $n = 1$ Reynolds equation is an ODE. Taking $O =]0, 1[$ and $\mathbf{V} = V \ \mathbf{e}_1$, it reads

$$6\mu (V \ h \ P)' = (h^3 \ P \ P')' \ , \quad 0 < x < 1 \tag{7}$$

$$h^2 \ P \ P' - 6\mu \ P \ V = 0 \ \ for \ \ x = 0, 1 \tag{8}$$

$$\int_0^1 h(x) \ P(x) \, dx = M \int_0^1 h(x) \, dx \ , \quad P \geq 0. \tag{9}$$

and it can be solved semi-explicitly. Thus, instead of just proving the existence of the solution, we give its construction.

We shall not suppose that V is a constant here, but we assume that it does not change the sign. In fact, we assume that $V \geq 0$. We define the number

$$d = \frac{6\mu \int_0^1 h(t) \int_0^t \frac{V(s) \, ds}{h^2(s)} \, dt}{\int_0^1 h(t)dt} \ .$$

In case $M \geq d$ the solution has a form

$$P(x) = 6\mu \int_0^x \frac{V(s)\,ds}{h^2(s)} + M - d. \tag{10}$$

It is obviously smooth and strictly positive (except in case $M = d$, when $P(0) = 0$).

In case $M < d$ we have a solution

$$P(x) = \begin{cases} 0 \ \text{ for } \ 0 \leq x \leq \overline{x} \\ 6\mu \int_{\overline{x}}^x \frac{V(s)\,ds}{h^2(s)} \ \text{ for } \ x \geq \overline{x} \end{cases} \tag{11}$$

where $\overline{x} \in {]}0, 1{[}$ is the unique solution to the equation (9), i.e. to the equation

$$6\mu \int_{\overline{x}}^1 h(s) \int_{\overline{x}}^s \frac{V(t)\,dt}{h^2(t)}\,ds = M \int_0^1 h(s)\,ds\,. \tag{12}$$

It remains to prove that such point $\overline{x}$ exists and is unique:

Lemma 1. *For $d > M > 0$ nonlinear equation (12) has a unique solution $\overline{x} \in {]}0, 1{[}$.*

Proof. The function

$$F(x) = 6\mu \int_x^1 h(s) \int_x^s \frac{V(t)\,dt}{h^2(t)}\,ds - M \int_0^1 h(s)\,ds$$

is smooth and, for $d > M$,

$$F(1) = -M \int_0^1 h(s)\,ds < 0$$

$$F(0) = 6\mu \int_0^1 h(s) \int_0^s \frac{V(t)\,dt}{h^2(t)}\,ds - M \int_0^1 h(s)\,ds >$$

$$> 6\mu \int_0^1 h(s) \int_0^s \frac{V(t)\,dt}{h^2(t)}\,ds - d \int_0^1 h(s)\,ds = 0.$$

Thus, there exists some $\overline{x} \in [0, 1]$ such that $F(\overline{x}) = 0$, which is equivalent to (12). To prove that $\overline{x}$ is unique it is sufficient to see that F is strictly monotone. Indeed, F is strictly decreasing on ${]}0, 1{[}$, since $F'(x) = -6\mu \frac{V(x)}{h^2(x)} \int_x^1 h(s)\,ds$ $\qquad\qquad \square$

Such solution, given semi-explicitly by (11) and (12), is not smooth and equals zero on an interval, i.e. vacuum appears on part of the domain. Physically, such situation appears when, due to the high pressure, the quantity of the fluid becomes insufficient to fill in the whole domain.

We are going to prove that such solution is the only solution to the problem (7)-(9). Our main result is:

Theorem 2. *The problem (7)-(9) has a unique solution described above.*

Proof. First of all, the problem (7)-(9) by integration reduces to a nonlinear ODE of 1st order

$$P(h^2 P' - 6\mu V) = 0,\ 0 < x < 1\,,\ \int_0^1 h(s)\,P(s)\,ds = M \int_0^1 h(s)\,ds\,,\ P \geq 0. \tag{13}$$

As the solution P is in $H^1([0,1])$, it is continuous. The equation (13) can also be written as

$$\frac{d}{dx} P^2 = 12\mu\, h^{-2}\, V\ P \geq 0\,. \tag{14}$$

Thus P^2 is non-decreasing and, consequently, P is non decreasing. We can now conclude that, if $P(c) = 0$ at some point $c \in [0,1]$, then $P = 0$ for all $x \leq c$. We denote by $\lambda = \sup\{x \in [0,1]\ ;\ P(x) = 0\}$. We notice that $\lambda = 1$ if and only if $M = 0$. Then $P(x) = 0$ on $[0,\lambda]$. If $\lambda = 1$ the proof is finished. Suppose that $\lambda < 1$. On $]\lambda, 1]$ we have $P > 0$. But then, due to the equation (13), we must have

$$h^2 P' = 6\mu V$$

on $]\lambda, 1]$ so that

$$P(x) = 6\mu \int_\lambda^x \frac{V(s)\,ds}{h^2(s)} + C$$

on $]\lambda, 1[$. As we are looking for an H^1 (i.e. continuous) solution, we must have $\lim_{x \to \lambda+} P(x) = 0$ so that we must choose $C = 0$. Thus, all possible solutions of (7) must have the form (11) i.e.

$$P(x) = \begin{cases} 0\ \text{ for }\ 0 \leq x \leq \lambda \\ 6\mu \int_\lambda^x \frac{V(s)\,ds}{h^2(s)}\ \text{ for }\ x \geq \lambda \end{cases}$$

and they can only differ in choice of λ. But since we have prescribed the total mass M, which can be expressed through the equation
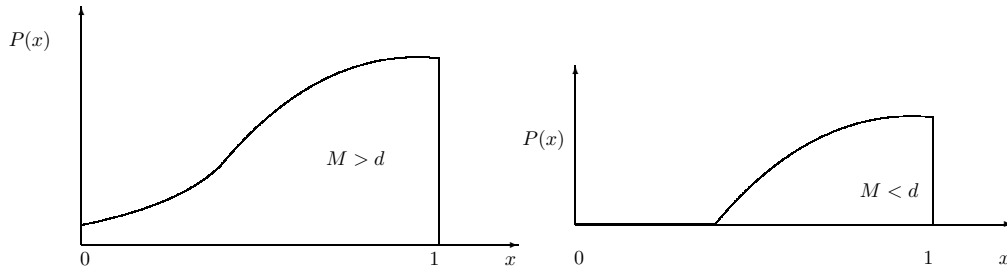
$$\int_0^1 h(s)\, P(s)\,ds = M \int_0^1 h(s)\,ds\,,$$

the number $\lambda \in]0,1[$ is determined by the equation (12), i.e., it must satisfy

$$6\mu \int_\lambda^1 h(s) \int_\lambda^s \frac{V(t)\,dt}{h^2(t)}\,ds = M \int_0^1 h(s)\,ds\,. \tag{15}$$

But, (15) has a unique solution due to the *Lemma 1*. $\qquad\square$

Remark 1. *As we have seen P is not a C^1 function but it's square P^2 is. Indeed, due to the continuity of P, the right-hand side in (14) is continuous. That makes $\frac{d}{dx} P^2$ continuous and P^2 a C^1 function.*

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